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Some Remarks to the author's paper
A new approach to the inversion of the Riesz potential operator,
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In the paper [1] , we gave a new formula for the inversion of the Riesz potential operator

$$I^\alpha \varphi = \frac{1}{\gamma_n(\alpha)} \int_{R^n} \frac{\varphi(y)\ dy}{|x-y|^{n-\alpha}} \ , \quad x \in R^n \ ,$$

$\gamma_n(\alpha)$ being the well known normalizing constant. The following theorem was proved.

Theorem A. *Let* $0 < \Re\alpha < 2m, \ m = 1, 2, ... \ , \alpha \neq 2, 4, 6, ...$ *Then the inversion of the Riesz potential operator* $f = I^\alpha \varphi, \varphi \in L_p(R^n), 1 \leq p < \frac{n}{\Re\alpha}$, *can be written in the form*

$$\varphi(x) = \frac{1}{\gamma_n(-\alpha)} \lim_{\varepsilon \to 0} \int_{R^n} \left[\frac{1}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}}} - \varepsilon A(y, \varepsilon) \right] f(x-y) dy \tag{1}$$

where

$$A(y, \epsilon) = \sum_{k=1}^{m} (-1)^{k-1} \frac{c_{m,k} \epsilon^{k-1}}{(|y|^2 + \epsilon^2)^{\frac{n+\alpha}{2}+k}}$$

with $c_{m,k} = \binom{m}{k} \frac{\left(\frac{n+1}{2}\right)_k}{\left(\frac{\alpha}{2}-m+1\right)_k}$. *The limit in (1) exists in the usual sense, for "nice" functions* $f(x)$, *and in the sense of* L_p*-convergence or almost everywhere, if* $f \in I^\alpha(L_p)$.

In particular, in the case $0 < \Re\alpha < 2$ *the inversion of the Riesz potential operator* $I^\alpha \varphi$ *may be taken in the form*

$$\varphi(x) = \frac{1}{\gamma_n(-\alpha)} \lim_{\varepsilon \to 0} \int_{R^n} \left[\frac{1}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}}} - \frac{n+\alpha}{\alpha} \frac{\varepsilon}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}+1}} \right] f(x-y) dy \ . \tag{2}$$

We use this opportunity to note a misprint in the formula (7.21) in [1]: the factor $\frac{n+1}{\alpha}$ there should be replaced by $\frac{n+\alpha}{\alpha}$, as in (2).

We wish to prove the following theorem.

Theorem. *The formula (1) is nothing else, but*

$$\varphi(x) = \frac{(-1)^m}{\gamma_n(2m-\alpha)} \lim_{\varepsilon \to 0} \int_{R^n} \Delta^m \left(\frac{1}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}-m}} \right) f(x-y) dy \tag{3}$$

where Δ *is the Laplace operator.*

Proof. We use notation from [1]. The formula (1) was obtained in [1] as a realization of the general inversion formula

$$\varphi(x) = \lim_{\substack{\epsilon\to 0 \\ (L_p)}} \frac{1}{\epsilon^{n+\alpha}} \int_{R^n} q_\alpha\left(\frac{y}{\epsilon}\right) f(x-y)dy \ , \tag{4}$$

given in [1], under the choice

$$q_\alpha(x) = \frac{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(m-\frac{\alpha}{2}\right)} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{\Gamma\left(\frac{n+\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}-m+k+1\right)} \frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}+k}} \ , \tag{5}$$

see (7.19) in [1]. Therefore, the only point we have to prove is that the function (5) is nothing else but

$$q_\alpha(x) = \frac{(-1)^m}{\gamma_n(2m-\alpha)} \Delta^m \left(\frac{1}{(|x|^2+1)^{\frac{n+\alpha}{2}-m}} \right), \quad x \in R^n. \tag{6}$$

In view of the relations (7.18) and (7.6) from [1] we have

$$\widehat{q}_\alpha(\xi) = \lambda(-1)^m a_m(\alpha)|\xi|^\alpha G_{n+2m-\alpha}(\xi) = \frac{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n-\alpha}{2}-m\right)}{\Gamma\left(m-\frac{\alpha}{2}\right)} |\xi|^{2m} \frac{G_{n+2m-\alpha}(\xi)}{|\xi|^{2m-\alpha}} \ , \ \xi \in R^n,$$

$G_\alpha(\xi)$ being the Fourier transform of the function $(1+|x|^2)^{-\frac{\alpha}{2}}$. Since

$$G_\alpha(\xi) = c_\alpha \frac{K_{\frac{n-\alpha}{2}}(|\xi|)}{|\xi|^{\frac{n-\alpha}{2}}} \quad \text{with} \quad c_\alpha = \frac{2^{1-\frac{\alpha+n}{2}}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \ , \tag{7}$$

where $K_\nu(r)$ is the McDonald function, we get

$$\widehat{q}_\alpha(\xi) = \frac{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n-\alpha}{2}-m\right)}{\Gamma\left(m-\frac{\alpha}{2}\right)} c_{n+2m-\alpha} \ |\xi|^{2m} \frac{K_{\frac{\alpha}{2}-m}(|\xi|)}{|\xi|^{\frac{m-\alpha}{2}}}$$

or

$$\widehat{q}_\alpha(\xi) = \frac{2^{1-m-\frac{\alpha}{2}}}{\Gamma\left(m-\frac{\alpha}{2}\right)} |\xi|^{2m} \frac{K_{m-\frac{\alpha}{2}}(|\xi|)}{|\xi|^{\frac{m-\alpha}{2}}}$$

because of the property $K_\nu(r) = K_{-\nu}(r)$ of the McDonald function.

Applying now the relation (7) in the reverse order with α replaced by $n-2m+\alpha$, we arrive at

$$\widehat{q}_\alpha(\xi) = \frac{2^{1-m-\frac{\alpha}{2}}}{\Gamma\left(m-\frac{\alpha}{2}\right) c_{n-2m+\alpha}} |\xi|^{2m} G_{n-2m+\alpha}(x) \ .$$

After easy calculation of the constant factor here and the passage to Fourier pre-images, we arrive at the formula (6), which was required.

Remark. The formula (3) is suggestive in the sense that it provides a clear idea of the direct realization of the approximative inverses construction to many potential type operators, especially to those which are negative fractional powers of differential operators.

In other words, the formula (2) prompts another idea to construct effectively fractional powers of differential operators.

Let us consider two examples: the Bessel potential operator and the heat parabolic operator. The first is defined as

$$B^\alpha \varphi = \int_{R^n} G_\alpha(x-y)\varphi(y)dy,$$

$G_\alpha(x)$ being the kernel (5), while the second is known to be defined as

$$(H^\alpha \varphi)(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_{R_+^{n+1}} \tau^{\frac{\alpha}{2}-1} W(y,\tau)\varphi(x-y,t-\tau)dyd\tau \ ,$$

(see f.e. [2], Sections 27 and 28), where $R_+^{n+1} = \{(x,t) : x \in R^n, t \in R_+^1\}$ and $W(y,\tau) = (4\pi\tau)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\tau}}$ is the Gauss-Weierstrass kernel.

The inversion of the Bessel potential operator $B^\alpha \varphi = f$ and that of the heat fractional operator $H^\alpha \varphi = f$ may be given in the form

$$\varphi(x) = \lim_{\varepsilon \to 0} \int_{R^n} B(y,\varepsilon) f(x-y)dy \tag{8}$$

and

$$\varphi(x,t) = \lim_{\varepsilon \to +0} \int_{R_+^{n+1}} C(y,\tau,\varepsilon) f(x-y,t-\tau)dyd\tau \ , \tag{9}$$

respectively, where

$$B(y,\varepsilon) = (I-\Delta)^m \left(\frac{|y|^{n+\alpha-2m} G_{2m-\alpha}(y)}{(|y|^2+\varepsilon^2)^{\frac{n+\alpha}{2}-m}} \right) \text{ and } C(y,\tau,\varepsilon) = \left(\frac{\partial}{\partial \tau} - \Delta_y \right)^m \left(\frac{\tau^{1+\frac{\alpha}{2}} W(y,\tau)}{(\tau+\varepsilon)^{1+\frac{\alpha}{2}-m}} \right), \tag{10}$$

Δ_y being the Laplace operator applied in the space variable $y \in R^n$.

One can now write down easily similar formulas for fractional powers of the wave operator, for that of the Schrödinger operator and so on.

The formulas (8)-(10) are easily justified on nice functions (f.e. on functions from the corresponding Lizorkin-type test function spaces). Their justification within the framework of the L_p-spaces will be given in another paper.

Bibliography

[1] Samko, S.G. A new approach to the inversion of the Riesz potential operator. *Fract. Calc. and Appl. Anal.* , vol. 1, no 3 (1998), 225-245.

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