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Bertram Ross,* University of New Haven, West Haven, CT, USA
Stefan G. Samko, Rostov State University, Rostov-on-Don, Russia, CIS
E. Russel Love, University of Melbourne, Parkville, Victoria, Australia

FUNCTIONS THAT HAVE NO FIRST ORDER DERIVATIVE MIGHT HAVE FRACTIONAL DERIVATIVES OF ALL ORDERS LESS THAN ONE

A question of classical mathematical analysis — the existence of a continuous non-differentiable function — is treated here in a detailed setting. We consider this question within the framework of fractional derivatives and show that there exist continuous functions $f(x)$ which nowhere have an ordinary first order derivative, but have continuous fractional derivatives of any order $\nu < 1$.

We begin with notation in Section 1 because there exist different forms of fractional integration and differentiation which do not necessarily coincide with each other. We use Riemann-Liouville, Liouville, Marchaud and Weyl fractional differentiation.

In the introductory Section 2 we specify the setting of the problem under consideration and give some comments. Section 3 contains statements of some known results we need. In Section 4 we deal with the Weyl, Liouville and Marchaud fractional derivatives of the well-known continuous but nowhere differentiable Weierstrass function. Section 5 is devoted to the Riemann-Liouville derivatives of this function.

In Section 6 we consider the Riemann-Liouville derivatives of the Riemann function that is almost everywhere nondifferentiable. Section 7 contains some generalizations, an Open Question for further research and Summary of results.

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1 Notation

We shall deal with the following well known (see [11], [9]) forms of fractional integration and differentiation:

a) an integral $\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1} f(t)\,dt,\ x > a$, with $\Re\nu > 0$, is known as the *Riemann-Liouville integral* of order ν. It is applied to functions defined for $x > a$. This integral defines integration to an arbitrary order and is sometimes denoted by the symbol ${}_a\mathrm{D}_x^{-\nu} f(x)$. The subscripts a and x are included in this case to avoid ambiguities in applications. We find it convenient to omit the subscript x on the operator (the operator itself cannot depend on x) and write

$$\mathrm{D}_a^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\,dt,\ \ x > a. \tag{1}$$

An alternative notation for (1) is $I_a^\nu f(x)$.

To define $\mathrm{D}_a^\nu f(x)$ with $\Re\nu > 0$, we write $\nu = m - p$, where m is the least integer greater than $\Re\nu$. We have then

$$\mathrm{D}_a^\nu f(x) = \mathrm{D}^m\,\mathrm{D}_a^{-p} f(x) = \frac{d^m}{dx^m}\frac{1}{\Gamma(p)} \int_a^x (x-t)^{p-1} f(t)\,dt,\ x > a,$$

by definition. If we restrict ν to be in $(0, 1)$, then $m = 1$ and

$$\mathrm{D}_a^\nu f(x) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dx} \int_a^x (x-t)^{-\nu} f(t)\,dt,\ \ x > a \tag{2}$$

b) *Liouville fractional integration* of order $\nu > 0$:

$$I^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_{-\infty}^x (x-t)^{\nu-1} f(t)\,dt,\ \ -\infty < x < \infty, \tag{3}$$

for functions defined on the whole line;

b$'$) the corresponding *Liouville fractional differentiation* of order $0 < \nu < 1$:

$$\mathrm{D}^\nu f(x) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dx} \int_{-\infty}^x \frac{f(t)\,dt}{(x-t)^\nu}. \tag{4}$$

We emphasize that the operators (3) and (4) can be applied to 2π-periodic functions φ (non-vanishing at infinity) if $\int_0^{2\pi} \varphi(x)\,dx = 0$ and the integrals in (3) and (4) are interpreted as conventionally convergent in the following sense:

$$\int_{-\infty}^x \frac{\varphi(t)\,dt}{(x-t)^{1-\nu}} = \lim_{\substack{n\to\infty \\ n=1,2,3,\ldots}} \int_{x-2n\pi}^x \frac{\varphi(t)\,dt}{(x-t)^{1-\nu}} \tag{5}$$

under the obligatory assumption $\int_0^{2\pi} \varphi(t)\,dt = 0$. See details and justification in [11], Section 19, Subsections 2 and 4.

c) *Weyl fractional integration* of order $\nu > 0$ for periodic functions:

$$I^{(\nu)} f(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi^{\nu}(t) f(x-t)\,dt \tag{6}$$

where

$$\Psi^{\nu}(t) = \sum_{n=-\infty}^{\infty}{}' \frac{e^{int}}{(in)^{\nu}} = 2\sum_{n=1}^{\infty} \frac{\cos(nt - \nu\pi/2)}{n^{\nu}}, \tag{7}$$

so that for Fourier series

$$f(t) \sim \sum_{n=-\infty}^{\infty} f_n e^{int} \tag{8}$$

we have

$$I^{(\nu)} f(t) \sim \sum_{n=-\infty}^{\infty}{}' (in)^{-\nu} f_n e^{int} \tag{9}$$

The strokes in (7) and (9) indicate that the term with $n = 0$ is omitted. This is the original definition due to H. Weyl [12]; see details in [16] and [11], section 19;

c$'$) *Weyl fractional differentiation* of order $0 < \nu < 1$ for periodic functions:

$$\mathrm{D}^{(\nu)} f(x) = \frac{d}{dx} I^{(1-\nu)} f(x) = \frac{1}{2\pi}\frac{d}{dx} \int_0^{2\pi} \Psi^{(1-\nu)}(t) f(x-t)\,dt \tag{10}$$

so that for (8) we have

$$\mathrm{D}^{(\nu)} f(t) \sim \sum_{n=-\infty}^{\infty}{}' (in)^{\nu} f_n e^{int} \tag{11}$$

Remark 1 Sometimes the Liouville fractional integral

$$I_{-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^{\infty} (x-t)^{\nu-1} f(t)\,dt \tag{3'}$$

is called the Weyl fractional integral. This is a historical misunderstanding. Liouville was the first who considered fractional integration just in the form

(3′), see [8], page 8, whereas Weyl dealt with periodic functions and the convergence of (3′) for periodic functions should be interpreted very specifically, as in (5). (See details in [11], Section 19). So, though paying homage to Weyl's profound ideas, we consider it more correct to name both (3) and (3′) as Liouville fractional integrals (See also [10], pages xxvii-xxviii in this connection).

d) *Marchaud fractional differentiation* of order $0 < \nu < 1$:

$$\mathrm{D}^{\nu} f(x) = \frac{\nu}{\Gamma(1-\nu)} \int_0^{\infty} \frac{f(x) - f(x-t)}{t^{1+\nu}}\, dt \tag{12}$$

which coincides with (4) for sufficiently good functions (see Lemma 3 below). Observe the use of D instead of D.

2 Introduction

The problem raised in the title may be generalized in the following way:

Problem *To find a continuous function $f(x)$ which has fractional derivatives of all orders $0 < \nu < \nu_0$, but has no derivative of order ν_0 (ν_0 may be either an integer or a non-integer).*

We shall discuss this generalization at the end of the paper, while in the main body of the paper we deal with the case $\nu_0 = 1$.

As we shall see, the solution of this problem depends, in a sense, on the type of fractional differentiation involved. But, what is much more important, it depends on terms under which the "non-existence" of the derivative of order ν_0 is treated. The typical situations for this "non-existence" are the following:

A) The derivative of order ν_0 does not exist at a finite number of points;

B) It does not exist on a set of measure zero;

C) It does not exist almost everywhere;

D) It does not exist at any point.

The case A) is the simplest one. The Problem in this case is in fact close to the following Open Question formulated by Prof. A. Erdélyi at the 1st Conference on Fractional Calculus, 1974, University of New Haven, USA (see [10], page 376):

(i) Let $f(t)$ be continuous for $t \geq a$ and let S be the set of all those non-negative ν for which the fractional derivative $\mathrm{D}_a^{\nu} f$ exists and is *continuous*. Does S have a largest element.

(ii) This problem is the same as that above, except the word "continuous" in italics above is replaced by "*locally integrable*".

The answer to (i) and (ii) is given in [11], page 456, and is in general negative: for $f(x) = x^\beta \ln x$, $\beta > 0$, we have $S = [0, \beta)$ and $S = [0, \beta + 1)$ respectively to the cases of continuity or integrability of $D_a^\nu f$. In the latter case we have $S = [0, \beta + 1)$ for $f(x) = x^\beta$, $\beta > 0$, also.

So, in the case $\nu_0 = 1$, the solution of the problem I treated in the sense A) is immediate: $f(x) = x \ln x$ for $x > 0$, $f(0) = 0$. This function is continuous and even Holderian of any order less than one. It has no derivative at the point $x = 0$ but it has the Riemann-Liouville fractional derivatives $D_0^\nu f$ of orders $\nu < 1$, all of them being continuous everywhere including the point $x = 0$. This follows from the well-known fact that any function $f(x)$ Holderian of order λ and vanishing at $x = a$ has continuous fractional derivatives $D_a^\nu f$ of any order $\nu < \lambda$ (see [7], page 239, Lemma 13.1). This also follows from the direct expression (see [11], page 41, formula (2.50)):

$$D_0^\nu (x \ln x) = \frac{x^{1-\nu}}{\Gamma(1-\nu)} [\psi(1) - \psi(2-\nu) + \ln x] \tag{13}$$

where $\psi(x)$ is the Psi-function.

Similarly, the function $f(x) = \prod_{k=0}^n (x - a_k) \ln |x - a_k|$, $x > a = a_0$ where $a = a_0 < a_1 < \cdots < a_n$, is an example of a continuous function which has no ordinary first order derivative at a finite number of points, but has fractional order derivatives $D_a^\nu f$ of order less than one (as a Lipschitzian function).

The series $f(x) = \sum_{n=1}^\infty \frac{1}{n!}(x - \frac{1}{n}) \ln^+(x - \frac{1}{n})$ where $\ln^+(x - \frac{1}{n}) = \ln(x - \frac{1}{n})$ if $x > \frac{1}{n}$ and $\ln^+(x - \frac{1}{n}) = 0$ if $x \leq \frac{1}{n}$, is an example of a function which has continuous fractional derivatives $D_0^\nu f$ of orders $\nu < 1$, but has no finite first order derivative at a countable set of points $x = 1, 1/2, 1/3, \ldots$ (The approach B)).

So, in the sequel, we shall consider the Problem stated above only in the senses C) and D). For this purpose it seems to be natural to appeal to the classical Riemann and Weierstrass functions:

$$R(x) = \sum_{n=1}^\infty \frac{\cos n^2 x}{n^2}, \tag{14}$$

$$W_\alpha(x) = \sum_{n=0}^\infty q^{-\alpha n} e^{iq^n x}, \tag{15}$$

where $\alpha > 0$ and $q > 1$.

We shall also consider the functions

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \tag{14'}$$

and

$$C(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \tag{14''}$$

which have better behaviour than (15).

It is usually ascribed to Riemann that he considered the function $R(x)$ as nowhere differentiable. It was G.H. Hardy [6], see also [7], who first gave an exact proof of the following assertions:

1. *The function $R(x)$ has no finite derivative at all points x such that $x/2\pi$ is irrational.*

2. *It has no finite derivative for all rational $x/2\pi$ of the form $\frac{x}{\pi} = \frac{2m}{4n+1}$ or $\frac{x}{\pi} = \frac{2m+1}{4n+2}$, m and n being integers.*

We note that the existence of finite first order derivative of the function $R(x)$ at some points was proved by J. Gerver [4], see also [5]. In particular, he showed that $R'(x) = -1/2$ at the points of the form $x = \pi\frac{2m+1}{2n+1}$. He also showed, in addition to Hardy's results, that $R(x)$ has no finite derivative at the points $x = \pi\frac{2m+1}{2^n}$, $n \geq 1$. We also mention the papers [12]-[13] by A. Smith who considered the remaining cases.

As regards the Weierstrass function $W_\alpha(x)$, $0 < \alpha \leq 1$, it is well known that it is continuous (and even satisfying the Holder condition, see Lemma 1 below), but nowhere differentiable if $q > 1$, see [6] or [7].

3 Preliminaries

We say that $f(x)$ satisfies the Holder condition of order λ, $0 < \lambda \leq 1$, on $[a, b]$, $-\infty \leq a < b \leq \infty$, or $f \in H^\lambda([a, b])$ if $|f(x+h) - f(x)| \leq c|h|^\lambda$ with c not depending on h and x; $x, x+h \in [a, b]$. In the case $\lambda = 1$ the function $f(x)$ is also called Lipschitzian.

We will need the following known facts.

Lemma 1 *The Weierstrass function $W_\alpha(x)$, $0 < \alpha \leq 1$, satisfies the Holder condition of order α if $\alpha < 1$ and of any order $\lambda < 1$ if $\alpha = 1$.*

The proof of this statement may be found in [6], page 103, for an arbitrary $q > 1$ and in [15], page 47, Theorem (4.9) and page 44, Theorem (3.4), for an integer q, where it is given for $W_\alpha(x)$ with $\cos(q^n x)$ instead of $\exp(iq^n x)$.

Lemma 2 *Let $f(x)$ be a 2π-periodic function and let $f(x) \in H^\lambda([0, 2\pi])$, $0 < \lambda \leq 1$. Then $f(x)$ has the Weyl fractional derivatives of all orders $\nu < \lambda$ and*

$$\mathrm{D}^{(\nu)} f(x) \in H^{\lambda-\nu}([0, 2\pi]).$$

See the proof in [11], page 365, Corollary of Theorem 19.7.

Lemma 2′ *Let $f(x) \in H^\lambda([a, b])$, $0 < \lambda \leq 1$. Then $f(x)$ has the Riemann-Liouville fractional derivatives of all orders $\nu < \lambda$ and*

$$\mathrm{D}_a^\nu f = \frac{f(a)}{\Gamma(1-\nu)(x-a)^\nu} + \psi(x)$$

where $\psi(x) \in H^{\lambda-\nu}([a, b])$.

See [11], page 239, Lemma 13.1, and page 242, Corollary of Lemma 13.2.

In the case of functions defined on the whole real line the following version of Lemma 2′ holds.

Lemma 2″ *Let $|f(x)| \leq c(1+|x|)^\gamma$, $\gamma < \nu$, and $|f(x+h) - f(x)| \leq A\, h^\lambda$ for all $x \in \mathbb{R}^1$ and $h > 0$. Then $f(x)$ has the Marchaud fractional derivative of any order $\nu < \lambda$ and $|\,\mathrm{D}^\nu f(x+h) - \mathrm{D}^\nu f(x)| \leq B\, h^{\lambda-\nu}$ with $B = \dfrac{2A\lambda}{\lambda-\nu}\dfrac{1}{\Gamma(1-\nu)}$.*

Proof. Since $\gamma < \nu$ and $\lambda > \nu$, the integral (12) converges absolutely. Therefore, $\mathrm{D}^\nu f(x)$ exists. Now,

$$\frac{\Gamma(1-\nu)}{\nu}|\,\mathrm{D}^\nu f(x+h) - \mathrm{D}^\nu f(x)| \leq \int_0^h \frac{|f(x+h) - f(x+h-t)|}{t^{1+\nu}}\,dt$$
$$+ \int_0^h \frac{|f(x) - f(x-t)|}{t^{1+\nu}}\,dt + \int_h^\infty \frac{|f(x+h) - f(x)|}{t^{1+\nu}}\,dt$$
$$+ \int_h^\infty \frac{|f(x+h-t) - f(x-t)|}{t^{1+\nu}}\,dt$$

whence the second assertion of the lemma follows.

Lemma 3 *Let $f(x)$ satisfy the assumptions of Lemma 2. Then all the forms (4), (10) and (12) of fractional differentiation of $f(x)$, (i.e. that of Liouville, Weyl and Marchaud) coincide with each other: $D^\nu f(x) = D^{(\nu)} f(x) = \mathrm{D}^\nu f(x)$, $0 < \nu < 1$.*

See [11], page 358, eq. (19.39).

Lemma 4 *If* $\sum_{n=-\infty}^{\infty} |f_n| < \infty$, *then the Weyl integration (6) can be applied termwise and* $I^{(\nu)} f(x) = \sum_{n=-\infty}^{\prime\infty} (in)^{-\nu} f_n \exp(inx)$.

The proof is easily derived from the definition.

Lemma 5 *Let* $f_k(t) \in C([a,b])$, $k = 1,2,3,\dots, -\infty < a < b < \infty$. *If the series* $\sum_{k=1}^{\infty} f_k(t)$ *converges uniformly on* $[a,b]$, *then* $I_a^\nu(\sum_{k=1}^{\infty} f_k) = \sum_{k=1}^{\infty} I_a^\nu f_k$ *and the series in the right hand side converges uniformly on* $[a,b]$ *as well.*

PROOF. We have

$$\Delta \stackrel{\text{def}}{=} \left| I_a^\nu f - \sum_{k=1}^{m} I_a^\nu (f_k) \right| \le \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \left| \sum_{k=m+1}^{\infty} f_k(t) \right| dt.$$

Hence, if $\left|\sum_{k=m+1}^{\infty} f_k(t)\right| < \varepsilon$, we have $\Delta < \frac{(b-a)^\nu}{\nu\Gamma(\nu)}\varepsilon$, which proves the lemma.

Lemma 6 *Let* $g(x) = \int_a^x f(t)\,dt$. *If* $f(t) \in L^p([a,b])$, $-\infty \le a < b \le \infty$, *for all* $p > 1$, *then* $|g(x+h) - g(x)| \le c|h|^{1-\varepsilon}$ *for all* x, $x+h \in [a,b]$, *however small* $\varepsilon > 0$ *is,* $c = c(p)$ *not depending on* x *and* h.

Proof is obvious.

We shall also need the following entire function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \tag{16}$$

known as the Mittag-Leffler function [2], page 210. It is known that

$$\frac{d}{dz} E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) + (1-\beta) E_{\alpha,\beta}(z)}{az} \tag{17}$$

The function $E_{1,\beta}(z)$ has the representation

$$E_{1,\beta}(\lambda x) = \frac{x^{1-\beta} e^{\lambda x}}{\Gamma(\beta - 1)} \int_0^x t^{\beta-2} e^{-\lambda t}\, dt, \ \beta > 1, \tag{18}$$

where $x > 0$ and λ may be complex, which may be verified by the termwise integration of the series expansion of $e^{\lambda t}$. By analytic continuation, we can derive the analogous representation for $0 < \beta \le 1$:

$$E_{1,\beta}(\lambda x) = \frac{e^{\lambda x}}{\Gamma(\beta)} + \frac{1-\beta}{\Gamma(\beta)} x^{1-\beta} e^{\lambda x} \int_0^x t^{\beta-2}(1 - e^{-\lambda t})\, dt, \ 0 < \beta < 1. \tag{19}$$

It is known that the Riemann-Liouville fractional integral of an exponential function can be expressed in terms of the Mittag-Leffler function $E_{1,\beta}$ ([11], page 173, eq.8):

$$I_a^\nu(e^{\lambda x}) = e^{\lambda a}(x-a)^\nu E_{1,1+\nu}(\lambda x - \lambda a), \quad x > a, \tag{20}$$

which is, in fact, the reformulation of (18).

4 The Weierstrass function and its Weyl, Liouville or Marchaud fractional derivatives

We begin with the consideration of the Marchaud derivative of the Weierstrass function $W_1(x)$ (=Weyl or Liouville derivative in the case of an integer q). In the next section we shall deal with the Riemann-Liouville derivative $\mathrm{D}_a^\nu W_1(x), \ x > a$, too. However, we would like to emphasize that for a function defined and studied on the whole line it is more natural to investigate its Liouville, Weyl or Marchaud derivative than that of Riemann-Liouville which is "tied" to a fixed point $x = a$.

Theorem 1 *For any $q > 1$ the Weierstrass function $W_1(x)$ has the Marchaud fractional derivative of any order $\nu < 1$:*

$$\mathrm{D}^\nu W_1(x) = i^\nu \sum_{n=0}^{\infty} q^{-n(1-\nu)} e^{iq^n x} \tag{21}$$

which is continuous and even satisfies the Holder condition of any order $\lambda < 1-\nu$, but $W_1(x)$ nowhere has the first order derivative. In the case of integer $q = 2, 3, 4, \dots$ the function $W_1(x)$ has also Liouville and Weyl derivatives (4) and (10) which coincide with (21).

PROOF. The fact that $\frac{d}{dx} W_1(x)$ exists nowhere is well known. As for the fractional derivatives we consider first the case of an integer q.

I. The case of an integer q. In this case the proof is simpler because $W_1(x)$ proves to be a 2π-periodic function. We consider the Weyl derivative. By the definition (10) and Lemma 4 we have $\mathrm{D}^{(\nu)} W_1(x) = \frac{d}{dx} I^{(1-\nu)} W_1(x) = i^{\nu-1} \frac{d}{dx} \sum_{n=0}^{\infty} q^{-n(2-\nu)} \exp(iq^n x)$. Here the termwise differentiation is possible since the series obtained after differentiation converges absolutely and uniformly. Therefore, we arrive at (21). From (21) we see that $\mathrm{D}^{(\nu)} W_1(x)$ is a bounded continuous function as a sum of uniformly convergent series. To show that

$$\mathrm{D}^{(\nu)} W_1(x) \in H^\lambda, \ \ \lambda < 1-\nu, \tag{22}$$

we refer to the 2π-periodocity of $W_1(x)$ and to the fact that $W_1(x) \in H^{1-\varepsilon}$ with an arbitrary small $\varepsilon > 0$, by Lemma 1. Then, by Lemma 2 the assertion (22) holds. It remains to refer to Lemma 3.

II. The case of an arbitrary q > 1. To manage with this case we use the Marchaud derivative (12) and consider the following truncated Marchaud derivative: $\mathrm{D}^{\nu}_{\varepsilon,N} f = \frac{\nu}{\Gamma(1-\nu)} \int_{\varepsilon}^{N} \frac{f(x)-f(x-t)}{t^{1+\nu}}\, dt$. We have

$$\mathrm{D}^{\nu}_{\varepsilon,N} W_1(x) = \frac{\nu}{\Gamma(1-\nu)} \sum_{n=0}^{\infty} q^{-n} e^{iq^n x} \int_{\varepsilon}^{N} \frac{1-e^{iq^n t}}{t^{1+\nu}}\, dt. \tag{23}$$

(The termwise integration in obtaining (23) is obviously justified). We have the estimate:

$$\int_{\varepsilon}^{N} \frac{|1-\exp(-iq^n t)|}{t^{1+\nu}}\, dt \le c q^{n\nu} \tag{24}$$

where c does not depend on ε, N and n. Really,

$$\int_{\varepsilon}^{N} \frac{|1-\exp(-iq^n t)|}{t^{1+\nu}}\, dt \le \int_0^{\infty} \frac{|1-\cos q^n t|}{t^{1+\nu}}\, dt + \int_0^{\infty} \frac{|\sin q^n t|}{t^{1+\nu}}\, dt$$

Hence, the change of the variable $q^n t = u$ yields (25) with

$$c = \int_0^{\infty} [1-\cos u + |\sin u|] u^{-1-\nu} du.$$

Since the series $\sum_{n=0}^{\infty} q^{-n} q^{n\nu}$ converges, by (24) we may pass to the limit in (23) termwise, which gives

$$\lim_{\substack{\varepsilon \Rightarrow 0 \\ N \Rightarrow \infty}} \mathrm{D}^{\nu}_{\varepsilon,N} W_1(x) = \frac{\nu}{\Gamma(1-\nu)} \sum_{n=0}^{\infty} q^{-n} e^{iq^n x} \int_0^{\infty} \frac{1-\exp(-iq^n t)}{t^{1+\nu}}\, dt. \tag{25}$$

We use the formula

$$\int_0^{\infty} \frac{1-\exp(-ixt)}{t^{1+\nu}}\, dt = \frac{\Gamma(1-\nu)}{\nu} (ix)^{\nu}, \quad 0 < \nu < 1, \tag{26}$$

which is easily obtained by integration by parts and using the known relation $\int_0^{\infty} t^{\nu-1} e^{-ixt}\, dt = \Gamma(\nu)/(-ix)^{\nu}$, see e.g. [11], page 138, eq.(7.6). By (26) and(25) we have $\mathrm{D}^{\nu} W_1(x) = \sum_{n=0}^{\infty} q^{-n} (iq^n)^{\nu} \exp(iq^n x)$ which coincides with (21). The Holder property for $\mathrm{D}^{\nu} W_1$ follows from Lemma 1 and Lemma 2$''$. The theorem is proved.

Remark 2 *We remind that in Theorem 1 the Liouville derivative (4) should be interpreted in accordance with (5).*

5 Riemann-Liouville derivatives of the Weierstrass function

The Riemann-Liouville derivative (2) even of a very good function $f(x)$ is infinite at the point $x = a$, if $f(a) \neq 0$. By this reason, considering the Riemann-Liouville derivative on $[a, \infty)$, we shall deal with the function

$$W(x) = W_1(x) - W_1(a) = \sum_{n=0}^{\infty} q^{-n}(e^{iq^n x} - e^{iq^n a}) \tag{27}$$

In the theorem below $E_{1,1-\nu}(x)$ is the Mittag-Leffler function, see (16).

Theorem 2 *The Weierstrass function $W(x)$, $x > a$, has continuous and bounded fractional Riemann-Liouville derivatives of all orders $\nu < 1$, which can be calculated by the formula $\mathrm{D}_a^\nu W(x) = \sum_{n=0}^{\infty} q^{-n} A_n(x)$, where*

$$A_n(x) = (x-a)^{-\nu} \exp(iaq^n) \left[E_{1,1-\nu}(iq^n(x-a)) - \frac{1}{\Gamma(1-\nu)}\right] \tag{28}$$

or

$$A_n(x) = \frac{\exp(iq^n x)}{\Gamma(1-\nu)} \left[\frac{1-\exp(-iq^n(x-a))}{(x-a)^n} + \nu q^{n\nu} \int_0^{q^n(x-a)} \frac{1-e^{-it}}{t^{1+\nu}} dt\right] \tag{29}$$

so that

$$|A_n(x)| \leq c q^{\nu n} \tag{30}$$

with c not depending on x and n.

PROOF. Since the series in (27) converges uniformly on $[a, \infty)$, by Lemma 5 we have $I_a^{1-\nu} W(x) = \sum_{n=0}^{\infty} q^{-n} I_a^{1-\nu}(e^{iq^n x} - e^{iq^n a})$. Hence, by (20) we obtain

$$I_a^{1-\nu} W(x) = (x-a)^{1-\nu} \sum_{n=0}^{\infty} q^{-n} e^{iaq^n} \left[E_{1,2-\nu}(iq^n(x-a)) - \frac{1}{\Gamma(2-\nu)}\right]. \tag{31}$$

The formal termwise differentiation of (31) yields

$$\frac{d}{dx} I_a^{1-\nu} W(x) = \sum_{n=0}^{\infty} q^{-n} A_n(x) \tag{32}$$

with

$$A_n(x) = e^{iaq^n} \frac{d}{dx} \left\{(x-a)^{1-\nu} \left[E_{1,2-\nu}(iq^n(x-a)) - \frac{1}{\Gamma(2-\nu)}\right]\right\}. \tag{33}$$

To justify the differentiation we shall prove that the series in the right hand side of (32) converges uniformly. Differentiating in (33) and applying the formula (17), we arrive at (28) after easy calculations. Applying the representation (19) in the right hand side of (28), we obtain (29).

Since the integral $\int_0^\infty t^{-1-\nu}|1-\exp(-it)|\,dt$ converges, the estimate (30) follows from (29). By (30) the series (32) converges uniformly and then $D_a^\nu W(x)$ is continuous and bounded. The theorem is proved.

6 Fractional derivatives of the functions (14′) and (14″) and of the Riemann function

I. Functions (14′) and (14″). We consider the functions $S(x)$ and $C(x)$ first. The function (14″) is known ([3], page 433) as an elementary function:

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{x^2}{4}-\frac{\pi}{2}|x|+\frac{\pi^2}{6},\quad -2\pi\le x\le 2\pi, \tag{34}$$

which may be obtained either by direct expansion of the right-hand side into Fourier series or by the integration of the well known relation $\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi-x}{2}$, for $0<x<2\pi$ (see [15], page 5), with the formula $\sum_{n=1}^{\infty}n^{-2}=\pi^2/6$ taken into account. We also remark that (34) may be rewritten as

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\pi^2B_2(\frac{x}{2\pi}),\quad 0\le x\le 2\pi, \tag{35}$$

where $B_2(x)$ is the Bernoulli polynomial of second degree. We remark that the relations (34)-(35) are particular cases of Fourier series expansions for the Bernoulli polynomials:

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^m}=(-1)^{-1+m/2}\frac{(2\pi)^m}{m!}B_m(\frac{x}{2\pi}),\ \text{if } m \text{ is even}$$

and

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^m}=(-1)^{-1+[m/2]}\frac{(2\pi)^m}{m!}B_m(\frac{x}{2\pi}),\ \text{if } m \text{ is odd,}$$

see e.g. [11], page 348, eq.(19.8) and (19.10).

As regards the function (14′), it has the following integral representation:

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}=\int_0^x\log\frac{1}{2|\sin t/2|}\,dt,\quad 0\le x\le 2\pi, \tag{36}$$

which follows from the known relation $\sum_{n=1}^{\infty}\frac{\cos nt}{n}=-\log 2|\sin t/2|$, $0<t<2\pi$, see [15], page 5.

Theorem 3 *Functions $C(x)$ and $S(x)$ have the Riemann-Liouville fractional derivatives $\mathrm{D}_0^\nu C(x)$ and $\mathrm{D}_0^\nu S(x)$ of any order $0 < \nu < 1$, which are continuous for all $x > 0$ and $x \geq 0$, respectively. For $0 < x < 2\pi$ they may be calculated as*

$$\mathrm{D}_0^\nu \left(\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \right) = \frac{1}{6\Gamma(3-\nu)} \frac{P_2(x)}{x^\nu} \tag{37}$$

with $P_2(x) = 3x^2 - 3\pi(2-\nu)x + \pi^2(2-\nu)(1-\nu)$, and

$$\mathrm{D}_0^\nu \left(\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \right) = -\frac{1}{\Gamma(1-\nu)} \int_0^x \frac{\log|2\sin t/2|}{(x-t)^\nu} dt. \tag{38}$$

The first order derivatives of $C(x)$ and $S(x)$ exist and are continuous for all $x > 0$ except the points $x = 2m\pi$, $m = 1, 2, 3, \ldots$

PROOF. The existence of the first order derivatives is seen directly from (34) and (36); in particular, we see from (36) that $\frac{d}{dx} S(x) = -\log 2|\sin x/2|$ does not exist at the points $x = 2m\pi$.

To consider the fractional derivatives we begin with the values $0 < x < 2\pi$. From (34) we have $\mathrm{D}_0^\nu C(x) = \frac{1}{4}\mathrm{D}_0^\nu(x^2) - \frac{\pi}{2}\mathrm{D}_0^\nu(x) + \frac{\pi^2}{6}\mathrm{D}_0^\nu(1)$ which yields (37) and proves the continuity of $\mathrm{D}_0^\nu C(x)$ for $0 < x \leq 2\pi$. As for $\mathrm{D}_0^\nu S(x)$, we have from (36)

$$\mathrm{D}_0^\nu S(x) = \mathrm{D}_0^\nu I_0^1 f(x) \tag{39}$$

where $f(x) = -\log 2|\sin t/2|$ and $I_0^1 f(x) = \int_0^x f(t)\,dt$. By the index law we have from (39): $\mathrm{D}_0^\nu S(x) = I_0^{1-\nu} f(x)$ which coincides with (38).

It remains to make sure that $\mathrm{D}_0^\nu S(x)$ and $\mathrm{D}_0^\nu C(x)$ are continuous for all x. We remark that $C(x)$ is a continuous periodic function. So, by (34) it is a continuous piece-wise differentiable function and, therefore, Lipschitzian. Then, by Lemma 2′ its fractional derivative $\mathrm{D}_0^\nu S(x)$ is continuous (and even Holderian or order $1-\nu$) beyond the point $x = 0$.

Now, by (36) and Lemma 6 the function $S(x)$ is Holderian of order $1 - \varepsilon$, $\varepsilon > 0$. Since $S(0) = 0$, by Lemma 2′ we have $\mathrm{D}_0^\nu f \in H^{1-\varepsilon-\nu}$, $0 < \varepsilon < 1-\nu$. The theorem is proved.

Remark 3 *As follows from the proof of Theorem 3, the fractional derivatives of the functions $C(x) - \pi^2/6$ and $S(x)$ are not only continuous, but even Holderian: $\mathrm{D}_0^\nu C(x) - \frac{\pi^2}{6\Gamma(1-\nu)x^\nu} \in H^{1-\nu}([0,b])$, $\mathrm{D}_0^\nu S(x) \in H^{1-\nu-\varepsilon}([0,b])$, whatever small ε is, $0 < \varepsilon < 1-\nu$; $b > 0$.*

II. Riemann's function. The case of the function (15) is more difficult and we can manage here only with orders $0 < \nu < 1/2$ of fractional differentiation. The following theorem is valid.

Theorem 4 *Let $0 < \nu < 1/2$. The fractional derivative $\mathrm{D}^{\nu} R(x)$ of Liouville, that of Weyl $\mathrm{D}^{(\nu)} R(x)$ and that of Marchaud $\mathrm{D}^{\nu} R(x)$ do exist, are continuous and coincide with each other. They may be calculated by the formula*

$$\mathrm{D}^{\nu} R(x) = \sum_{n=1}^{\infty} \frac{\cos(n^2 x + \nu\pi/2)}{n^{2-2\nu}}. \tag{40}$$

However, if $\nu > 3/4$, the Weyl derivative $\mathrm{D}^{(\nu)} R(x)$ does not exist at least for all irrational values of x/π.

PROOF. We shall show that

$$R(x) \in H^{1/2}([0, 2\pi]). \tag{41}$$

Following [15], page 47, proof of Theorem (4.9), we have

$$\begin{aligned} |R(x+h) - R(x)| &= 2\left|\sum_{n=1}^{\infty} \frac{\sin(n^2 x + n^2 h/2 + \nu\pi/2)\sin(n^2 h/2)}{n^2}\right| \\ &= 2\left|\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right| = 2|P+Q| \end{aligned} \tag{42}$$

where $N = [(2/h)^{1/2}]$ is the largest integer such that $N^2 h/2 \le 1$, $h > 0$. So,

$$|P| \le \sum_{n=1}^{N} \frac{1}{n^2}\frac{n^2 h}{2} = Nh/2 \le (h/2)^{1/2} \tag{43}$$

and

$$|Q| \le \sum_{n=N+1}^{\infty} \frac{1}{n^2} \le \frac{1}{N} = \frac{1}{[(2/h)^{1/2}]}.$$

Taking $h \le 1/6$, we have $[(2/h)^{1/2}] \ge (2/h)^{1/2} - 1 \ge (1/h)^{1/2}$, so that

$$|Q| \le h^{1/2}. \tag{44}$$

Therefore, (43) and (44) prove (41) via (42).

In view of (41) and Lemmas 2 and 3 we see that the three fractional derivatives exist for $\nu < 1/2$ and are equal. To prove (40) we remark that for the real form of Fourier series $f(x) \sim \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ the Weyl definition (11) of a fractional derivative is the following:

$$\mathrm{D}^{(\nu)} f(x) \sim \sum_{n=1}^{\infty} n^{\nu}[a_n \cos(nx + \nu\pi/2) + b_n \sin(nx + \nu\pi/2)] \tag{45}$$

which is easily derived from (11). So

$$\mathrm{D}^{(\nu)} \left(\sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^2} \right) \sim \sum_{n=1}^{\infty} \frac{\cos(n^2 x + \nu\pi/2)}{n^{2-2\nu}}. \tag{46}$$

Since the series in (46) converges absolutely and uniformly (for $0 < \nu < 1/2$), the sign $\sim$ in (46) may be replaced by $=$ and we arrive at (40).

To show that $\mathrm{D}^{(\nu)} R(x)$ does not exist almost everywhere if $\nu > 3/4$, we refer to the following result due to G.H. Hardy ([6], page 323):

Neither of the functions $\sum_{n=1}^{\infty} \frac{\cos n^2 x}{n^{\beta}}$, $\sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^{\beta}}$, *where* $\beta < 5/2$, *is differentiable for any irrational multiple of* π.

By (46) $I^{(1-\nu)} R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x + \nu\pi/2)}{n^{4-2\nu}}$. So, $\mathrm{D}^{(\nu)} R(x) = \frac{d}{dx} I^{(1-\nu)} R(x)$ does not exist at irrational multiples of π by the above Hardy's result since $4 - 2\nu < 5/2$ if $\nu > 3/4$. The theorem is proved.

7 On Problem stated in Section 2.

Here we discuss shortly the Problem formulated in Section 2 for arbitrary ν_0, not necessarily $\nu_0 = 1$. This is just the Weierstrass function $W_{v_0}(x)$ defined in (15) which is a solution of this problem. The theorem below generalizes Theorem 1. We use here the standard extension of fractional derivatives (4), (10) and (12) to the case of orders $\nu > 1$, see [11], page 95, 348 and 118, respectively. When ν is an integer, either of these derivatives is an ordinary derivative of order ν.

Theorem 5 *For any* $q > 1$ *the Weierstrass function* $W_{\nu_0}(x)$, $\nu_0 > 0$, *has the Marchaud and Liouville (and Weyl, if q is an integre) fractional derivative of any order* $\nu < \nu_0$. *They coincide with each other and are equal to*

$$\mathrm{D}^{\nu} W_{\nu_0}(x) = i^{\nu} \sum_{n=0}^{\infty} q^{-n(\nu_0 - \nu)} \exp(iq^n x). \tag{47}$$

Besides, $\mathrm{D}^{\nu} W_{\nu_0}(x)$ *is Holderian oforder* $\lambda < \nu_0 - \nu$. *However,* $W_{\nu_0}(x)$ *nowhere has the Liouville fractional derivative of order* ν_0.

PROOF. The assertion on the existence of the derivative of order $\nu < \nu_0$ and the relation (47) are obtained following the same lines as those of Theorem 1. We remark that (47) means that

$$\mathrm{D}^{\nu} W_{\nu_0}(x) = i^{\nu} W_{\nu_0 - \nu}(x) \tag{48}$$

for all $-\infty < \nu < \nu_0$, $0 < \nu_0 < \infty$, the series in the right-hand side being absolutely convergent. To demonstrate that $W_{\nu_0}(x)$ nowhere has fractional Liouville derivative of order ν_0, we remark that $\mathrm{D}^{\nu_0} = \frac{d}{dx}\mathrm{D}^{\nu_0 - 1}$ (independently of the sign of $\nu_0 - 1$). Then, $\mathrm{D}^{\nu_0} W_{\nu_0}(x) = \frac{d}{dx}\mathrm{D}^{\nu_0-1} W_{\nu_0}(x)$, if exists, should coincide, by (48), with $i^{\nu_0 - 1}\frac{d}{dx}W_1(x)$, which is impossible.

Remark 4 *The results similar to those stated in this paper for the Weierstrass function (15) are valid for its real valued version* $W_\alpha(x) = \sum_{n=0}^{\infty} q^{-\alpha n}\cos(q^n x)$

Remark 5 *The following generalization*

$$\sum_{n=0}^{\infty} \varepsilon_k a^k \exp(ib^k x), \;\; \varepsilon_k = \pm 1, \tag{49}$$

of the Weierstrass function, with some restrictions on a and b, can be similarly studied. (See [1], page 361, concerning the nowhere differentiability of the function (49)).

Finally, we put as open the following question inspired by Theorem 4.

Open question. *Does the Riemann function* $\sum_{n=1}^{\infty}\frac{\cos n^2 x}{n^2}$ *have continuous fractional derivatives of order* $1/2 \le \nu \le 3/4$*?*

For reader's convenience we give the following

Summary of results

1. *The Weierstrass function $W_1(x)$ has continuous Liouville fractional derivatives $D^\nu W_1$ of any order $\nu < 1$, but nowhere has the first order derivative.*

2. *The Weierstrass function $W_1(x) - W_1(a)$ for all $x \ge a$ has continuous Riemann-Liouville fractional derivatives $D_a^\nu[W_1(x) - W_1(a)]$ of any order $\nu < 1$, but nowhere has the first order derivative.*

3. *The functions $C(x)$ and $S(x)$ have continuous Riemann-Liouville fractional derivatives $D_0^\nu C(x)$ and $D_0^\nu S(x)$, $x > 0$, while the first order derivatives do not exist at points $x = 2m\pi$.*

4. *Riemann's function $R(x)$ has continuous Liouville fractional derivatives of any order $\nu < 1/2$, but fractional derivatives of order $\nu > 3/4$ do not exist almost everywhere.*

5. *The Weierstrass function* $W_{\nu_0}(x)$, $\nu > 0$, *has continuous Liouville fractional derivatives of any order* $\nu < \nu_0$, *but nowhere has the derivative of order* ν_0.

The above results are given in Theorems 1-5, respectively, where more detailed statements can also be found.

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