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ON EQUIVALENT NORMS IN FRACTIONAL ORDER FUNCTION SPACES OF CONTINUOUS FUNCTIONS ON THE UNIT SPHERE

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Dedicated to Rudolf Gorenflo,
Professor Emeritus of the Free University of Berlin,
on the occasion of his 70-th birthday

Abstract

The detailed consideration of fractional order function spaces of the type $C^{\lambda}(S^{n-1})$ and $H^{\lambda}(S^{n-1}), \lambda > 0$, is presented. The former is defined in terms of fractional differentiation on the base of the space $C(S^{n-1})$, the latter is the Hölder type space. The averaged type Hölder spaces $H^{\lambda}(S^{n-1})$ are also dealt with. It is shown that that in case when $\lambda \geq 1$, all these spaces coincide, or on the contrary are different, depending on the fact what kind of definition of the differentiation of integer order is used: in terms invariant with respect to rotations or in terms connected with differentiation with respect to Cartesian coordinates

Key Words and Phrases: function spaces on unit sphere, fractional smoothness, Hölder spaces, spherical harmonics, spherical multipliers, potential operators, singular operators

1 Introduction

The main goal of the paper is to compare different approaches to function spaces of fractional smoothness on the unit sphere, constructed on the base of the space $C(S^{n-1}),\ n \geq 2$. The spaces $C^{\lambda}(S^{n-1})$ and the Hölder type spaces $H^{\lambda}(S^{n-1}), \lambda > 0$, are considered. When $\lambda > 1$ (or $\lambda > 2$ in the case of averaged Hölder spaces), the crucial point is the choice of the definition of differentiability of integer order for functions on the unit sphere. This definition may be given following two principally different approaches:

I) in inner terms of the sphere,

II) in terms of continuation, for instance, the homogeneous continuation to $R^h\backslash\{0\}$ and posterious differentiation with respect to Cartesian coordinates.

This will be explained in detail in Section 2. What should be emphasized immediately, is the following. The space $C^1(S^{n-1})$ defined by **I)** does not coincide with the same space via the approach **II)**. This happens because via **I)** we obtain functions representable by the first order potential with a continuous density, and after the differentiation of this potential with respect to Cartesian coordinates we arrive at a singular integral operator (in case $n \geq 2$), which does not preserve the space of continuous functions, as is known. Therefore, the definition of fractional order function spaces of the type $C^\lambda(S^{n-1})$ or $H^\lambda(S^{n-1})$ in case $\lambda \geq 1$ depends on the choice of the notions of differentiation used. We pay a special attention to the Hölder type spaces $H^\lambda(S^{n-1})$ in case of integer λ.

The paper represents a development and refinement of the results given in the preprint-type publication [22], see also the brief communication [23].

Section 2 contains exact settings of the problems under consideration and formulations of the final statements. In particular, the coincidence of the usual and averaged Hölder spaces is formulated there, and it is also stated that the "inner" and "outer" definitions of the space $C^1(S^{n-1})$ gives different spaces, indeed. (Roughly speaking, "inner" definitions are those which are invariant with respect to rotations). Section 3 includes some preliminaries and auxiliary results, while in Section 4 there are given the proofs of the main statements.

N o t at i o n

We use the standard notation of harmonic analysis, see e.g. Stein [19] or Stein and Weiss [20]; in particular, $Y_{m\mu}(\sigma), \sigma \in S_{n-1}$, denote the basis spherical harmonics on the unit sphere S^{n-1} in R^n, $|S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is its area;
$e_1 = (1, 0, ..., 0), \ ... \ , \ e_n = (0, ..., 0, 1)$;
an operator K invariant with respect to rotations is called spherical convolution operator; its action on spherical harmonics $Y_{m\mu}(\sigma)$ is reduced to multiplication by some numbers k_m; the sequence $\{k_m\}_{m=0}^{\infty}$ is known as a Laplace-Fourier multiplier;
in the case of a "nice" multiplier, the spherical convolution operator has the form $K\varphi = \int_{S^{n-1}} k(x \cdot \sigma) d\sigma, x \in S^{n-1}$;
δ is the Beltrami-Laplace differentiation operator on the unit sphere with the Laplace-Fourier multiplier $\{m(m+n-2)\}_{m=0}^{\infty}$;
δ^α is defined by the multiplier $\{[m(m+n-2)]^\alpha\}_{m=0}^{\infty}$;
$f_{m\mu} = (f, Y_{m\mu}), \qquad (f,g) = \int_{S^{n-1}} f(\sigma) g(\sigma) \, d\sigma$;
$C^\infty(S^{n-1})$ stands for the class of functions $f(\sigma), \sigma \in S^{n-1}$, such that $f\left(\frac{x}{|x|}\right) \in C^\infty(R^n \backslash \{0\})$;
$j = (j_1, ..., j_n)$ denotes a multi-index with integer components, $|j| = j_1 + ... + j_n$;
$D_k f = |x| \frac{\partial}{\partial x_k} f\left(\frac{x}{|x|}\right), k = 1, ..., n$, denote the differentiation operators preserving the order of homogeneity; $D = (D_1, ..., D_n)$, $D^j = D_1^{j_1} \cdots D_n^{j_n}$;
$[\lambda]$ is the entire part of the number $\lambda > 0$;
notation $X \longrightarrow Y$ means that X is continuously embedded into Y.

2 Formulation of the main statements

We start with the space $C^1(S^{n-1})$. It is possible to define it by many ways. Along the approach **I)** (see Introduction), the following variants of definition are possible:

1) *to suppose existence of* $\delta^{\frac{1}{2}}$ *in the strong sense*, that is, $C^1(S^{n-1})$ may be defined as

a closure of $C^\infty(S^{n-1})$ with respect to the norm $\|f\|_C + \|\delta^{\frac{1}{2}} f\|_C$;

2) *to close* $C^\infty(S^{n-1})$ *with respect to the norm* $\|f\|_C + \|\mathbb{D}f\|_C$, where $\mathbb{D}$ is any operator invariant with respect to rotations, whose multiplier $\{d_m\}_{m=0}^\infty$ admits the asymptotics

$$d_m = am + a_0 + \frac{a_1}{m} + ... + \frac{a_{[\frac{n+1}{2}]}}{m^{[\frac{n+1}{2}]}} + O\left(\frac{1}{m^{\frac{n}{2}+\varepsilon}}\right), \quad a \neq 0; \tag{2.1}$$

3) *to suppose that there exists* $\delta^{\frac{1}{2}} f$ *(or* $\mathbb{D}f$*, where* $\mathbb{D}$ *has a multiplier with the asymptotics (2.1)) in the weak sense*, that is, there exists a function $g(\sigma) \in C(S^{n-1})$ such that $(g, \omega) = (f, \delta^{\frac{1}{2}}\omega)$ for all $\omega \in C^\infty(S^{n-1})$;

4) *to suppose that there exist continuous limiting values* $\lim_{r\to 1} \frac{d}{dr} P_r f$ *of the radial derivative of harmonic continuation of the function* f, P_r being the Poisson operator

$$(P_r f)(x') = \frac{1}{c_n} \int_{S^{n-1}} \frac{1 - r^2}{|rx' - \sigma|^n} f(\sigma)\, d\sigma, \quad r = |x|,\ x' = \frac{x}{|x|} \in S^{n-1}; \tag{2.2}$$

5) *to define the space* $C^1(S^{n-1})$ *as a set of functions representable by the first order potential of a continuous function*:

$$f(x) = K^1\varphi = \frac{1}{\gamma_{n-1}(1)} \int_{S^{n-1}} \frac{\varphi(\sigma) d\sigma}{|x - \sigma|^{n-2}}, \quad \varphi(\sigma) \in C(S^{n-1}), \quad n \geq 3, \tag{2.3}$$

where $x \in S^{n-1}$, and $\gamma_{n-1}(1)$ is the known normalizing constant, see (2.12); in the case $n = 2$ this potential is introduced as

$$K^1\varphi = \frac{1}{\pi} \int_{S^1} \varphi(\sigma) ln \frac{1}{|x - \sigma|}\, d\sigma = \frac{1}{2\pi} \int_{S^1} \varphi(\sigma) ln \frac{1}{2(1 - x\cdot\sigma)}\, d\sigma\ , \tag{2.4}$$

6) *to define* $C^1(S^{n-1})$ *as the set of functions for which there converges the spherical hypersingular integral of order 1:*

$$\mathfrak{D}^1 f = \frac{1}{\gamma_{n-1}(-1)} \lim_{\substack{\varepsilon\to 0 \\ (C(S^{n-1}))}} \int_{|x'-\sigma|>\varepsilon} \frac{f(\sigma) - f(x')}{|x' - \sigma|^n}\, d\sigma \ \in\ C(S^{n-1}) \tag{2.5}$$

known as Riesz spherical differentiation operator (of order 1), see (2.22) below; we refer also to Lemma 3.13 in connection with the approach 6).

As regards the approach **II)**, one may suggest the following definitions:

7) *to suppose that the homogeneous continuation* $f\left(\frac{x}{|x|}\right)$ *of a function* $f(\sigma)$, $\sigma \in S^{n-1}$ *has usual partial derivatives, continuous in* $R^n\backslash\{0\}$;

8) *to suppose that the same derivatives exist in the weak sense*, which means that there exist continuous functions $g_k(\sigma) \in C(S^{n-1})$, such that

$$(f, D_k\omega) = (n-1)(A_k f, \omega) - (g_k, \omega) \tag{2.6}$$

where

$$(A_k f)(\sigma) = \sigma_k f(\sigma) \tag{2.7}$$

(we justify the interpretation (2.6) of weak derivatives $D_k f$ below at the end of this section);

9) *local approach* is possible: to suppose that the "projection" of a function $f(x), x \in S^{n-1}$, onto the tangent plane is differentiable in a neighborhood of the point x, for any $x \in S^{n-1}$.

One can also treat $C^1(S^{n-1})$ by considering differentiability along geodesic lines (as in Nikol'skii and Lizorkin [11]), but we do not dwell on this approach here

By elementary arguments it is shown that the classes $C^1(S^{n-1})$, defined by the definitions 7) and 9) coincide.

Definition 2.1. By $W_{\lambda,N}$ with $\lambda \in R^1,\ N = 1, 2, 3, ...$ we denote the class of spherical Laplace-Fourier multipliers $\{k_m\}_{m=0}^{\infty}$, which have the following asymptotics at infinity

$$k_m = \sum_{i=0}^{N} c_i m^{-\lambda - i} \ + \ O\left(m^{-\lambda-N-\varepsilon}\right), \quad c_0 \neq 0,$$

for some $\varepsilon > 0$.

An easily verified sufficient condition for $\{k_m\}_{m=0}^{\infty}$ to be in $W_{\lambda,N}$ is that k_m has the form

$$k_m = \frac{a(m)}{m^{\lambda}}, \quad m = 1, 2, 3, ... \tag{2.8}$$

where the function $a(r)$ is defined for all $r > 0$ and $a\left(\frac{a}{r}\right) \in C^N([0, \delta])$ for some $\delta > 0$.

Theorem A. *The following spaces*
1) the closure of $C^{\infty}(S^{n-1})$ *with respect to the norm* $\|f\|_C + \|\delta^{\frac{1}{2}} f\|_C$*;*
2) the closure of $C^{\infty}(S^{n-1})$ *with respect to the norm* $\|f\|_C + \|\mathbb{D}f\|_C$*, where* $\mathbb{D}$ *is any spherical convolution operator with a multiplier in* $W_{1,N}, N > \frac{n}{2}$ *;*
3) the space of functions $f \in C(S^{n-1})$ *which have* $\delta^{\frac{1}{2}} f$ *(or* $\mathbb{D}f$*) in the weak sense, that is, there exists* $g \in C(S^{n-1})$ *such that* $(g, \omega) = (f, \delta^{\frac{1}{2}}\omega)$ *for all* $\omega \in C^{\infty}(S^{n-1})$, $\|f\| = \|f\|_C + \|g\|_C$*;*
4) $\left\{f \in C(S^{n-1}) : g(\sigma) = \lim_{r\to 1} \left(\frac{d}{dr} P_r f\right)(\sigma) \in C(S^{n-1})\right\}, \quad \|f\| = \|f\|_C + \|g\|_C$*;*
5) $\{f(x) : f(x) = K^1\varphi,\ \varphi \in C(S^{n-1})\}, \quad \|f\| = \|\varphi\|_C$*;*
6) $\{f(x) : \|f\| = \|f\|_C + \|\mathfrak{D}^1 f\|_C < \infty\}$*;*
coincide up to equivalence of norms.

Remark 2.2. *The equivalence of the spaces 3) and 4) may be treated in the form: for* $f \in C(S^{n-1})$*, the following statements are equivalent:* ***a)*** *there exists a function* $g \in C(S^{n-1})$ *such that* $m f_{m\mu} = g_{m\mu}$*, and* ***b)*** *there exists a function* $g \in C(S^{n-1})$ *such that* $\lim_{r\to 1} \left\|\frac{P_r f - f}{1-r} - g\right\| = 0$*. This equivalence was proved in Butzer and Johnen [4], p. 245.*

Theorem B. *The spaces of functions* $f(\sigma) \in C(S^{n-1})$*, for which*
7) $\|f\| = \|f\|_C + \sum_{k=1}^{n} \|D_k f\|_C < \infty$*, and*
8) there exist functions $g_k \in C(S^{n-1})$ *satisfying the relation (2.6),* $\|f\| = \|f\|_C + \sum_{k=0}^{n} \|g_k\|_C$*,*
coincide up to equivalence of norms.

Theorems A and B allow us to give the following definitions below.

Definition 2.3. *By $\mathbb{C}^1(S^{n-1})$ we denote any of the spaces 1) - 6) and by $\mathcal{C}^1(S^{n-1})$ - any of the spaces 7)-8). Similarly, $\mathcal{C}^r(S^{n-1}),\ r = 1,2,3,...$ will denote the space of functions on S^{n-1} with homogeneous continuation $f\left(\frac{x}{|x|}\right)$ continuously differentiable up to the order r:*

$$\mathcal{C}^r(S^{n-1}) \ = \ \left\{ f \in C(S^{n-1}) : D^j\ f\left(\frac{x}{|x|}\right) \in C\left(\{x : \frac{1}{2} \leq |x| \leq 2\}\right)\right\}\ . \qquad (2.9)$$

Theorem C. $\mathbb{C}^1(S^{n-1}) \neq \mathcal{C}^1(S^{n-1})$. *Moreover,* $\mathbb{C}^1(S^{n-1}) \nsubseteq \mathcal{C}^1(S^{n-1})$ and $\mathcal{C}^1(S^{n-1}) \nsubseteq \mathbb{C}^1(S^{n-1})$.

Definition 2.4. *By $\mathbb{C}^\lambda(S^{n-1}), \lambda > 0$ we denote the closure of $C^\infty(S^{n-1})$ with respect to the norm $\|f\|_C + \|\delta^{\frac{\lambda}{2}} f\|_C$.*

Remark 2.5. *The definition of the space $\mathbb{C}^\lambda(S^{n-1})$ remains equivalent if we replace the operator $\delta^{\frac{\lambda}{2}}$ by any spherical convolution operator $\mathbb{D}^\lambda$ in the class $W_{\lambda,N}, N > \frac{n}{2}$. This may be shown as in the proof of the part 1) $\longleftrightarrow$ 2) of Theorem A.*

The space $\mathbb{C}^\lambda(S^{n-1})$ is related to the range of the spherical Riesz potentials ($|x| = 1$):

$$K^\lambda\varphi = \frac{1}{\gamma_{n-1}(\lambda)}\int_{S^{n-1}} \frac{\varphi(\sigma)d\sigma}{|x-\sigma|^{n-1-\lambda}}, \quad \lambda > 0,\ \lambda \neq n-1, n+1, n+3, ... \qquad (2.10)$$

$$K^\lambda\varphi = \frac{1}{\gamma_{n-1}(\lambda)}\int_{S^{n-1}} \frac{\ln\frac{\sqrt{2}}{|x-\sigma|}}{|x-\sigma|^{n-1-\lambda}}\varphi(\sigma)d\sigma, \qquad \lambda = n-1, n+1, n+3, ... \qquad (2.11)$$

where the normalizing constant $\gamma_{n-1}(\lambda)$ is given by the formula

$$\gamma_{n-1}(\lambda) = \frac{2^\lambda \pi^{\frac{n-1}{2}}\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{n-1-\lambda}{2}\right)} \qquad (2.12)$$

when $\lambda - n \neq -1, 1, 3, ...$ and $\gamma_{n-1}(\lambda) = (-1)^k 2^{\lambda-1}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\lambda}{2}\right)k!$, when $\lambda = n-1-2k, k = 0,1,2,...$, see Samko [17], Ch. 6 . It is also known that in the case $\lambda \neq n-1+2k, k = 0,1,2,...$, the Laplace-Fourier multiplier of the operator K^λ is equal to

$$\{k_m^\lambda\}_{m=0}^\infty = \left\{\frac{\Gamma\left(m + \frac{n-1-\lambda}{2}\right)}{\Gamma\left(m+\frac{n-1+\lambda}{2}\right)}\right\}_{m=0}^\infty \ \in\ W_{\lambda,N}. \qquad (2.13)$$

In the case when $\lambda = n-1+2k, k = 0,1,2,...$ the multiplier has same expression for all $m \geq k$, but for a finite number of indices $m = 1,2,...,k$ the expression is different:

$$k_m^\lambda = \frac{(-1)^{k-m}}{(k-m)!(n+k+m-2)!}\left[\psi(n+k+m-1)\right.$$
$$\left. +\psi(k-m+1) - \psi(k+1) - \psi\left(k+\frac{n-1}{2}\right) - ln\ 2\right], \quad m = 0,1,2,...,k. \qquad (2.14)$$

In particular, when $n = 2$ and $\lambda = 1$, one has

$$\{k_m^\lambda\}_{m=0}^{\infty} = \begin{cases} \frac{1}{m}, & m = 1, 2, 3, ... \\ ln\ 2\ , & m = 0\ . \end{cases} \tag{2.15}$$

The inclusion into $W_{\lambda,N}$ in (2.13) is valid for an arbitrary integer N, which is easily obtained by means of the asymptotic relation for $\frac{\Gamma(z+a)}{\Gamma(z+b)}$, see e.g. Luke [9], p. 20 (Russian ed.).

We note that

$$k_m^\lambda \neq 0 \quad \text{for all} \quad m = 0, 1, 2, ... \quad \text{and} \quad \lambda > 0.$$

This is obvious in the case (2.13), while in the case (2.14) it suffices to observe that the expression in the brackets in (2.14) has the form $(-1)^n \ln\ 2\ +$ a rational number. The last statement can be easily obtained by means of the known properties of the ψ-function, see Gradstein and Ryzhik [7], N 8.365.4 and N 8.366.3.

Sometimes there are considered other spherical potential operators, the kernels of which may be expressed in terms of some special functions, but which have "nice" multipliers, see for instance Samko [17], Ch. 6, Section 2.3. For example, spherical potential operators

$$\mathbb{K}^\lambda \varphi = \frac{1}{\Gamma(\lambda)} \int_0^1 \left(ln\ \frac{1}{r} \right)^{\lambda - 1} (P_r \varphi)(x) dr, \quad \lambda > 0, \tag{2.16}$$

and

$$\mathfrak{K}^\lambda \varphi = \frac{1}{\Gamma(\lambda)} \int_0^1 (1-r)^{\lambda-1} (P_r \varphi)(x) dr, \quad \lambda > 0, \tag{2.17}$$

where $P_r\varphi$ is spherical Poisson integral (2.2), have the Fourier-Laplace multipliers equal to

$$k_m^\lambda = \frac{1}{(m+1)^\lambda} \ \text{in the case of} \ \mathbb{K}^\lambda \ \text{and} \ k_m^\lambda = \frac{\Gamma(m+1)}{\Gamma(m+1+\lambda)} \ \text{in the case of} \ \mathfrak{K}^\lambda. \tag{2.18}$$

Theorem D. *For any* $\lambda > 0$

$$\mathbb{C}^\lambda(S^{n-1}) = \mathbb{K}^\lambda \left[C(S^{n-1}) \right] = \mathfrak{K}^\lambda \left[C(S^{n-1}) \right] = K^\lambda \left[C(S^{n-1}) \right]; \tag{2.19}$$

and a function $f(x) \in C(S^{n-1})$ *is in* $\mathbb{C}^\lambda(S^{n-1})$ *if and only if there exists a function* $\varphi(x) \in C(S^{n-1})$ *such that*

$$k_m^\lambda f_{m\mu} = \varphi_{m\mu} \quad \text{for all} \ \ m = 1, 2, 3, ... \tag{2.20}$$

where k_m *is one of the multipliers (2.18).*

In the case $0 < \lambda < 2$, *the space* $\mathbb{C}^\lambda(S^{n-1})$ *may be also characterized as the subspace of functions* $f(x) \in C(S^{n-1})$ *for which there exists the limit of the type (2.5):*

$$\mathfrak{D}^\lambda f = \lim_{\substack{\varepsilon \to 0 \\ (C(S^{n-1}))}} \mathfrak{D}_\varepsilon^\lambda f \tag{2.21}$$

where

$$\mathfrak{D}_\varepsilon^\lambda f = \frac{1}{\gamma_{n-1}(-\lambda)} \int_{|x'-\sigma|>\varepsilon} \frac{f(\sigma) - f(x')}{|x' - \sigma|^{n-1+\lambda}} \, d\sigma, \quad x \in S^{n-1}. \tag{2.22}$$

Definition 2.6. *By* $H^\lambda(S^{n-1})$, $0 < \lambda \le 1$, *we denote the space of functions* $f(x)$ *satisfying the usual Hölder condition on the sphere*:

$$\|f\|_{H^\lambda} := \|f\|_C \ + \ \sup_{t>0} \frac{\omega(f,t)}{t^\lambda} < \infty, \tag{2.23}$$

where

$$\omega(f,t) = \sup_{|x-\sigma|\le t;\ x,\sigma\in S^{n-1}} |f(x) - f(\sigma)|. \tag{2.24}$$

We shall also deal with the averaged Hölder condition. Let S_h be the the averaged shift operator :

$$(S_h f)(x) = \frac{1}{|S^{n-2}|\sin^{n-2} h} \int_{x\cdot\sigma=\cos h} f(\sigma)d\sigma, \quad n \ge 3, \tag{2.25}$$

known also as spherical means, see Berens, Butzer and Pawelke [3] and Pawelke [14], which in the case $n = 2$ should be replaced by $(S_h f)(x) = \frac{f(x_h^+)+f(x_h^-)}{2}$, where $x_h^\pm = (x_1 \cos h \pm x_2 \sin h, x_1 \cos h \mp x_2 \sin h)$.

Let

$$\omega_*(f,t) = \sup_{0<h\le t} \sup_{x\in S^{n-1}} |f(x) - (S_h f)(x)| \tag{2.26}$$

so that $\omega_*(f,t) \le \omega(f,t)$.

Definition 2.7. *By* $H_*^\lambda(S^{n-1})$, $0 < \lambda \le 2$, *we denote the Hölder type space of functions* $f(x)$ *for which*

$$\|f\|_{H_*^\lambda} := \|f\|_C \ + \ \sup_{t>0} \frac{\omega_*(f,t)}{t^\lambda} < \infty. \tag{2.27}$$

The spaces $H^\lambda(S^{n-1})$ and $H_*^\lambda(S^{n-1})$ of arbitrary order $\lambda > 0$ may be introduced via subsequent differentiation. This may be done in two different ways according to the approaches **I)** and **II)**, see Introduction. Basing on Definitions 2.3 and 2.4, we introduce the following definition.

Definition 2.8. *Let* $\lambda > 0$. *1). By* $\mathbb{H}^\lambda(S^{n-1})$ *we denote the space of functions in* $\mathbb{C}^{[\lambda]}(S^{n-1})$ *such that*

$$\delta^{\frac{[\lambda]}{2}} f \in H^{\lambda-[\lambda]}(S^{n-1}), \quad \text{if } \ \lambda \ne 1,2,3,... \tag{2.28}$$

and $\delta^{\frac{\lambda-1}{2}} f \in H^1(S^{n-1})$, *if* $\lambda = 1,2,3,....$

2). We say that $f(x) \in \mathcal{H}^\lambda(S^{n-1})$, *if* $f(x) \in \mathcal{C}^{[\lambda]}(S^{n-1})$ *and* $D^j f \in H^{\lambda-[\lambda]}(S^{n-1})$ *for all the multi-indices* j *with* $|j| \le [\lambda]$, $\lambda \ne 1,2,3,...$; *in case* $\lambda = 1,2,3,...$ *we require that* $D^j f \in H^1(S^{n-1})$ *for all* $|j| \le \lambda - 1$.

By this definition, $\mathbb{H}^\lambda(S^{n-1}) = \mathcal{H}^\lambda(S^{n-1}) = H^\lambda(S^{n-1})$ in the case $0 < \lambda < 1$.

Remark 2.9. *It may be shown that in Definition 2.8 the requirement (2.28) is equivalent to* $\mathfrak{D}^{[\lambda]} f \in H^{\lambda-[\lambda]}(S^{n-1})$, *where* $\mathfrak{D}^{[\lambda]}$ *is any spherical convolution operator with the multiplier* $\{k_m\}$ *in the class* $W_{\lambda,N}$, $N > \frac{n}{2}$.

The "averaged" Hölder spaces for $\lambda > 2$ are introduced by the following definition.

Definition 2.10. *Let* $\lambda > 0$. *1). By* $\mathbb{H}_*^\lambda(S^{n-1})$ *we denote the space of functions* $f(x)$ *in* $\mathbb{C}^{2\left[\frac{\lambda}{2}\right]}(S^{n-1})$ *such that*

$$\delta^{\left[\frac{\lambda}{2}\right]} f \in H_*^{\lambda-2\left[\frac{\lambda}{2}\right]}(S^{n-1}), \quad \lambda \neq 2,4,6,... \tag{2.29}$$

which should be replaced by $\delta^{\frac{\lambda-2}{2}} f \in H_*^2(S^{n-1})$ *when* $\lambda = 2,4,6,....$

2). By $\mathcal{H}_*^\lambda(S^{n-1})$ *we denote the space of functions* $f(x) \in \mathcal{C}^{2\left[\frac{\lambda}{2}\right]}(S^{n-1})$ *for which* $D^j f \in H_*^{\lambda-2\left[\frac{\lambda}{2}\right]}(S^{n-1})$ *for all the multi-indices* j *with* $|j| \leq 2\left[\frac{\lambda}{2}\right]$, $\lambda \neq 2,4,6,...$ *with a similar change in the case when* $\lambda = 2,4,6,...$

Norms in the spaces $\mathbb{H}_*^\lambda(S^{n-1})$ and $\mathcal{H}_*^\lambda(S^{n-1})$ are introduced in the natural way.

Theorem E. *Let* $\lambda > 0, \lambda \neq 1,2,3,...$ *Then*

$$\mathbb{C}^\lambda(S^{n-1}) \longrightarrow \mathbb{H}^\lambda(S^{n-1}), \quad \mathbb{C}^\lambda(S^{n-1}) \neq \mathbb{H}^\lambda(S^{n-1}) \tag{2.30}$$

and

$$\mathbb{H}^\lambda(S^{n-1}) = \mathcal{H}^\lambda(S^{n-1}) = \mathbb{H}_*^\lambda(S^{n-1}) = \mathcal{H}_*^\lambda(S^{n-1}), \tag{2.31}$$

up to equivalence of norms in (2.31). When $\lambda = 1,2,3,...$, *the spaces* $\mathbb{H}^\lambda(S^{n-1})$ *and* $\mathcal{H}^\lambda(S^{n-1})$ *are different from each other, and are not embedded one into another, but*

$$\mathcal{H}^\lambda(S^{n-1}) \longrightarrow \mathcal{H}_*^\lambda(S^{n-1}).$$

Finally, the following lemma gives a justification of the definition (2.6) for weak derivatives.

Lemma 2.11. *Let* $f\left(\frac{x}{|x|}\right) \in C^1(R^n\backslash\{0\})$. *Then*

$$(f, D_k\omega) = (n-1)(A_k f, \omega) - (D_k f, \omega), \quad \omega \in C^\infty(S^{n-1}), \quad k = 1,...,n. \tag{2.32}$$

Proof. We have

$$\int_{S^{n-1}} (D_k\omega)(\sigma) f(\sigma)\, d\sigma = (n-1)\int_{|x|<1} f\left(\frac{x}{|x|}\right) \frac{\partial}{\partial x_k}\left[\omega\left(\frac{x}{|x|}\right)\right]\, dx$$

$$= (n-1)\left\{\int_{|x|<1} \frac{\partial}{\partial x_k}\left[f\left(\frac{x}{|x|}\right)\omega\left(\frac{x}{|x|}\right)\right]\, dx - \int_{|x|<1} \frac{\partial}{\partial x_k}\left[f\left(\frac{x}{|x|}\right)\right]\omega\left(\frac{x}{|x|}\right)\, dx\right\}.$$

Applying the Gauss-Ostrogradski formula, we arrive at the relation (2.32). □

3 Preliminaries and auxiliary statements

a). Differentiability of spherical means with respect to the parameter. We find it convenient to introduce another notation for spherical means (2.25) in the following way:

$$(T_t f)(x) = \frac{1}{|S^{n-2}|(1-t^2)^{\frac{n-2}{2}}} \int_{x\cdot\sigma=t} f(\sigma) d\sigma, \quad |x| = 1, \quad -1 < t < 1 \tag{3.1}$$

when $n \geq 3$, and similarly for $n = 2$. Obviously, $T_t f = S_{arccos\, t} f$ and $S_h f = T_{\cos h} f$. Spherical means in the form (3.1) were essentially used in Samko [16], see also [17], Chapter 4. We are now interested in the knowledge of the behaviour of $\frac{\partial}{\partial t} T_t f$as $t \to 1$. As is well known, the operator T_t is invariant with respect to rotations and its Laplace-Fourier multiplier is the generalized Legendre polynomial with respect to the parameter t:

$$(T_t Y_m)(x) = P_m(t) Y_m(x), \quad m = 0, 1, 2, ..., \quad x \in S^{n-1} \tag{3.2}$$

where $P_m(t) = \frac{C_m^{\frac{n-2}{2}}(t)}{C_m^{\frac{n-2}{2}}(1)}$ in the case $n \geq 3$, $C_m^{\frac{n-2}{2}}(t)$ being the Gegenbauer polynomial, and $P_m(t)$ is the Chebyshev polynomial in the case $n = 2$. We observe the formula

$$(T_t f, \varphi) = (f, T_t \varphi) \tag{3.3}$$

for all $f, \varphi \in C(S^{n-1})$.

Lemma 3.1. *Let $f \in C(S^{n-1})$ be differentiable in the sense 7) (see Section 2). The formula is valid*

$$\frac{\partial}{\partial t} T_t f = \frac{1}{1 - t^2} x \cdot (T_t\ grad\ f)(x), \quad |x| = 1, \tag{3.4}$$

where $grad\ f = (D_1 f, ..., D_n f)$ and $T_t\ grad\ f = (T_t\ D_1 f, ..., T_t\ D_n f)$.

This formula was proved in [16], p.166, in the form

$$\frac{\partial}{\partial t} T_t f = \frac{1}{1 - t^2} \sum_{k=1}^{n} \left[\frac{x_k}{|x|} (T_t D_k f)(x) - t T_t x_k (D_k f)(x) \right],$$

where the last sum in reality disappears, since $\sum_{k=1}^{n} t T_t x_k (D_k f)(x) = T_t(x \cdot grad\ f(x)) = 0, |x| = 0$, by the Euler equation for homogeneous functions.

The next lemma provides another formula for differentiation of the means with respect to the parameter.

Lemma 3.2. *Let $f \in C(S^{n-1})$ be differentiable in the sense 7). Then*

$$\frac{\partial}{\partial t} T_t f = \frac{(n-1) t T_t f - \sum_{k=1}^{n} D_k T_t A_k f}{1 - t^2}, \tag{3.5}$$

where $(A_k f)(x) = x_k f(x)$, $|x| = 1$.

Proof. We differentiate the equality in (3.3) with respect to the parameter t, taking $\varphi(\sigma) \in C^\infty(S^{n-1})$ and applying formula (3.4), which yields

$$\frac{d}{dt} (T_t f, \varphi) = (f, \frac{\partial}{\partial t} T_t \varphi) = \frac{1}{1 - t^2} \sum_{k=1}^{n} (A_k f, T_t D_k \varphi). \tag{3.6}$$

Hence, by (3.3) and (2.32),

$$\frac{d}{dt} (T_t f, \varphi) = \frac{1}{1 - t^2} \sum_{k=1}^{n} [(n-1)(A_k f, T_t A_k \varphi) - (D_k T A_k f, \varphi)]$$

Obviously,

$$\sum_{k=1}^{n} A_k T_t A_k f = tT_t f. \tag{3.7}$$

Therefore,

$$\frac{d}{dt}(T_t f, \varphi) = \frac{1}{1-t^2}\left((n-1)tT_t f - \sum_{k=1}^{n} D_k T_t A_k f, \varphi\right) \tag{3.8}$$

and then to obtain (3.4), it remains to refer to Remark 3.3 given below. □

Remark 3.3. Let $f_t(\sigma), f_t(\sigma) \in C((-1,1) \times S^{n-1})$. If $\frac{d}{dt}(f_t, \varphi) = (g_t, \varphi)$ for all $\varphi \in C^\infty(S^{n-1})$, then $f_t(\sigma)$ is differentiable in t and $\frac{\partial}{\partial t} f_t(\sigma) = g_t(\sigma)$.

The proof is obvious.

Corollary. *For functions f differentiable in the sense 7), the relation is valid*

$$\sum_{k=1}^{n} (A_k T_t D_k + D_k T_t A_k) f = (n-1)tT_t f. \tag{3.9}$$

Indeed, (3.9) is obtained by comparison of (3.4) and (3.5).

In the next lemma we extend the formula (3.4) to the case when the information about the usual differentiability is replaced by the weak one.

Lemma 3.4. *Let $f \in C(S^{n-1})$ have weak first order derivative $g_k = D_k f,\ k = 1, ..., n$, in the sense (2.6). Then*

$$\frac{\partial}{\partial t} T_t f = \frac{1}{1-t^2} \sum_{k=1}^{n} A_k T_t g_k. \tag{3.10}$$

Proof. Let $\varphi(x) \in C^\infty(S^{n-1})$. By (3.3) and (3.5), we obtain

$$\frac{d}{dt}(T_t f, \varphi) = \frac{(n-1)t}{1-t^2}(f, T_t\varphi) - \frac{1}{1-t^2}\sum_{k=1}^{n}(f, D_k T_t A_k \varphi).$$

Since $T_t A_k \varphi \in C^\infty(S^{n-1})$, by the definition in (2.6) and the property (3.7) we have

$$\frac{d}{dt}(T_t f, \varphi) = \frac{1}{1-t^2}\sum_{k=1}^{n}(g_k, T_t A_k \varphi)$$

whence (3.10) follows in view of Remark 3.3 and arbitrariness of $\varphi \in C^\infty(S^{n-1})$. □

Lemma 3.5. *Let $f \in C(S^{n-1})$ have weak first order derivative $g_k = D_k f,\ k = 1, ..., n$, in the sense (2.6). Then the same is true for the functions $A_j f,\ j = 1, ..., n$.*

The proof is direct.

We need below also the following formulas (see Samko [16], p. 167, or [17], Ch.4, Lemma 4.13):

$$\int_{S^{n-1}} f(x' \cdot \sigma)\varphi(\sigma)\, d\sigma = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} f(t)(T_t\varphi)(x')(1-t^2)^{\frac{n-3}{2}}\, dt, \tag{3.11}$$

$$(T_t\sigma_k)(x') = t\frac{x_k}{|x|}, \quad (T_t\sigma_k^2)(x') = \frac{1}{n} - \frac{1}{n}P_2(t)\left(1 - n\frac{x_k^2}{|x|^2}\right), \tag{3.12}$$

where $x' = \frac{x}{|x|} \in S^{n-1}$ and $P_2(t)$ is the generalized Legendre polynomial, see (3.2), of order 2.

b) On continuity moduli. Besides the continuity moduli $\omega(f,t)$ and $\omega_*(f,t)$, the following continuity modulus is also known to be considered on the sphere:

$$\omega_\rho(f,t) = \sup_{\substack{x,y\in S^{n-1}\\ \rho(x,y)\le t}} |f(x) - f(y)|,$$

where $\rho(x,y)$ is the distance between the points $x, y \in S^{n-1}$ measured along the geodesic line passing through these points. The inequalities are valid

$$\omega_\rho(f,t) \le \omega(f,t) \le \omega_\rho(f,kt), \quad k = \frac{\pi}{2}, \tag{3.13}$$

which is obtained from the bounds $1 \le \frac{\rho(x,y)}{|x-y|} \le k$. Evidently,

$$\omega_*(f,t) \le \omega(f,t). \tag{3.14}$$

The following properties of the moduli ω, ω_* and ω_ρ are known:

$$\omega_\rho(f,\lambda t) \le (1+\lambda)\omega_\rho(f,t), \tag{3.15}$$

$$\omega(f,\lambda t) \le (1+k\lambda)\omega(f,t), \tag{3.16}$$

$$\omega_*(f,\lambda t) \le A(1+\lambda)^2\omega_*(f,t), \tag{3.17}$$

where $\lambda > 0$ and $A > 0$ is some absolute constant not depending on λ and f. The inequality (3.15) follows from its validity for periodic functions of one variable, see Lizorkin and Nikol'skii [11]; (3.16) is derived from (3.15) in view of (3.13), while (3.17) was proved in Pawelke [14].

c) Identity approximations on the sphere. The spherical convolution operator

$$f_\varepsilon(x) = \int_{S^{n-1}} k_\varepsilon(x\cdot\sigma)\, f(\sigma)\, d\sigma, \quad |x| = 1, \tag{3.18}$$

is known to be called the *identity approximation* in the space X of functions on S^{n-1}, if $\lim_{\varepsilon\to 0}\|f_\varepsilon - f\|_X = 0$.

The following statement is known, see Berens, Butzer and Pawelke [3], p.210.

Lemma 3.6. *Let the kernel $k_\varepsilon(t)$ satisfy the conditions*
a) $|S^{n-2}|\int_{-1}^1 k_\varepsilon(t)(1-t^2)^{\frac{n-3}{2}}\,dt = 1$, *for all* $0<\varepsilon<\varepsilon_0$,
b) $\int_{-1}^1 |k_\varepsilon(t)|(1-t^2)^{\frac{n-3}{2}}\,dt \le M < \infty$, $0<\varepsilon<\varepsilon_0$, *where M does not depend on ε,*
c) $\lim_{\varepsilon\to 0}\int_{-1}^{1-\delta} |k_\varepsilon(t)|(1-t^2)^{\frac{n-3}{2}}\,dt = 0$ *for any* $0<\delta<1$.

Then f_ε is an identity approximation in the spaces $X = L_p(S^{n-1})$, $1\le p<\infty$, and $X = C(S^{n-1})$ and $\|f_\varepsilon\|_X \le M\|f\|_X$.

Corollary 1. *Let $f(x) \in C(S^{n-1})$. Under the conditions a)-c) of Lemma 3.6,*

$$\lim_{\substack{\varepsilon\to 0 \\ C(S^{n-1})}} |S^{n-2}| \int_{-1}^{1} k_\varepsilon(t)(T_t f)(x)(1-t^2)^{\frac{n-3}{2}}\, dt = f(x) \tag{3.19}$$

for any $f(x) \in C(S^{n-1})$.

Corollary 2. *Let $\varphi(t) \in C([-1,1])$. Under the conditions a)-c) of Lemma 3.6,*

$$\lim_{\varepsilon\to 0} |S^{n-2}| \int_{-1}^{1} k_\varepsilon(t)\varphi(t)(1-t^2)^{\frac{n-3}{2}}\, dt = \varphi(1). \tag{3.20}$$

To obtain (3.20), it suffices to note that

$$\lim_{\varepsilon\to 0} \int_{S^{n-1}} k_\varepsilon(e_1\cdot\sigma)\psi(\sigma)\, d\sigma = \psi(e_1) \tag{3.21}$$

for any $\psi(\sigma) \in C(S^{n-1})$, because of (3.11) and (3.19).

A well known example of the approximating kernel is the *Jackson kernel*

$$k_\varepsilon(t) = k_\varepsilon^s(t) = \frac{1}{\varkappa_\nu}\mathcal{D}_\nu^s(arccos\ t), \qquad \mathcal{D}_\nu^s(u) = \left(\frac{\sin\frac{\ell}{2}u}{\sin\frac{u}{2}}\right)^{2s} \tag{3.22}$$

where $\nu = \frac{1}{\varepsilon},\ s = 1,2,3,...,\ell = 2,3,4,...,\ \nu = s(\ell-1)$ and the normalizing constant

$$\varkappa_\nu = |S^{n-2}| \int_0^\pi D_\nu^s(u)\sin^{n-2}u\, du$$

is chosen in correspondence with the condition a) of Lemma 3.6.

Everywhere below $k_\varepsilon(t)$ will be the Jackson kernel (3.22).

For the integrals

$$J_\nu^r = \frac{1}{\varkappa_\nu}\int_0^\pi u^r D_\nu^s(u)\sin^{n-2}u\, du, \quad r \geq 0,$$

the estimate

$$J_\nu^r \leq \frac{c}{\nu^{\min(r,2s-n+1)}}, \quad r \neq 2s-n+1 \tag{3.23}$$

is known, see Lizorkin and Nikol'skii [8], p. 216-217.

The following statement is well known (Djafarov [6], Lizorkin and Nikol'skii [11], Berens, Butzer and Pawelke [3]) as Jackson type theorems.

Lemma 3.7. *Let $f(x) \in C(S^{n-1})$ and $f_\varepsilon(x)$ be the approximation (3.18) with the Jackson kernel. Then*

$$|f(x) - f_\varepsilon(x)| \leq A\omega(f,\varepsilon), \quad 2s > n, \tag{3.24}$$

$$|f(x) - f_\varepsilon(x)| \leq A\omega_*(f,\varepsilon), \quad 2s > n+1, \tag{3.25}$$

where the constant $A > 0$ does not depend on ε and $f(x)$.

P r o o f. Inequalities (3.24)-(3.25) are known in various versions, see references above, but we present the proof for the completeness of the presentation. By the condition a) of Lemma 3.6, we have

$$f(x) - f_\varepsilon(x) = \int_{S^{n-1}} k_\varepsilon^s(x\cdot\sigma)[f(x) - f(\sigma)]\, d\sigma \tag{3.26}$$

so that

$$|f(x) - f_\varepsilon(x)| \le \int_{S^{n-1}} k_\varepsilon^s(x\cdot\sigma)\omega(f, |x-\sigma|)\, d\sigma.$$

Since $|x-\sigma| = 2(1 - x\cdot\sigma)$ for $x, \sigma \in S^{n-1}$, by (3.11) we obtain

$$|f(x) - f_\varepsilon(x)| \le |S^{n-2}| \int_{-1}^{1} k_\varepsilon^s(t)\omega(f, \sqrt{2}\sqrt{1-t})(1-t^2)^{\frac{n-3}{2}}\, dt.$$

Hence, in view of (3.16)

$$|f(x) - f_\varepsilon(x)| \le |S^{n-2}|\, \omega(f,\varepsilon)(I_\varepsilon^1 + I_\varepsilon^2) \tag{3.27}$$

where

$$I_\varepsilon^1 = \int_{-1}^{1} k_\varepsilon^s(t)(1-t^2)^{\frac{n-3}{2}}\, dt = \frac{1}{\varkappa_\nu}\int_0^\pi \mathcal{D}_\nu^s(u)\ sin^{n-2}u\, du = \frac{1}{|S^{n-2}|},$$

$$I_\varepsilon^1 = \frac{k\sqrt{2}}{\varepsilon}\int_{-1}^{1} k_\varepsilon^s(t)(1-t)^{\frac{1}{2}}(1-t^2)^{\frac{n-3}{2}}\, dt = \frac{2k\nu}{\varkappa_\nu}\int_0^\pi \mathcal{D}_\nu^s(u)\ sin\frac{u}{2} sin^{n-2}u\, du \le const,$$

the latter boundedness with respect to ε following from (3.23). Then (3.24) follows from (3.27).

To prove (3.25), we we use the formula (3.11) directly in (3.26), which gives:

$$f(x) - f_\varepsilon(x) = \frac{|S^{n-2}|}{\varkappa_\nu}\int_0^\pi \mathcal{D}_\nu^s(u)\,[f(x) - (S_u f)(x)] \sin^{n-2} u\, du. \tag{3.28}$$

Hence

$$|f(x) - f_\varepsilon(x)| \le \frac{|S^{n-2}|}{\varkappa_\nu}\int_0^\pi \mathcal{D}_\nu^s(u)\omega_*(f,u) \sin^{n-2} u\, du. \tag{3.29}$$

Making use of (3.17), we arrive at

$$|f(x) - f_\varepsilon(x)| \le \frac{c}{\varkappa_\nu}\omega_*(f,\varepsilon)\int_0^\pi \mathcal{D}_\nu^s(u)\left(\frac{u}{\varepsilon} + 1\right)^2 \sin^{n-2} u\, du. \tag{3.30}$$

Then application of the estimate (3.23) with $2s > n+1$ completes the proof. □

For spherical polynomials $P_m = \sum_{k=1}^m Y_k(x')$, the following Bernstein type inequality is known:

$$\|\delta^{\frac{1}{2}} P_m\|_{C(S^{n-1})} \le c(n) m \|P_m\|_{C(S^{n-1})} \tag{3.31}$$

where $c(n)$ is a constant depending only on the dimension n, see Pawelke [14], Nikol'skii and Lizorkin [12] and others for Bernstein type inequalities on the sphere, or Rustamov [15], p. 23, just in the form (3.31).

Following the standard arguments known in the approximation theory (see e.g. Daugavet [5], p. 76), the following lemma is proved by means of the Bernstein inequality (3.31).

Lemma 3.8. *Let $g(x) \in C(S^{n-1})$ and $E_k(g)$ be the best approximation of g in $C(S^{n-1})$ of order k. If for some integer $\ell = 1, 2, 3, ...$ there converges the series $\sum_{k=1}^{\infty} k^{\ell-1} E_k(g)$, then $\delta^{\frac{\ell}{2}} g \in C(S^{n-1})$ and*

$$E_m(\delta^{\frac{\ell}{2}} g) \leq c(n, \ell) \left[m^\ell E_m(g) + \sum_{k=m+1}^{\infty} k^{\ell-1} E_k(g) \right]$$

where the constant $c(n, \ell)$ does not depend on $m = 1, 2, 3, ...$ and the function g.

d) On spherical convolution whose multiplier has a power behaviour at infinity.

Lemma 3.9. *Let $\{k_m\}_{m=0}^{\infty} \in W_{\lambda,N}$, $\lambda > 0, \lambda > n - 1 + 2r, r = 0, 1, 2, ...,$ $\lambda + N > \frac{n}{2}$. Then there exists a kernel $k(t) \in L_1([-1, 1]; (1 - t^2)^{\frac{n-3}{2}})$ such that the operator*

$$K\varphi = \int_{S^{n-1}} k(x \cdot \sigma)\varphi(\sigma) d\sigma$$

has the spherical multiplier $\{k_m\}$. If $N + \lambda > n - 1$, $0 < \lambda < n - 1$, then $k(t)$ has the form

$$k(t) \ = \ (1 - t)^{\frac{\lambda - n + 1}{2}} \ell(t) \tag{3.32}$$

where $\ell(t) \in C([-1, 1])$ and $\ell(1) = \frac{2^{\frac{\lambda+1-n}{2}}}{\gamma_{n-1}(\lambda)} c_0$ with $c_0 = \lim_{m\to\infty} m^\lambda k_m$.

Statements of the type of Lemma 3.9 are well known, see Askey and Wainger [1] and [2] for more general situation; the proof of this statement just in the form of Lemma 3.9 may be found in Vakulov [21] or Samko [17], Lemma 6.21.

Corollary. If $\{k_m\}_{m=0}^{\infty} \in W_{\lambda,N}$ with $\lambda \geq 0, \lambda \neq n-1+2k, k = 0, 1, 2, ...,$ and $\lambda + N > \frac{n}{2}$, then the operator K is bounded in $C(S^{n-1})$.

We shall need also the following lemma.

Lemma 3.10. *1). Let $\{k_m\}_{m=0}^{\infty} \in W_{\lambda,N}$ and $k_m \neq 0$ for all $m = 0, 1, 2, ...$ Then $\left\{\frac{1}{k_m}\right\}_{m=0}^{\infty} \in W_{-\lambda,N}$.*
2). Let $\{k_m\}_{m=0}^{\infty} \in W_{\lambda,N}$ and $\{\ell_m\}_{m=0}^{\infty} \in W_{\mu,N}$. Then $\{k_m \ell_m\}_{m=0}^{\infty} \in W_{\lambda+\mu,N}$.

Proof. The statement 2) is obvious. To see 1), it suffices to observe that $k_m = c_0 m^\lambda (1 + A_m)$ with $A_m = \frac{c_1}{c_0}\frac{1}{m} + ... + \frac{c_{N-1}}{c_0}\frac{1}{m^{N-1}} + \frac{c_{N-1}}{c_0} O\left(\frac{1}{m^N}\right)$ so that in the fraction $\frac{1}{1+A_m}$ we have $|A_m| < 1$ for large values of m. □

e) On spherical Riesz potential from $L_\infty(S^{n-1})$ into $H^\lambda(S^{n-1})$. In the proof of Lemma 3.11 below we follow Sobolev [18], where a similar statement was proved for regions in R^n.

Lemma 3.11. *The integral*

$$J_{\lambda,\mu,\nu}(x,y)=\int_{S^{n-1}}\frac{(|x-\sigma|+|y-\sigma|)^{\lambda}}{|x-\sigma|^{\mu}|y-\sigma|^{\nu}}\,d\sigma, \tag{3.33}$$

where $x,y\in S^{n-1}$ *and* $0<\mu<n-1, 0<\nu<n-1,\ \lambda\in R^1$, *admits the estimate*

$$J_{\lambda,\mu,\nu}(x,y)\le A\begin{cases}|x-y|^{-\gamma}, & \gamma>0\\ ln\,\frac{2}{|x-y|}, & \gamma=0\\ 1, & \gamma<0\end{cases} \tag{3.34}$$

where $\gamma=\mu+\nu-\lambda-n+1$ *and* A *does not depend on* x *and* y.

Proof. The change of variables $\sigma=x-\tau|x-y|$ yields

$$J_{\lambda,\mu,\nu}(x,y)=|x-y|^{-\gamma}\int_{S\left(\frac{x}{|x-y|},\frac{1}{|x-y|}\right)}\frac{(|\tau|+|\tau-e|)^{\lambda}}{|\tau|^{\mu}|\tau-e|^{\nu}}\,d\tau \tag{3.35}$$

where $S(x,r)$ stands for the sphere of the radius r centered at x and we denoted $e=\frac{x-y}{|x-y|}$ for brevity. We split the sphere $S\left(\frac{x}{|x-y|},\frac{1}{|x-y|}\right)$ into two parts

$$S^{*}=B(0,2)\bigcap S\left(\frac{x}{|x-y|},\frac{1}{|x-y|}\right)\quad \text{and}\quad S^{**}=S\left(\frac{x}{|x-y|},\frac{1}{|x-y|}\right)\Big\backslash S^{*}$$

where $B(0,2)$ is the ball of the radius 2 centered at the origin. In the representation

$$J_{\lambda,\mu,\nu}(x,y)=|x-y|^{-\gamma}\left[\int_{S^{*}}\cdots d\sigma+\int_{S^{**}}\cdots d\sigma\right]=|x-y|^{-\gamma}(J_1+J_2)$$

the integral J_1 is bounded since the integrand is bounded beyond the singular points $\tau=0$ and $\tau=e$, at which the singularities are weak. In the integral J_2 the integrand is equivalent to $|\tau|^{\lambda-\mu-\nu}$, so that $J_2\le c\int_{S^{**}}|\tau|^{\lambda-\mu-\nu}d\tau$. Hence, after the inverse change of variables $\tau=\frac{\sigma}{|x-y|}-\frac{x}{|x-y|}$ we obtain

$$J_2\le c|x-y|^{\gamma}\int_{S_*(x,y)}\frac{d\sigma}{|x-\sigma|^{\mu+\nu-\lambda}} \tag{3.36}$$

where $S_*(x,y)=\{\sigma:|\sigma|=1,|x-\sigma|\ge 2|x-y|\}$. The inequality is valid:

$$\Lambda(x,y):=\int_{S_*(x,y)}\frac{d\sigma}{|x-\sigma|^{n-1-\alpha}}\le\begin{cases}|x-y|^{\alpha}, & \alpha<0\\ ln\,\frac{2}{|x-y|}, & \alpha=0\\ 1, & \alpha>0\end{cases} \tag{3.37}$$

Indeed, let $\xi=\omega_x(\sigma)$ be any rotation on the sphere such that $\omega_x(e_1)=x, x\in S^{n-1}$. Then

$$\Lambda(x,y)=2^{\frac{\alpha-n+1}{2}}\int_{\substack{|\xi|=1\\ 1-\xi-1>2|x-y|^2}}(1-\xi_1)^{\frac{\alpha-n+1}{2}}\,d\xi$$

and then the required bounds for $\Lambda(x,y)$ are obtained by passing to the integration over the corresponding part of the ball in R^{n-1} and easy estimations. □

Lemma 3.12. *The operator* $K^\lambda, 0<\lambda<1$, *is bounded from* $L_\infty(S^{n-1})$ *into* $H^\lambda(S^{n-1})$.

Proof. By the inequality

$$\left|\frac{1}{a^\alpha}-\frac{1}{b^\alpha}\right| \le k\frac{|a-b|(a+b)^{\alpha-1}}{(ab)^\alpha}, \tag{3.38}$$

where $a>0, b>0$, and $k>0$ does not depend on a and b, we have

$$\left|(K^\lambda f)(x)-(K^\lambda f)(y)\right| \le c\|f\|_{L_\infty}|x-y|\int_{S^{n-1}}\frac{(|x-\sigma|+|y-\sigma|)^{n-2-\lambda}}{|x-\sigma|^{n-1-\lambda}|y-\sigma|^{n-1-\lambda}}\,d\sigma.$$

Then the application of Lemma 3.11 completes the proof. □

Corollary. $K^\lambda(C(S^{n-1})) \longrightarrow H^\lambda(S^{n-1})$ and $K^\lambda(C(S^{n-1})) \ne H^\lambda(S^{n-1}), 0<\lambda<1$.

f) Spherical hypersingular integrals. The spherical hypersingular operator (2.21) has the Laplace-Fourier expansion

$$(\mathfrak{D}^\lambda f)(x)=\sum_{m,\mu}\left[\frac{\Gamma\left(m+\frac{n-1+\lambda}{2}\right)}{\Gamma\left(m+\frac{n-1-\lambda}{2}\right)}-\frac{\Gamma\left(\frac{n-1+\lambda}{2}\right)}{\Gamma\left(\frac{n-1-\lambda}{2}\right)}\right]f_{m\mu}Y_{m\mu}(x),$$

see Samko [17], Lemma 6.26. As is known (Pavlov and Samko [13]; Samko [17], Theorem 6.32), the operator inverse to the spherical Riesz potential operator K^α is constructed in terms of the hypersingular operator, see Lemma 3.13 below, in which $X=L_p(S^{n-1}), 1\le p<\infty$ or $X=C(S^{n-1})$ and

$$c=\begin{cases}\frac{\Gamma\left(\frac{n-1+\lambda}{2}\right)}{\Gamma\left(\frac{n-1-\lambda}{2}\right)}, & \text{if} \quad n\ge 3 \ \text{ or } \ n=2 \ \text{ and } \ \lambda\ne 1\\ \frac{1}{ln\, 2}, & \text{if} \quad n=2 \ \text{ and } \ \lambda=1\ .\end{cases} \tag{3.39}$$

Lemma 3.13. *Let* $0<\lambda<2$ *and*

$$B^\lambda f = cf(x)+\lim_{\substack{\varepsilon\to 0\\(X)}}\mathfrak{D}^\lambda_\varepsilon f$$

where $\mathfrak{D}^\lambda_\varepsilon f$ *is the truncated integral (2.22). Then*

$$B^\lambda K^\lambda\varphi=\varphi, \quad \varphi\in X. \tag{3.40}$$

Moreover, $f(x)\in K^\lambda(X)$, *if and only if* $f\in X$ *and there exists the limit* $\lim_{\substack{\varepsilon\to 0\\(X)}}\mathfrak{D}^\lambda_\varepsilon f$.

g) Spherical Riesz-type transformation. We introduce the singular integral over the unit sphere

$$R_k\varphi=\int_{S^{n-1}}\frac{x_k-\sigma_k}{|x-\sigma|^n}\varphi(\sigma)d\sigma, \quad |x|=1, \quad k=1,2,...,n, \tag{3.41}$$

similar to the well known Riesz transformation in R^n. This integral, treated as the limit of the corresponding integral over $S^{n-1}_\varepsilon(x) = \{\sigma \in S^{n-1} : |\sigma - x| > \varepsilon\}$, may be also represented as

$$R_k\varphi = |S^{n-2}|x_k\varphi(x) + \int_{S^{n-1}} \frac{x_k - \sigma_k}{|x-\sigma|^n}\Big(\varphi(\sigma) - \varphi(x)\Big)d\sigma, \quad |x| = 1, \tag{3.42}$$

where the integral converges absolutely, for example when $\varphi(x) \in H^\lambda(S^{n-1})$.

Lemma 3.14. *The operator R_k is bounded in $H^\lambda(S^{n-1}), 0 < \lambda < 1$, and from $H^1(S^{n-1})$ into $\widetilde{H}^1(S^{n-1}) = \{f \in C(S^{n-1}) : \omega(f,t) \le Atln\ \frac{2}{t}\}$. It is not bounded in $C(S^{n-1})$: there exists a function $\psi_0 \in C(S^{n-1})$ such that $R_k\psi_0 \notin C(S^{n-1})$.*

The statement of the lemma is known: the boundedness within the framework of the spaces H^λ is even known in the general case of singular integrals over a Lyapunov manifold, see for example Mikhlin [10], p.56.

h) On differentiation of the spherical Riesz potential. Let $K^1\varphi$ be the Riesz potential (2.3) of order 1 and let $(K^1\varphi)(x')$ be its homogeneous continuation to R^n.

Lemma 3.15. *Let $\varphi \in C(S^{n-1})$. Then the formula is valid*

$$|x|\frac{\partial}{\partial x_i}(K^1\varphi)(x') = \frac{2-n}{\gamma_{n-1}(1)}R_i\varphi + \frac{2-n}{2}x'_i(K^1\varphi)(x'),\ i = 1,...,n,\ n \ge 3, \tag{3.43}$$

where R_i is the spherical Riesz singular operator (3.41). In the case $n = 2$ the corresponding relation involves the Hilbert transform:

$$|x|\frac{\partial}{\partial x_1}(K^1\varphi)(x') = \frac{\sin\psi}{2\pi}\int_{-\pi}^{\pi} \frac{\widetilde{\varphi}(\theta)\ d\theta}{tg\ \frac{\psi-\theta}{2}} \tag{3.44}$$

with $x = (cos\psi, \sin\psi)$, $\sigma = (cos\theta, \sin\theta)$ and $\widetilde{\varphi}(\theta) = \varphi(cos\theta, \sin\theta)$.

Proof. The direct differentiation gives

$$\frac{\partial}{\partial x_i}\ \frac{1}{\left|\sigma - \frac{x}{|x|}\right|^{n-2}} = \frac{n-2}{|x|^3}\ \frac{|x|^2\sigma_i - x_i(x\cdot\sigma)}{\left|\sigma - \frac{x}{|x|}\right|^n}\ . \tag{3.45}$$

Therefore, the differentiation under the integral sign gives the relation

$$|x|\frac{\partial}{\partial x_i}(K^1\varphi)(x') = \frac{2-n}{\gamma_{n-1}(1)}\int_{S^{n-1}} \frac{x'_i(x\cdot\sigma) - \sigma_i}{|x'-\sigma|^n}\varphi(\sigma)d\,\sigma \tag{3.46}$$

which is nothing else but (3.43). However, this differentiation under the integral sign needs justification, because we arrive at the singular integral. We give this justification in the Appendix **A**. □

4 Proofs of the main statements

Proof of Theorem A.

Proof. To prove the equivalence 1)$\longleftrightarrow$ 2), we have to show the equivalence of norms defined in 1) and 2) on functions in $C^\infty(S^{n-1})$. Let $\{d_m\}_{m=0}^\infty \in W_{-1,N}, N > \frac{n}{2}$, be the multiplier of the operator $\mathbb{D}$ and let $a = \lim_{m\to\infty} \frac{d_m}{m}$. The multiplier of the difference $\delta^{\frac{1}{2}} - \frac{1}{a}\mathbb{D}$ is in $W_{0,N-1}$. Then this difference is the identity operator plus an operator bounded in $C(S^{n-1})$ by Corollary to Lemma 3.9, whence the required equivalence becomes obvious.

The equivalence of the spaces 2)-4) will be given in the following direction: 2) $\longrightarrow$ 4)$\longrightarrow$ 3) $\longrightarrow$ 2). Let f be in the space defined by 2) and $\mathbb{D}$ an operator with the multiplier $k_m = m$. By 2), there exists a function $g \in C(S^{n-1})$ and a sequence $f_k \in C^\infty(S^{n-1})$ such that $\|f - f_k\|_C + \|g - \mathbb{D}f_k\|_C \to 0$ as $k \to \infty$. Hence we conclude that the harmonic continuation (2.2) of the function g has the form $(P_r g)(x) = \sum_{m,\mu} m r^m f_{m\mu} Y_{m\mu}(x),\ x \in S^{n-1}$. Then $\frac{d}{dr} P_r f = \sum_{m,\mu} m r^{m-1} f_{m\mu} Y_{m\mu}(x) = \frac{1}{r} P_r g$. Therefore, there exists $\lim_{r\to 1} \frac{d}{dr} P_r f \in C(S^{n-1})$. The conclusion 2)$\longrightarrow$ 4) has been proved.

To prove 4)$\longrightarrow$ 3), we suppose that there exists $g(x) = \lim_{r\to 1} \frac{d}{dr} P_r f \in C(S^{n-1})$. Then for $\omega(\sigma) \in C^\infty(S^{n-1})$ we have

$$(g, \omega) = \lim_{r\to 1} \left(\frac{d}{dr} P_r f, \omega \right) = \lim_{r\to 1} \left(f, \frac{d}{dr} P_r \omega \right). \tag{4.1}$$

The relation $\frac{d}{dr} P_r \omega = \frac{1}{r} P_r \mathbb{D}\omega$, where $\mathbb{D}$ is the operator with the multiplier $d_m = m$, is evidently valid for $\omega(\sigma) \in C^\infty(S^{n-1})$. Therefore, $(g, \omega) = (f, \mathbb{D}\omega)$, that is, 4)$\longrightarrow$ 3).

Finally, to prove the passage 3)$\longrightarrow$ 2), we assume that there exists the weak derivative $g = \mathbb{D}f$ in the sense 3). Let us show that $\mathbb{D}f_\varepsilon \to g$, where $f_\varepsilon(\sigma)$ is the approximation (3.18) with the Jackson kernel (3.22). Since $\mathbb{D}_x k(x\cdot\sigma) = \mathbb{D}_\sigma k(x \cdot \sigma)$ for any nice function $k(t)$, we have

$$\mathbb{D}f_\varepsilon = \int_{S^{n-1}} \mathbb{D}_x k_\varepsilon(x\cdot\sigma) f(\sigma) d\sigma = \int_{S^{n-1}} f(\sigma) \mathbb{D}_\sigma k_\varepsilon(x\cdot\sigma) d\sigma = \int_{S^{n-1}} k_\varepsilon(x\cdot\sigma) \mathbb{D}f(\sigma) d\sigma$$

the last passage being made by the definition 3) itself, with the inclusion $k_\varepsilon(x\cdot\sigma) \in C^\infty(S^{n-1})$ for all $x \in S^{n-1}$ taken into account. Therefore, 3)$\longrightarrow$ 2).

To complete the proof, it remains to show the equivalence 2)$\longleftrightarrow$ 5) $\longleftrightarrow$ 6). This equivalence is a particular case of Theorem D when $\lambda = 1$ and we observe that the proof of Theorem D is independent of the proof of our theorem. □

Proof of Theorem B.

Proof. The passage 7)$\longrightarrow$8) is valid by Lemma 2.11. Let us prove that 8)$\longrightarrow$7). Let $f(x)$ be a function having the weak derivatives $g_i = D_i f, i = 1, ..., n$, in the sense of the definition 8) and let $f_\varepsilon(x)$ be its approximation (3.18) with the Jackson kernel. Since the space $\mathcal{C}^1(S^{n-1})$ with the norm defined by 7) is complete, it suffices to prove that

$$\lim_{\varepsilon\to 0} \left\| \frac{\partial f_\varepsilon}{\partial x_i} - g_i \right\|_{C(S^{n-1})} = 0.$$

To calculate

$$\frac{\partial f_\varepsilon}{\partial x_i} = \int_{S^{n-1}} \frac{\partial}{\partial x_i} k_\varepsilon(x'\cdot\sigma) f(\sigma)\, d\sigma, \qquad x' = \frac{x}{|x|},$$

we observe that $|x|\frac{\partial}{\partial x_i} k_\varepsilon(x'\cdot\sigma) = k'_\varepsilon(x'\cdot\sigma)[\sigma_i - x'_i(x'\cdot\sigma)]$ and similarly for $|\sigma|\frac{\partial}{\partial \sigma_i} k_\varepsilon(x'\cdot\sigma)$, so that

$$|x|\frac{\partial}{\partial x_i} k_\varepsilon(x'\cdot\sigma) = -|\sigma|\frac{\partial}{\partial \sigma_i} k_\varepsilon(x'\cdot\sigma) + k'_\varepsilon(x'\cdot\sigma)(1 - x'\cdot\sigma')(x'_i + \sigma'_i).$$

Therefore,

$$|x|\frac{\partial}{\partial x_i} f_\varepsilon(x') = -\int_{S^{n-1}} f(\sigma)\frac{\partial}{\partial \sigma_i} k_\varepsilon(x'\cdot\sigma)\, d\sigma$$

$$+\int_{S^{n-1}} k'_\varepsilon(x'\cdot\sigma)(\sigma_i + x'_i)(1 - x'\cdot\sigma) f(\sigma)\, d\sigma \;:=\; -\mathcal{L}_\varepsilon(x') + \mathcal{I}_\varepsilon(x'). \tag{4.2}$$

To the term $\mathcal{L}_\varepsilon(x')$ the relation (2.32) is applicable, which is easy and will be considered at the end of the proof. The main point is to calculate $\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon(x')$. We shall prove that

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon(x') = (n-1)x'_k f(x'). \tag{4.3}$$

We have

$$\mathcal{I}_\varepsilon(x') = x'_i\int_{S^{n-1}} k'_\varepsilon(x'\cdot\sigma)(1 - x'\cdot\sigma) f(\sigma)\, d\sigma$$

$$+\int_{S^{n-1}} \sigma_i k'_\varepsilon(x'\cdot\sigma)(1 - x'\cdot\sigma) f(\sigma)\, d\sigma := \;\mathcal{I}_{1,\varepsilon}(x') + \mathcal{I}_{2,\varepsilon}(x').$$

For $\mathcal{I}_{1,\varepsilon}(x')$, by (3.11) we have

$$\mathcal{I}_{1,\varepsilon}(x') = x'_i|S^{n-2}|\int_{-1}^{1}\frac{dk_\varepsilon(t)}{dt}(1-t)^{\frac{n-1}{2}}(1+t)^{\frac{n-3}{2}}(T_t f)(x')\, dt.$$

Integrating by parts and making use of formula (3.10), we obtain

$$\mathcal{I}_{1,\varepsilon}(x') = x'_i|S^{n-2}|\frac{n-1}{2}\int_{-1}^{1} k_\varepsilon(t)(1-t^2)^{\frac{n-3}{2}} T_t f(x')\, dt$$

$$+\, x'_i|S^{n-2}|\frac{n-3}{2}\int_{-1}^{1} k_\varepsilon(t)(1-t)^2(1-t^2)^{\frac{n-5}{2}} T_t f(x')\, dt$$

$$-\, x'_i\,|S^{n-2}|\sum_{j=1}^{n} x'_j\int_{-1}^{1} k_\varepsilon(t)(1-t)(1-t^2)^{\frac{n-5}{2}} T_t g_j(x')\, dt \;=:\; A_\varepsilon(x') + B_\varepsilon(x') + C_\varepsilon(x'), \tag{4.4}$$

with zero non-integral terms. The latter is obvious when $n > 3$, while in the cases $n = 2$ and $n = 3$ one may refer to the property $k_\varepsilon(t) \le c(1-t)$ (with c depending on ε).

By Corollary 1 to Lemma 3.6, we see that

$$\lim_{\varepsilon\to 0} A_\varepsilon(x') = \frac{n-1}{2}x'_i f(x'). \tag{4.5}$$

We wish to show that

$$\lim_{\varepsilon\to 0} B_\varepsilon(x') = \lim_{\varepsilon\to 0} C_\varepsilon(x') = 0. \tag{4.6}$$

For $B_\varepsilon(x')$ we have

$$|B_\varepsilon(x')| \le c\|f\|_{C(S^{n-1})}\frac{1}{\varkappa_\nu}\int_{-1}^{1}\mathcal{D}_\nu^s(arccos\ t)(1-t)^2(1-t^2)^{\frac{n-5}{2}}\ dt$$

$$= \frac{c_1}{\varkappa_\nu}\int_0^\pi \mathcal{D}_\nu^s(u)\sin^{n-4}u(1-\cos u)^2 du \le \frac{c_2}{\varkappa_\nu}\int_0^\pi u^2\mathcal{D}_\nu^s(u)\sin^{n-2}u du. \tag{4.7}$$

Making use of the estimate from (3.23), we see that $|B_\varepsilon(x')| \le c_3\varepsilon^{min(2,2s-n+1)}$.

To show that $\lim_{\varepsilon\to 0} C_\varepsilon(x') = 0$, we consider the functionals

$$V_\varepsilon\varphi = \int_{-1}^{1} k_\varepsilon(t)(1-t)(1-t^2)^{\frac{n-5}{2}}\varphi(t)dt$$

generated by this term. We wish to show that

$$\lim_{\varepsilon\to 0} V_\varepsilon\varphi = 0 \tag{4.8}$$

for any

$$\varphi(t) \in C_0([-1,1]) = \{\varphi(t) : \varphi(t) \in C([-1,1]),\ \varphi(1) = 0\}.$$

The functionals $V_\varepsilon\varphi$ are uniformly bounded in $C([-1,1])$:

$$|V_\varepsilon\varphi| \le \frac{c}{\varkappa_\nu}\|\varphi\|_C\int_0^\pi \mathcal{D}_\nu^s(u)sin^{n-2}u\ du \le c\|\varphi\|_C$$

in view of (3.23). Then by the Banach-Steinhauss theorem, it suffices to check the passage (4.8) on a dense set $C_0^\infty([-1,1])$. For such functions we have

$$|V_\varepsilon\varphi| \le \int_{-1}^{1} k_\varepsilon(t)(1-t^2)^{\frac{n-3}{2}}\left|\frac{\varphi(t)-\varphi(-1)\frac{1-t}{2}}{1+t}\right|\ dt$$

$$+\frac{1}{2}|\varphi(-1)|\int_{-1}^{1} k_\varepsilon(t)(1-t)^2(1-t^2)^{\frac{n-5}{2}}\ dt.$$

The first term tends to zero by the property (3.20). The tendency to zero of the second term was in fact shown in (4.7).

To apply (4.8) in the case of the term $C_\varepsilon(x')$, we have to show that the function

$$\varphi(t) = \sum_{j=1}^{n} x'_j(T_t g_j)(x') \tag{4.9}$$

tends to zero as $t\to 1$. Evidently, $\lim_{t\to 1}\varphi(t) = \sum_{j=1}^n x'_j g_j(x')$. The equality $\sum_{j=1}^n x'_j g_j(x') = 0$ is a consequence of the Euler equation for homogeneous functions. Its justification for our case when the derivatives g_j are treated in the weak sense, is easily done via (2.6). Then (4.8) is applicable and we get $\lim_{\varepsilon\to 0} C_\varepsilon(x') = 0$. Then from (4.4), (4.5) and (4.6)

$$\lim_{\varepsilon\to 0} \mathcal{I}_{1,\varepsilon}(x') = \frac{n-1}{2} x'_k f(x'). \tag{4.10}$$

In a similar way, with Lemma 3.5 taken into account, one may obtain also that

$$\lim_{\varepsilon\to 0} \mathcal{I}_{2,\varepsilon}(x') = \frac{n-1}{2} x'_k f(x'). \tag{4.11}$$

Hence (4.3) follows.

Returning to (4.2), we apply the passage (2.6) in the term $\mathcal{L}_\varepsilon(x')$ and make use of (4.3), which yields

$$\lim_{\varepsilon\to 0} |x| \frac{\partial}{\partial x_i} f_\varepsilon(x') = \lim_{\varepsilon\to 0} \int_{S^{n-1}} \left[-(n-1)\sigma_i f(\sigma) + g_i(\sigma)\right] k_\varepsilon(x'\cdot\sigma)\, d\sigma \tag{4.12}$$
$$+(n-1)x'_k f(x').$$

Applying Lemma 3.6, we obtain that $\lim_{\varepsilon\to 0} D_i f_\varepsilon(x') = g_i(x')$. The theorem has been proved. □

Proof of Theorem C.

Proof. We start proving that $\mathbb{C}^1(S^{n-1}) \nsubseteq \mathcal{C}^1(S^{n-1})$. Suppose that on the contrary,

$$\mathbb{C}^1(S^{n-1}) \subseteq \mathcal{C}^1(S^{n-1}). \tag{4.13}$$

By the statement 5) of Theorem A, any $f(x) \in \mathbb{C}^1(S^{n-1})$ is representable as

$$f(x) = K^1\varphi, \quad \varphi \in C(S^{n-1}), \tag{4.14}$$

where K^1 is the Riesz potential operator (2.3) of order 1. We apply Lemma 3.15. From the assumption made in (4.13) and formula (3.43) it follows that $R_i\varphi \in C(S^{n-1})$ for any $\varphi \in C(S^{n-1})$ which is not possible by Lemma 3.14. Therefore, (4.13) cannot be valid.

Inversely, let us show that

$$\mathcal{C}^1(S^{n-1}) \nsubseteq \mathbb{C}^1(S^{n-1}). \tag{4.15}$$

By Theorem A and Lemma 3.13, the space $\mathbb{C}^1(S^{n-1})$ is characterized by the condition that there exists the limit

$$\mathfrak{D}^1 f \;=\; \lim_{\substack{\varepsilon\to 0\\(C)}} \mathfrak{D}^1_\varepsilon f. \tag{4.16}$$

So to prove (4.15), we show that there exists a function $f \in \mathcal{C}^1(S^{n-1})$ for which the limit (4.16) is not in $C(S^{n-1})$.

To this end, we observe that the hypersingular integral $\mathfrak{D}^1 f$ of a function $f \in \mathcal{C}^1(S^{n-1})$ may be represented in the form

$$\mathfrak{D}^1 f \;=\; \lim_{\varepsilon\to 0} \sum_{i=1}^n \int_{S^{n-1}_\varepsilon(x')} \frac{\sigma_i - x'_i}{|x'-\sigma|^n} \int_0^1 \left(\frac{\partial}{\partial\sigma_i} f^*\right)(x' + u(\sigma - x'))\,du\,d\sigma \tag{4.17}$$

where $S^{n-1}_\varepsilon(x') = \{\sigma \in S^{n-1} : |\sigma - x'| \le \varepsilon\}$ and we denoted $f^*(x) = f\left(\frac{x}{|x|}\right)$. To get (4.17), it suffices to apply the Taylor formula to the difference $f(\sigma) - f(x)$ in the hypersingular integral $\mathfrak{D}^1 f$.

Let $g(t)$ be a differentiable function of a single variable $t \in [-1, 1]$ with the value $g(0) = 0$. We take $f(x')$ as $f(x') = g\left(\frac{x_1}{|x|}\right)$, meaning to choose $g(t)$ later in such a way that $\mathfrak{D}^1 f \notin C(S^{n-1})$.

From (4.17) after some calculation we arrive at the following representation for $(\mathfrak{D}^1 f)(x)$, $|x| = 1$:

$$(\mathfrak{D}^1 f)(x) = \lim_{\varepsilon\to 0} \int_{S^{n-1}_\varepsilon} \frac{\sigma_1 - x_1'}{|x' - \sigma|^n} d\sigma \int_0^1 \frac{2 - u|x-\sigma|^2}{|x + u(\sigma - x)|^3} g'\left(\frac{x_1 + u(\sigma_1 - x_1)}{|x + u(\sigma_1 - x)|}\right) du$$

$$+ x_1 \int_{S^{n-1}} \frac{d\sigma}{|x-\sigma|^{n-2}} \int_0^1 \frac{\frac{1}{2} - u}{|x + u(\sigma - x)|^3} g'\left(\frac{x_1 + u(\sigma_1 - x_1)}{|x_1 + u(\sigma_1 - x_1)|}\right) du := A(x) + B(x).$$

The term $B(x)$ is a function continuous on S^{n-1}. To treat the term $A(x)$ as $x \to e_n$, we observe that

$$\int_0^1 \frac{2 - u|e_n - \sigma|^2}{|e_n + u(\sigma - e_n)|^3} g'\left(\frac{u\sigma_1}{|e_n + u(\sigma_1 - e_n)|}\right) du = 2\, \frac{g(\sigma_1)}{\sigma_1},$$

because

$$2 \frac{d}{du}\left(\frac{u\sigma_1}{|e_n + u(\sigma_1 - e_n)|}\right) = \frac{2 - u|e_n - \sigma|^2}{|e_n + u(\sigma - e_n)|^3}\ .$$

Therefore,

$$A(e_n) = 2 \lim_{\varepsilon \to 0} \int_{|\sigma - e_n| > \varepsilon} \frac{g(\sigma_1)}{|\sigma - e_n|^n}\, d\sigma\ . \tag{4.18}$$

It remains to show that it is possible to choose a differentiable function $g(t)$ in such a way that the singular integral $A(e_n)$ does not exists. This is shown in Appendix **B**. □

Proof of Theorem D.

Proof. To prove the coincidence (2.19), we observe that the space

$$K^\lambda(C) = \mathbb{K}^\lambda(C) = \mathfrak{K}^\lambda(C) \tag{4.19}$$

is the closure of $C^\infty(S^{n-1})$ with respect to the norm

$$\|f\|_{K^\lambda(C)} = \|f\|_C + \|B^\lambda f\|_C,$$

where B^λ is the inverse to any of the potential operators $\mathbb{K}^\lambda$, $\mathfrak{K}^\lambda$ or K^λ (We do not need to know the construction itself of the inverse operator; it suffices to know the behaviour of its Fourier multiplier at infinity). The coincidence (4.19) follows from the fact that the operators $\mathbb{K}^\lambda$, $\mathfrak{K}^\lambda$ and K^λ differ one from another by a factor which is a continuous operator boundedly invertible in the space $C(S^{n-1})$, the latter being a consequence of "nice" behaviour at infinity of quotients of the Fourier multipliers of the operators $\mathbb{K}^\lambda$, $\mathfrak{K}^\lambda$ and K^λ. The reference to Lemmas 3.9 and 3.10 is also relevant.

Now, to prove that the space $\mathbb{C}^\lambda(S^{n-1})$ coincides with the range (4.19), we refer to the same reason: by Definition 2.4, the space $\mathbb{C}^\lambda(S^{n-1})$ is the closure of $C^\infty(S^{n-1})$ with respect to the norm $\|f\|_{\mathbb{C}^\lambda} = \|f\|_C + \|\delta^{\frac{\lambda}{2}} f\|_C$. Therefore, it remains to show the equivalence of norms, which again may be obtained via the investigation of the quotients of multipliers.

The relation (2.19) being proved, the last statement of the theorem follows immediately by Lemma 3.13.

The description in (2.20) in terms of the coefficients is obtained in the standard way. Let $f(x) \in \mathbb{C}^\lambda(S^{n-1})$. Then $f(x) = K^\lambda \varphi$ with $\varphi \in C(S^{n-1})$, by (2.19). Therefore, $f_{m\mu} = k_m^\lambda \varphi_{m\mu}$, where k_m^λ is the Fourier multiplier of the potential K^λ.

□

Proof of Theorem E.

Proof. In the case $0 < \lambda < 1$, the statements in (2.30) have already been obtained, see Corollary to Lemma 3.12.

Let $\lambda > 1$. To obtain the imbedding in (2.30), we observe that according to Theorem A and Remark 2.9, we may construct a spherical differentiation operator $\mathfrak{D}^k$ equivalent to the operator $\delta^{\frac{k}{2}}$, with the multiplier in the class $Wk, N,\ N \geq \left[\frac{n-1}{2}\right]$, which reduces the order of the potential, that is, $\mathfrak{D}^k K^\lambda = K^{\lambda-k}$. Taking $k = [\lambda]$, we easily arrive at the required statement by reducing to the case $\lambda - [\lambda] < 1$.

The equalities in (2.31) are easy in the case $0 < \lambda < 1$. The first equality fulfills by definition. Further, the imbedding $H^\lambda(S^{n-1}) \longrightarrow H_*^\lambda(S^{n-1})$ is obvious by (3.14). The inverse imbedding follows from Jackson type statement (3.25) for the averaged continuity modulus $\omega_*(f,t)$ and the Bernstein type theorem (see Berens, Butzer and Pawelke [3], p. 216) for the usual continuity modulus $\omega(f,t)$.

Let $\lambda > 1$. The proof of the equalities in (2.31) will be given according to the following scheme:

$$\mathbb{H}_*^\lambda(S^{n-1}) \longrightarrow \mathbb{H}^\lambda(S^{n-1}) \longrightarrow \mathcal{H}^\lambda(S^{n-1}) \longrightarrow \mathcal{H}_*^\lambda(S^{n-1}) \longrightarrow \mathbb{H}_*^\lambda(S^{n-1}). \tag{4.20}$$

When $1 < \lambda < 2$, the imbedding $\mathbb{H}_*^\lambda(S^{n-1}) \longrightarrow \mathbb{H}^\lambda(S^{n-1})$ follows from Lemma 3.8. Indeed, let $f \in \mathbb{H}_*^\lambda(S^{n-1})$ so that $\omega_*(f,t) \leq t^\lambda$. Then, as is known, (see e.g. Lizorkin and Nikol'skii [8], p.209), $E_m(f) \leq \frac{c}{m^\lambda}$, $1 < \lambda < 2$, and by Lemma 3.8,

$$E_m(\delta^{\frac{1}{2}} f) \leq c\left[\frac{1}{m^\lambda} + \sum_{k=m+1}^{\infty} \frac{1}{k^\lambda}\right].$$

Therefore, $E_m(\delta^{\frac{1}{2}} f) \leq \frac{c}{m^{\lambda-1}}$ and by the Bernstein type theorem (Lizorkin and Nikol'skii [8], p. 216) we obtain $\delta^{\frac{1}{2}} f \in \mathbb{H}^{\lambda-1}(S^{n-1})$. The proof for $\lambda > 2$ follows by induction.

Passing to the second imbedding in (4.20), we take $f \in \mathbb{H}^\lambda(S^{n-1})$ and $1 < \lambda < 2$. Taking Remark 2.5 into account, we have $B^1 f \in \mathbb{H}^{\lambda-1}(S^{n-1})$ where B^1 is the operator inverse to the operator K^1 (It has the multiplier $\frac{\Gamma\left(m+\frac{n}{2}\right)}{\Gamma\left(m+\frac{n}{2}-1\right)}$, with the corresponding modification in the case $n = 2, m = 0$). Therefore, $f = K^1\varphi$ with $\varphi = B^1 f \in \mathbb{H}^{\lambda-1}(S^{n-1})$ (see Lemma 3.13). Making use of formula (3.43) and Lemma 3.14, we obtain

$$\left\|\frac{\partial}{\partial x_k}f\right\|_{(\lambda)} \leq c\|B^1 f\|_{(\lambda)},$$

where $\|f\|_{(\lambda)} = \sup_{x,\sigma\in S^{n-1}} \frac{|f(x)-f(\sigma)|}{|x-\sigma|^\lambda}$. Therefore, $\mathbb{H}^\lambda(S^{n-1}) \longrightarrow \mathcal{H}^\lambda(S^{n-1}), 1 < \lambda < 2$.

When $\lambda > 2$, the idea of consideration is the same, so we omit the details, only sketching main points. In the case $2 < \lambda < 3$ we have to estimate the partial derivatives of order 2. Suppose that $f \in \mathbb{H}^\lambda(S^{n-1})$. Then

$$f = K^{[\lambda]}\varphi \quad \text{ with } \quad \varphi \in \mathbb{H}^{\lambda-[\lambda]}(S^{n-1}) \tag{4.21}$$

where $[\lambda] = 2$ in the considered case.

From (4.21) and (3.46) we have

$$|x|^2\frac{\partial^2}{\partial x_k^2}f = \frac{3-n}{\gamma_{n-1}(2)}|x|^2\frac{\partial}{\partial x_k}\int_{S^{n-1}} \frac{\frac{x'_k}{|x|}(x'\cdot\sigma) - \frac{\sigma_k}{|x|}}{|x'-\sigma|^{n-1}}\varphi(\sigma)\, d\sigma$$

and similarly to the previous case we arrive at some singular operator bounded in $\mathbb{H}^\lambda(S^{n-1})$. The case of mixed derivatives is similar, with easier calculations.

When $3 < \lambda < 4$, similar actions for $|x|^3\frac{\partial^3}{\partial x_i^3}f$ lead also to a singular operator, etc.

Now, the second embedding in (4.20) being over, we have to prove the next step:

$$\mathcal{H}^\lambda(S^{n-1}) \longrightarrow \mathcal{H}_*^\lambda(S^{n-1})\ .$$

We consider the case $1 < \lambda < 2$, the general case $\lambda > 2$ may be treated by induction. Let $f \in \mathcal{H}^\lambda(S^{n-1})$. The space $\mathcal{H}_*^\lambda(S^{n-1})$ being defined in terms of the shift (3.1), we consider the difference

$$f(x) - T_t f(x) = \frac{1}{|S_{n-2}|}\int_{S^x_{n-2}} \left[f(x) - f(xt + \tau\sqrt{1-t^2})\right]\, d\tau \tag{4.22}$$

where $S_x^{n-2} = \{\tau \in S^{n-1} :\ \tau\cdot x = 0\}$. Since $f \in \mathcal{H}^\lambda(S^{n-1})$, there exist

$$D_k f = |x|\frac{\partial}{\partial x_k}f\left(\frac{x}{|x|}\right) \quad \text{ and } \quad D_k f \in H^\lambda(S^{n-1}), \qquad k = 1, ..., n,$$

which enables us to use the Taylor formula, as in (4.17), and we obtain

$$f(x) - T_t f(x) = \frac{\sqrt{1-t}}{|S_{n-2}|}\sum_{k=1}^n \int_{S_x^{n-2}} \left(x'_k\sqrt{1-t} - \tau_k\sqrt{1+t}\right)\, d\tau \int_0^1 \frac{(D_k f)[\theta'_{x',t}(u,\tau)]}{|\theta_{x',t}(u,\tau)|}\, du, \tag{4.23}$$

where the vector $\theta_{x',t}(u,\tau)$ is defined by

$$\theta_{x',t}(u,\tau) = x' + \sqrt{1-t}\ u\left(x'\sqrt{1-t} - \tau\sqrt{1+t})\right)$$

and as usual, $\theta'_{x',t}(u,\tau) = \frac{\theta_{x',t}(u,\tau)}{|\theta_{x',t}(u,\tau)|}$.

We represent further the difference (4.23) as

$$f(x) - T_t f(x) = \frac{\sqrt{1-t}}{|S_{n-2}|} \sum_{k=1}^{n} \int_{S_x^{n-2}} \left(x'_k \sqrt{1-t} - \tau_k \sqrt{1+t} \right) d\tau \int_0^1 \frac{D_k f[\theta'_{x',t}(u,\tau)] - D_k f(x')}{|\theta_{x',t}(u,\tau)|} \, du \tag{4.24}$$

$$- \frac{1}{|S_{n-2}|} \left\{ \sum_{k=1}^{n} D_k f(x') \int_{S_x^{n-2}} (x'_k t + \tau_k \sqrt{1-t^2}) \, d\tau \int_0^1 \frac{du}{|\theta_{x',t}(u,\tau)|} - \int_{S_x^{n-2}} \int_0^1 \frac{du d\tau}{|\theta_{x',t}(u,\tau)|} \sum_{k=1}^{n} x'_k D_k f(x') \right\}.$$

Since $x' \cdot \tau = 0$, we obtain

$$|\theta_{x',t}(u,\tau)| = \sqrt{2(1-t)u^2 + 2(1-t)u + 1} \quad \text{and} \quad \int_0^1 \frac{du}{|\theta_{x',t}(u,\tau)|} = \frac{1}{\sqrt{1-t}} \, ln \, \frac{\sqrt{2} + \sqrt{1-t}}{\sqrt{2} - \sqrt{1-t}}.$$

Taking also into account that

$$(T_t \sigma_k)(x') = \frac{1}{|S_{n-2}|} \int_{S_x^{n-2}} (x'_k t + \tau_k \sqrt{1-t^2}) \, d\tau = t \cdot x'_k$$

by the first of the formulas (3.12) and making use of the Euler equation $x \cdot grad \, f(x') = 0$ for functions homogeneous of degree 0, from (4.24) we obtain the representation

$$f(x) - T_t f(x) = \frac{\sqrt{1-t}}{|S_{n-2}|} \sum_{k=1}^{n} \int_{S_x^{n-2}} \left(x'_k \sqrt{1-t} - \tau_k \sqrt{1+t} \right) d\tau \int_0^1 \frac{D_k f[\theta'_{x',t}(u,\tau)] - D_k f(x')}{|\theta_{x',t}(u,\tau)|} \, du. \tag{4.25}$$

Since the derivatives $D_k f$ satisfy the Hölder condition of order $\lambda - 1$, we get

$$|f(x) - T_t f(x)| \le c(1-t)^{\frac{\lambda}{2}} \frac{ln \, \frac{\sqrt{2}+\sqrt{1-t}}{\sqrt{2}-\sqrt{1-t}}}{\sqrt{1-t}} \int_{S_x^{n-2}} \left| x'_k \sqrt{1-t} - \tau_k \sqrt{1+t} \right|^{1+\lambda} d\tau.$$

Hence the estimate

$$|f(x) - S_h f(x)| \le c h^{\lambda}$$

easily follows, which implies $\omega_*(f,h) \le ch\lambda$.

The final imbedding $\mathcal{H}_*^{\lambda}(S^{n-1}) \longrightarrow \mathbb{H}_*^{\lambda}(S^{n-1})$ in (4.20) is obvious since

$$\delta^m f = |x|^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \left[\cdots \left[|x|^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f \right] \cdots \right].$$

It remains to consider the integer values of λ. When $\lambda = 1, 2, 3, ...$, the spaces $\mathcal{H}^{\lambda}(S^{n-1})$ and $\mathbb{H}_*^{\lambda}(S^{n-1})$ do not coincide with each other because of Theorem C, and $\mathbb{H}^{\lambda}(S^{n-1}) \nsubseteq \mathcal{H}_*^{\lambda}(S^{n-1})$ and also $\mathcal{H}^{\lambda}(S^{n-1}) \nsubseteq \mathbb{H}_*^{\lambda}(S^{n-1})$, since the arising corresponding singular operators are unbounded in $C(S^{n-1})$.

The embedding $\mathcal{H}^1(S^{n-1}) \longrightarrow H^1_*(S^{n-1})$ is easily derived from (3.14). The embedding $\mathcal{H}^{2m+1}(S^{n-1}) \longrightarrow \mathbb{H}^{2m+1}(S^{n-1})$ follows by induction.

□

Appendix A. Justification of formula (3.46).

Because of the singular integral at which we arrive after the differentiation under the integral sign in (3.46), it is convenient to deal with the weak interpretation of the derivative $\frac{\partial}{\partial x_i}$, which is possible by Theorem B. By the relation (2.32),

$$\left(|x|\frac{\partial}{\partial x_i}(K^1\varphi)(x'),\psi\right) = (n-1)\left(x_i'K^1\varphi,\psi\right) - \left(\varphi, K^1D_i\psi\right) \tag{4.26}$$

where $\psi \in C^\infty(S^{n-1})$. Since

$$\int_{S^{n-1}} f(\sigma)d\sigma = (n+\gamma)\int_{B(0,1)} f(y)dy \tag{4.27}$$

for any function homogeneous of degree γ, we obtain

$$(K^1D_i\psi)(x') = \frac{n-1}{\gamma_{n-1}(1)}\lim_{\varepsilon\to 0}\int_{B\setminus K_\varepsilon} \frac{\frac{\partial}{\partial y_i}\psi\left(\frac{y}{|y|}\right)}{\left|\frac{y}{|y|}-x'\right|^{n-2}}\,dy$$

where $B = B(0,1)$ and $K_\varepsilon = B(0,1)\bigcap V_\varepsilon$ where $V_\varepsilon = \left\{y : \left|\frac{y}{|y|}-x'\right| \le \varepsilon\right\}$ is a conic neighborhood of the point x'. Hence

$$(K^1D_i\psi)(x') = \frac{n-1}{\gamma_{n-1}(1)}\lim_{\varepsilon\to 0}\left\{\int_{B\setminus K_\varepsilon}\frac{\partial}{\partial y_i}\left[\frac{\psi\left(\frac{y}{|y|}\right)}{\left|\frac{y}{|y|}-x'\right|^{n-2}}\right]dy\right.$$

$$\left. - \int_{B\setminus K_\varepsilon}\frac{\partial}{\partial y_i}\left[\frac{1}{\left|\frac{y}{|y|}-x'\right|^{n-2}}\right]\psi\left(\frac{y}{|y|}\right)\,dy\right\}.$$

Making use of the Gauss-Ostrogradski formula, we obtain

$$(K^1D_i\psi)(x') = \frac{n-1}{\gamma_{n-1}(1)}\lim_{\varepsilon\to 0}\int_{S^{n-1}_\varepsilon}\frac{\sigma_i\psi(\sigma)}{|\sigma-x'|^{n-2}}\,d\sigma$$

$$+\lim_{\varepsilon\to 0}\left\{\int_{\widetilde{K}_\varepsilon}\frac{\cos(\overrightarrow{n},\sigma_i)}{\left|\frac{y}{|y|}-x'\right|^{n-2}}\psi\left(\frac{y}{|y|}\right)\,dy - \int_{B\setminus K_\varepsilon}\frac{\partial}{\partial y_i}\left[\frac{1}{\left|\frac{y}{|y|}-x'\right|^{n-2}}\right]\psi\left(\frac{y}{|y|}\right)\,dy\right\}$$

where $S_\varepsilon^{n-1} = \{\sigma \in S^{n-1} : |\sigma - x'| \geq \varepsilon\}$ and $\widetilde{K}_\varepsilon = \{y \in B : |y - x'| = \varepsilon\}$ and $\overrightarrow{n}$ is the normal vector to the cone surface $\widetilde{K}_\varepsilon$. Then from (4.26), with (3.45) taken into account, we conclude that

$$\left(|x|\frac{\partial}{\partial x_i}(K^1\varphi)(x'), \psi\right) = \frac{1}{\gamma_{n-1}(1)}\lim_{\varepsilon\to 0}\left\{(n-2)\left(\varphi, \int_{S_\varepsilon^{n-1}} \frac{x_i' - \sigma_i(\sigma\cdot x')}{|\sigma - x'|^n}\psi(\sigma)\,d\sigma\right)\right. \tag{4.28}$$

$$\left. -(n-1)\left(\varphi, \int_{\widetilde{K}_\varepsilon} \frac{\cos\left(\overrightarrow{n}, \frac{y_k}{|y|}\right)}{\left|x - \frac{y}{|y|}\right|^{n-2}}\psi(y)\,dS_y\right)\right\}.$$

We have to show that in reality the second term disappears. We denote

$$N_\varepsilon\psi := \frac{n-1}{\gamma_{n-1}(1)}\int_{\widetilde{K}_\varepsilon} \frac{\cos\left(\overrightarrow{n}, \frac{y_k}{|y|}\right)}{\left|x - \frac{y}{|y|}\right|^{n-2}}\psi(y)\,dS_y.$$

Since ψ is bounded and $\left|\cos\left(\overrightarrow{n}, \frac{y_k}{|y|}\right)\right| \leq 1$, we obtain that

$$|N_\varepsilon\psi| \leq c\int_{\widetilde{K}_\varepsilon} \frac{1}{\left|x - \frac{y}{|y|}\right|^{n-2}}\,dS_y = c\int_{\widetilde{K}_\varepsilon} \frac{1}{\varepsilon^{n-2}}\,dS_y = c_1\varepsilon \to 0$$

as $\varepsilon \to 0$. Then from (4.28) we arrive at (3.46).

Appendix B. The integral $A(e_n)$ has the form

$$A(e_n) = \lim_{\varepsilon\to 0}\int_{\substack{|\sigma - e_n|\geq\varepsilon \\ \sigma_n \geq 0}} \frac{g(\sigma_1)}{|\sigma - e_n|^n}\,d\sigma + \int_{\sigma_n\geq 0}\frac{g(\sigma_1)}{|\sigma + e_n|^n}\,d\sigma.$$

The second term here always exists, whatever the continuous function $g(t)$ is. As regards the first term, we project it onto the $(n-1)$-dimensional unit ball $B^{n-1}(0,1)$ which is the base of the semisphere $\{\sigma : \sigma \in S^{n-1}, \sigma_n \geq 0\}$, according to the formula:

$$\int_{S^{n-1},\sigma_n\geq 0} f(\sigma)\,d\sigma = \int_{B^{n-1}(0,1)} \frac{f(y, \sqrt{1-|y|^2})}{\sqrt{1-|y|^2}}\,dy \tag{4.29}$$

where $y = (y_1, y_2, ..., y_{n-1}) \in R^{n-1}$, and obtain the integral

$$\lim_{\varepsilon\to 0}\frac{1}{2^{\frac{n}{2}}}\int_{0\leq\sigma_n\leq 1-\varepsilon}\frac{g(\sigma_1)}{(1-\sigma_n)^{\frac{n}{2}}}\,d\sigma = \frac{1}{2^{\frac{n}{2}}}\lim_{\varepsilon\to 0}\int_{B^{n-1}(0,1)}\frac{g(y_1)\,dy}{\left(1-\sqrt{1-|y|^2}\right)^{\frac{n}{2}}\sqrt{1-|y|^2}}$$

$$= \frac{1}{2^{\frac{n}{2}}}\lim_{\delta\to 0+}\int_{\delta\leq|y|\leq 1}\frac{g(y_1)\left(1+\sqrt{1-|y|^2}\right)^{\frac{n}{2}}}{|y|^n\sqrt{1-|y|^2}}dy\ .$$

Then it is clear that it suffices to show that there exists a differentiable function $g(y_1)$ such that the singular integral

$$\lim_{\delta\to 0+}\int_{\delta\le|y|\le 1}\frac{g(y_1)}{|y|^n}\,dy$$

does not exist, which is already easily reduced to the one-dimensional case. Indeed,

$$\lim_{\delta\to 0+}\int_{\delta\le|y|\le 1}\frac{g(y_1)}{|y|^n}=\lim_{\delta\to 0+}\int_{-1}^{1}g(y_1)dy_1\int_{\delta\le|\widetilde{y}|^2+y_1^2\le 1}\frac{d\widetilde{y}}{(|\widetilde{y}|^2+y_1^2)^{\frac{n}{2}}}$$

where $\widetilde{y}=(y_2,...y_{n-1})$. An easy calculation of the inner integral yields

$$\lim_{\delta\to 0+}\int_{\delta\le|y|\le 1}\frac{g(y_1)}{|y|^n}\,dy=\lim_{\delta\to 0+}\int_{\delta\le|y_1|\le 1}\frac{g(y_1)A(y_1)}{y_1^2}\,dy_1$$

where the function $A(t)=|S^{n-3}|\int_{|t|}^{1}\xi(1-\xi^2)^{\frac{n-4}{2}}d\xi$ is non vanishing and having the bounded derivative near the origin. (We take $n\ge 3$, the remaining case $n=2$ being easier).

We take the differentiable function $g(t)$ as $g(t)=th(t)$ with $h(t)=\frac{1}{ln\,\frac{2}{|t|}}$ for $0\le|t|\le 1$ and $h(t)=\frac{1}{ln\,\frac{3}{|t|}}$ for $-1\le|t|\le 0$, so that $g(t)\in C^1([-1,1])$ and the singular integral $\int_{-1}^{1}\frac{h(t)dt}{t}$ diverges.

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