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Singular integral equations on the real line with homogeneous kernels and the inversion shift

Abstracts

Fredholmness criterion and formula for the index are given for a class of singular type integral equations which are generated by singular integral operator, operators with kernels homogeneous of degree -1 and the inversion shift operator $Q\varphi = \varphi\left(\frac{1}{x}\right)$. The equation is studied in the weighted space $L_p(R^1, |x|^\gamma)$ with the power weight.

1 Introduction

Fredholmness of singular integral equations on the real line, perturbed by terms with homogeneous kernels, is known, we refer for example to [4] or [1]. In this paper we consider such equations with the shift $\alpha(x) = \frac{1}{x}$ which suits well for homogeneous kernels:

$$K\varphi := a(x)\varphi(x) + \frac{b(x)}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(t)dt}{t-x} + c(x)\int_{-\infty}^{\infty}k(x,y)\varphi(y)dy$$

$$+\mathfrak{a}(x)\varphi\left(\frac{1}{x}\right) + \frac{\mathfrak{b}(x)}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(t)dt}{t-\frac{1}{x}} + \mathfrak{c}(x)\int_{-\infty}^{\infty}\ell\left(\frac{1}{x},y\right)\varphi(y)dy, \quad x \in R^1, \tag{1.1}$$

where $k(x,y)$ and $\ell(x,y)$ are kernels homogeneous of degree -1, satisfying some standard integrability conditions, see (2.3). The case when the equation (1.1) does not contain singular terms, that is, $b(x) \equiv \mathfrak{b}(x) \equiv 0$, was considered in [6] in the case of equations on R^1 and in [3] in the case of equations on R^1_+. Fredholmness of equations without the shift and singular terms, that is, the case $b(x) = \mathfrak{a}(x) = \mathfrak{b}(x) = \mathfrak{c}(x) \equiv 0$ were investigated long ago, see [7] and also the recent survey [5] on integral equations with homogeneous kernels. Both in [6] and in this paper we apply the general approach to investigation of Fredholmness of equations with involutive operators developed in [4], [6]. A possibility to cover Fredholmness of the equation (1.1) in full is due to more delicate realization of that general approach.

Applying the general approach of [4], [6], one may also cover equations of the form (1.1) containing also terms with complex conjugation, but we do not dwell on such more general situation.

Since from equation (1.1) we naturally pass to a system of equations without the shift $\alpha(x) = \frac{1}{x}$, in Section 2 we preliminarily give the Fredholmness results for systems of singular equations perturbed by matrix terms with homogeneous kernels, which is a modified and extended version of the results from [4]. The main results are given in Section 4.

2 Singular integral operators perturbed by integral operators with homogeneous kernels

In this section we describe the nature of normal solvability and calculate the index of singular integral operators perturbed by non compact integral operators with a homogeneous kernel of degree -1:

$$N\varphi \equiv a(x)\varphi(x) + \frac{b(x)}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(y)\,dy}{y-x} + c(x)\int_{-\infty}^{\infty} k(x,y)\varphi(y)\,dy = f(x), \quad x \in R^1, \tag{2.1}$$

where $k(\lambda x, \lambda y) = \lambda^{-1}k(x,y),\ \lambda > 0$. We treat the operator (2.1) in the weighted space

$$L_p^\gamma(R^1) = \{\varphi : \int_{-\infty}^{\infty} |x|^\gamma |\varphi(x)|^p\,dx < \infty\}, \quad 1 < p < \infty, \quad -1 < \gamma < p-1. \tag{2.2}$$

We assume that $a(x)$, $b(x)$, $c(x) \in C(\dot{R}^1)$ and

$$\int_{-\infty}^{\infty} |k(\pm 1, y)| \frac{dy}{|y|^{\frac{1+\gamma}{p}}} < \infty, \tag{2.3}$$

the latter guaranteeing the boundedness of the last term on the left-hand side of (2.1) in the space $L_p^\gamma(R^1)$. Below we will have to impose also another condition on the kernel, namely

$$\int_{-\infty}^{\infty} \frac{dy}{|y|^{\frac{1+\gamma}{p}}} \left| \int_{-\infty}^{\infty} \frac{k(t,y)}{t \pm 1}\,dt \right| < \infty. \tag{2.4}$$

2.1 Preliminaries on equations with a homogeneous kernel

a) Scalar case. Let

$$K\varphi \ :\equiv\ \lambda\varphi(x) - \sum_{j=1}^{n} c_j(x)\int_{-\infty}^{\infty} k_j(x,y)\varphi(y)\ dy = \ f(x)\ ,\ x \in R^1, \tag{2.5}$$

where the kernels $k_j(x,y)$ are homogeneous of order -1 : $k_j(tx,ty) = t^{-1}k_j(x,y),\ x,y \in R^1,\ t > 0$, and the coefficients $c_j(x) \in L_\infty(R^1)$ are assumed to have values $c_j(\pm 0)$ and $c_j(\pm\infty)$ understood in the following sense

$$\lim_{N\to\infty} \underset{0<x<\frac{1}{N}}{esssup} |c_j(\pm x) - c_j(\pm 0)| = 0, \quad \lim_{N\to\infty} \underset{x>N}{esssup} |c_j(\pm x) - c_j(\pm\infty)| = 0 \tag{2.6}$$

under the respective choice of the signs. Let

$$\mathcal{K}^j_{\pm\pm}(z) = \int_0^\infty k_j(\pm 1, \pm y) y^{z-1}\, dy \tag{2.7}$$

denote the Mellin transforms of the kernels in the correspondent quadrants.

Theorem 2.1. *Let $c_j(x) \in L_\infty(R^1)$ have the values $c_j(\pm 0)$ and $c_j(\pm\infty)$, $j = 1, 2, ..., n$ in the sense of the definition (2.6). Then the operator K is Fredholm in $L_p(R^1, |x|^\gamma)$, $1 \leq p \leq \infty$, $-1 < \gamma < p-1$, if and only if*

$$\det \sigma_0\left(i\xi + 1 - \frac{1+\gamma}{p}\right) \neq 0 \quad \text{and} \quad \det \sigma_\infty\left(i\xi + 1 - \frac{1+\gamma}{p}\right) \neq 0\ , \quad \xi \in \dot{R}^1\ , \tag{2.8}$$

where

$$\sigma_0(z) = \begin{pmatrix} \lambda + \sum\limits_{j=1}^n c_j(+0)\mathcal{K}^j{}_{++}(z) & \sum\limits_{j=1}^n c_j(+0)\mathcal{K}^j{}_{+-}(z) \\ \\ \sum\limits_{j=1}^n c_j(-0)\mathcal{K}^j{}_{-+}(z) & \lambda + \sum\limits_{j=1}^n c_j(-0)\mathcal{K}^j{}_{--}(z) \end{pmatrix},$$

and

$$\sigma_\infty(z) = \begin{pmatrix} \lambda + \sum\limits_{j=1}^n c_j(+\infty)\mathcal{K}^j{}_{++}(z) & \sum\limits_{j=1}^n c_j(+\infty)\mathcal{K}^j{}_{+-}(z) \\ \\ \sum\limits_{j=1}^n c_j(-\infty)\mathcal{K}^j{}_{-+}(z) & \lambda + \sum\limits_{j=1}^n c_j(-\infty)\mathcal{K}^j{}_{--}(z) \end{pmatrix}.$$

Under the conditions (2.8)

$$Ind_{L_p(R^1,|x|^\gamma)}\ K = ind\ \frac{\det \sigma_\infty(i\xi + 1 - \frac{1+\gamma}{p})}{\det \sigma_0(i\xi + 1 - \frac{1+\gamma}{p})}\ . \tag{2.9}$$

b) Matrix case. For further goals we give also a matrix version of Theorem 2.1 for the case of systems of equations with homogeneous kernels:

$$N\varphi \equiv A(x)\varphi(x) + C(x)\int_{-\infty}^\infty K(x,y)\varphi(y)\, dy = F(x), \quad x \in R^1, \tag{2.10}$$

where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_m)$ and $F = (f_1, f_2, ..., f_m)$ are vector-functions, $A(x), C(x)$ and $K(x,y)$ are $(m \times m)$-matrices. We assume that the matrix kernel

$$K(x,y) = (k_{ij}(x,y))_{i,j=1}^m$$

has the entries $k_{ij}(x,y)$ satisfying the conditions (2.4) and for simplicity suppose that the entries of the matrices $A(x)$ and $C(x)$ are continuous on $\dot{R}^1$.

Let

$$\mathcal{K}_{\pm\pm}(z) = (\mathcal{K}_{ij,\pm\pm}(z))_{i,j=1}^m \tag{2.11}$$

where

$$\mathcal{K}_{ij,\pm\pm}(z) = \int_0^\infty k_{ij}(\pm 1, \pm y) y^{z-1} dy. \tag{2.12}$$

and

$$\sigma_0(z) = \begin{pmatrix} \sigma_0^{11}(z) & \sigma_0^{12}(z) \\ \\ \sigma_0^{21}(z) & \sigma_0^{22}(z) \end{pmatrix}, \quad \sigma_\infty(z) = \begin{pmatrix} \sigma_\infty^{11}(z) & \sigma_\infty^{12}(z) \\ \\ \sigma_\infty^{21}(z) & \sigma_\infty^{22}(z) \end{pmatrix}, \tag{2.13}$$

where the $(m \times m)$-blocs $\sigma_0^{kj}(z)$ and $\sigma_\infty^{kj}(z)$ have the form:

$$\begin{aligned} \sigma_0^{11}(z) = A(0) + C(0)\mathcal{K}_{++}(z), \qquad & \sigma_0^{12}(z) = C(0)\mathcal{K}_{+-}(z), \\ \sigma_0^{21}(z) = C(0)\mathcal{K}_{-+}(z), \qquad & \sigma_0^{22}(z) = A(0) + C(0)\mathcal{K}_{--}(z) \end{aligned} \tag{2.14}$$

and similarly for $\sigma_\infty^{kj}(z),\ k,j = 1,2$ with $A(0)$ and $C(0)$ replaced by $A(\infty)$ and $C(\infty)$, respectively.

Theorem 2.2. *Let the entries of the matrices $A(x)$ and $C(x)$ be in $C(\dot{R}^1)$ and the entries of the matrix $K(x,y)$ satisfy the conditions (2.4). The operator of the form (2.10) is Fredholm in the space $L_p^m(R^1;|x|^\gamma),\ 1 \le p \le \infty,\ -1 < \gamma < p-1$, if and only if* $\det A(x) \neq 0,\ x \in \dot{R}^1$ *and*

$$\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0\ , \quad \det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0$$

for all $\xi \in \dot{R}^1$. Under these conditions

$$Ind\ N \ = \ ind \, \frac{\det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p} \right)}{\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right)} \ .$$

2.2 Some necessary conditions

Lemma 2.3. *Let $1 < p < \infty, \quad -1 < \gamma < p-1$ and assumptions (2.3), be satisfied. If the operator (2.1) is Fredholm in the space $L_p^\gamma(R^1)$, then its "characteristic" part $a(x)I + b(x)S$ is also Fredholm in $L_p^\gamma(R^1)$, so that the conditions*

$$a(x) \pm b(x) \neq 0, \quad x \in \dot{R}^1 \tag{2.15}$$

are necessary for the operator (2.1) to be Fredholm in $L_p^\gamma(R^1)$.

Proof. By homogeneity of the kernels of the integral operators in (2.1), we can introduce the weight into the operator, so that one may consider the operator N in the space $L_p(R^1)$ instead of $L_p^\gamma(R^1)$.

Let, on the contrary, the operator N be Fredholm, but $a(x_0) + b(x_0) = 0$.

1. Let $x_0 = \infty$, so that $a(\infty) + b(\infty) = 0$. Because of the stability property for Fredholm operators, we can approximate our operator N in such a way that $a(x) + b(x)$ is a finite function supported on some interval $(-m, m)$ and the operator N is still Fredholm. Let $\omega(x) \in C^\infty(R^1)$ be any smooth step-function, which is equal to zero in the interval $(m+2, m+3)$, to 1 in the intervals $(-\infty, m+1)$ and $(m+4, \infty)$ and has values between 0 and 1 in the intervals $(m+1, m+2)$ and $(m+3, m+4)$.

We have $N = (a+b)P_+ + (a-b)P_- + cH$, where $P_\pm = \frac{1}{2}(I \pm S)$ and

$$H\varphi = \int_{-\infty}^{\infty} k(x,y)\varphi(y)\,dy. \tag{2.16}$$

Hence

$$N(1-\omega)P_+ = (a+b)(1-\omega)P_+ + cH(1-\omega)P_+ + T,$$

where T is a compact operator. Evidently, $(a+b)(1-\omega) \equiv 0$. It is also clear that $H(1-\omega)$ is a compact operator. Therefore, the operator $N(1-\omega)P_+ = T_1$ is also compact and we obtain

$$N = N((1-\omega+\omega)P_+ + P_-) = N(\omega P_+ + P_-) + T_2$$

with a compact operator T_2. Consequently, the operator $\omega P_+ + P_-$ is Fredholm operator, which is not possible.

2. Let now $x_0 \neq \infty$ and for simplicity $x_0 = 0$. The arguments are similar to those above. Indeed, we can assume that $a(x) + b(x) \equiv 0$ in the interval $\left(-\frac{1}{m}, \frac{1}{m}\right)$ and introduce a smooth step-function $\beta(x) \in C_0^\infty(R^1)$ such that $\beta(x) \equiv 0$ when $|x| \geq \frac{1}{2m}$ and $|x| \leq \frac{1}{16m}$ and $\beta(x) \equiv 1$ for $\frac{1}{8m} \leq |x| \leq \frac{1}{4m}$. Then $(a+b)\beta \equiv 0$ and $N\beta P_+ = T$, where T is a compact operator. Hence

$$N = N((1-\beta)P_+ + P_-) + T_2$$

and we obtain that operator $P_- + (1-\beta)P_+$ is Fredholm, which is not possible. □

2.3 Reduction to a system of pair convolution equations

As is well known, see for example [5], equations with homogeneous kernels satisfying the condition (2.3) are easily reduced to convolution type equations by means of the direct exponential change of variables. We may treat the same idea for the operator N. However, this operator includes not only the homogeneous kernel satisfying the condition (2.3), but also a singular homogeneous kernel $\frac{1}{x-y}$ which does not satisfy such a condition. By this reason, it is more convenient to exclude first the singular operator S, basing on Lemma 2.3.

Lemma 2.4. *Let $1 < p < \infty, -1 < \gamma < p-1$. Under the assumptions (2.3)-(2.4) the operator N is Fredholm in $L_p^\gamma(R^1)$ simultaneously with the operator*

$$(a^2 - b^2)I + acH - bcH^1 \ , \tag{2.17}$$

where H is the operator (2.16) and the operator

$$H^1\varphi = SH\varphi = \int_{-\infty}^{\infty} k^1(x,y)\varphi(y)\,dy$$

has the homogeneous kernel

$$k^1(x,y) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{k(t,y)}{t-x}\,dt = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{k(t, sign\ y)}{t|y|-x}\,dt\ . \tag{2.18}$$

Proof. We have

$$(aI - bS)N = (a^2 - b^2)I + acH - bcH^1 + T, \tag{2.19}$$

where T is a compact operator. Then the statement of Lemma 2.4 follows from that of Lemma 2.3. □

Remark 2.5. *The condition (2.4) imposed on the kernel $k(x,y)$ means that the operator H^1 has the same nature as the operator H, that is, it belongs to the algebra of operators H with a homogeneous kernel satisfying the condition (2.3). (It may be shown that there exist kernels homogeneous of degree -1, satisfying the conditions (2.3), but not satisfying the conditions (2.4)).*

The main statement of this subsection is given by Theorem 2.7 below, in which we use the following notation:

$$\sigma_0(z) = \begin{pmatrix} 1 + \lambda_0\mathcal{K}_{++}(z) - \mu_0\mathcal{K}^1_{++}(z) & \lambda_0\mathcal{K}_{+-}(z) - \mu_0\mathcal{K}^1_{+-}(z) \\ \lambda_0\mathcal{K}_{-+}(z) - \mu_0\mathcal{K}^1_{-+}(z) & 1 + \lambda_0\mathcal{K}_{--}(z) - \mu_0\mathcal{K}^1_{--}(z) \end{pmatrix},$$

and

$$\sigma_\infty(z) = \begin{pmatrix} 1 + \lambda_\infty\mathcal{K}_{++}(z) - \mu_\infty\mathcal{K}^1_{++}(z) & \lambda_\infty\mathcal{K}_{+-}(z) - \mu_\infty\mathcal{K}^1_{+-}(z) \\ \lambda_\infty\mathcal{K}_{-+}(z) - \mu_\infty\mathcal{K}^1_{-+}(z) & 1 + \lambda_\infty\mathcal{K}_{--}(z) - \mu_\infty\mathcal{K}^1_{--}(z) \end{pmatrix},$$

where

$$\lambda_0 = \frac{a(0)c(0)}{a^2(0) - b^2(0)}, \quad \mu_0 = \frac{b(0)c(0)}{a^2(0) - b^2(0)},$$

$$\lambda_\infty = \frac{a(\infty)c(\infty)}{a^2(\infty) - b^2(\infty)}, \quad \mu_\infty = \frac{b(\infty)c(\infty)}{a^2(\infty) - b^2(\infty)},$$

and

$$\mathcal{K}_{\pm\pm}(z) = \int_0^\infty k(\pm 1, \pm y)y^{z-1}dy \quad \text{and} \quad \mathcal{K}^1_{\pm\pm}(z) = \int_0^\infty k^1(\pm 1, \pm y)y^{z-1}dy \tag{2.20}$$

are the Mellin transforms of the kernels $k(\pm 1, \pm y)$ and $k^1(\pm 1, \pm y)$, the latter being defined in (2.18).

Lemma 2.6. *Under the condition (2.3), the Mellin transforms $\mathcal{K}_{\pm\pm}(z)$ converge absolutely for $z = i\xi + 1 - \frac{1+\gamma}{p}$, $-\infty < \xi < \infty$. If the condition (2.4) is also satisfied, then the Mellin transforms $\mathcal{K}^1_{\pm\pm}(z)$ converge absolutely for the same z.*

The functions $\mathcal{K}^1_{\pm\pm}(z)$ are expressed in terms of the functions $\mathcal{K}_{\pm\pm}(z)$ by means of the formulas

$$\mathcal{K}^1_{++}(z) = \frac{i}{sin\ z\pi}[\mathcal{K}_{++}(z)\ cos\ z\pi + \mathcal{K}_{-+}(z)], \tag{2.21}$$

$$\mathcal{K}^1_{-+}(z) = -\frac{i}{sin\ z\pi}[\mathcal{K}_{-+}(z)\ cos\ z\pi + \mathcal{K}_{++}(z)], \tag{2.22}$$

$$\mathcal{K}^1_{+-}(z) = \frac{i}{sin\ z\pi}[\mathcal{K}_{+-}(z)\ cos\ z\pi + \mathcal{K}_{--}(z)], \tag{2.23}$$

$$\mathcal{K}^1_{--}(z) = -\frac{i}{sin\ z\pi}[\mathcal{K}_{--}(z)\ cos\ z\pi + \mathcal{K}_{+-}(z)]. \tag{2.24}$$

Proof. The convergence of the Mellin transforms for $z = i\xi + 1 - \frac{1+\gamma}{p}$ is evident. Let us verify, for instance, the first of the formulas (2.21) - (2.24). We have

$$\mathcal{K}^1_{++}(z) = \frac{1}{\pi i}\int_0^\infty y^{z-1}\,dy \int_{-\infty}^\infty \frac{k(t,1)}{yt-1}\,dt = \frac{1}{\pi i}\int_{-\infty}^\infty \frac{k(t,1)}{t}\,dt \int_0^\infty \frac{y^{z-1}}{y-\frac{1}{t}}\,dy.$$

Using the formula

$$\int_0^\infty \frac{y^{z-1}dy}{y+a} \ = \ \frac{\pi|a|^{z-1}}{\sin\ \pi z}\left\{ \begin{array}{cc} 1, & a>0 \\ -cos\ \pi z, & a<0 \end{array} \right. , \tag{2.25}$$

see [2], N 3.222.2, we obtain

$$\mathcal{K}^1_{++}(z) = i\ ctg\ z\pi \int_0^\infty t^{-z}k(t,1)\,dt + i\ cosec\ z\pi \int_0^\infty t^{-z}k(-t,1)\,dt,$$

which coincides with the right hand side in (2.21) after easy transformations. □

Theorem 2.7. *Let $a(x)$, $b(x)$, $c(x) \in C(\dot{R}^1)$ and let the conditions (2.3) and (2.4) be satisfied. The operator N is Fredholm in the space $L_p^\gamma(R^1)$, $1<p<\infty$, $-1<\gamma<p-1$, if and only if $a(x)\pm b(x) \neq 0$, $x \in \dot{R}^1$ and*

$$\det\sigma_0\left(i\xi+1-\frac{1+\gamma}{p}\right)\neq 0, \qquad \det\sigma_\infty\left(i\xi+1-\frac{1+\gamma}{p}\right)\neq 0$$

for all $\xi \in \dot{R}^1$. Under these conditions

$$Ind_{L^p_\gamma}\ N = ind\ \frac{a(x)-b(x)}{a(x)+b(x)} + ind\ \frac{\det\sigma_\infty\left(i\xi+1-\frac{1+\gamma}{p}\right)}{\det\sigma_0\left(i\xi+1-\frac{1+\gamma}{p}\right)}\ . \tag{2.26}$$

Proof. By Lemma 2.4, we may deal with the operator (2.17) instead of the operator N. Applying Theorem 2.1, after direct calculations we arrive at the statement of the theorem. □

2.4 Systems of singular integral equations perturbed by integrals with homogeneous kernels

The result of the previous Subsection given in Theorem 2.7 may be extended to the matrix case, that is, to the case of the operator

$$N\varphi \equiv A(x)\varphi(x) + B(x)(S\varphi)(x) + C(x)\int_{-\infty}^{\infty} K(x,y)\varphi(y)\,dy = f(x) \tag{2.27}$$

where $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$, $A(x)$, $B(x)$, $C(x)$ are $(m\times m)-$ matrices with entries continuous on $\dot{R}^1$, and $K(x,y)$ is a matrix kernel with entries satisfying the familiar conditions (2.3) and (2.4), and S stands for the diagonal $(m\times m)$-matrix with the singular operator at the diagonal.

The arguments being analogous to those in the previous subsection, we only sketch briefly the main points.

Similarly to Lemma 2.3 it is shown that Fredholmness of the matrix singular operator $AI + BS$ is necessary for that of the operator N. By this reason, we assume that the matrices $A\pm B$ are normal: $\det[A(x)\pm B(x)] \neq 0$, $x\in \dot{R}^1$. The regularizer of the operator $AI + BS$ has the form $R = A_1 I + B_1 S$ (see [8], p.414), where

$$A_1 = \frac{1}{2}\left[(A+B)^{-1} + (A-B)^{-1}\right] = (A+B)^{-1}A(A-B)^{-1}, \tag{2.28}$$

and

$$B_1 = \frac{1}{2}\left[(A+B)^{-1} - (A-B)^{-1}\right] = -(A+B)^{-1}B(A-B)^{-1}\ . \tag{2.29}$$

Applying the regularizer R to the operator N and passing afterwards to the corresponding equations separately on each half-axis, we arrive at the following system of $2m$ equations, up to compact terms $T_j\varphi_\pm,\ j = 1,2,3,4,$

$$\begin{cases} \varphi_+(x) + A_1(x)C(x)\int_0^\infty K(x,y)\varphi_+(y)\,dy + \\ +A_1(x)C(x)\int_0^\infty K(x,-y)\varphi_-(-y)\,dy + B_1(x)C(x)\int_0^\infty K^1(x,y)\varphi_+(y)\,dy + \\ +B_1(x)C(x)\int_0^\infty K^1(x,-y)\varphi_-(-y)\,dy + T_1\varphi_+ + T_2\varphi_- = f_+(x), \quad x>0; \\ \varphi_-(-x) + A_1(-x)C(-x)\int_0^\infty K(-x,y)\varphi_+(y)\,dy + \\ +A_1(-x)C(-x)\int_0^\infty K(-x,-y)\varphi_-(-y)\,dy + B_1(-x)C(-x)\int_0^\infty K^1(-x,y)\varphi_+(y)\,dy + \\ +B_1(-x)C(-x)\int_0^\infty K^1(-x,-y)\varphi_-(-y)\,dy + T_3\varphi_+ + T_4\varphi_- = f_-(-x), \quad x>0, \end{cases}$$

where $\varphi_\pm(x) = \frac{1}{2}\left(1\pm \mathrm{sign}x\right)\varphi(x)$ and

$$K^1(x,y) = \left(k^1_{ij}(x,y)\right)_{i,j=1}^m$$

with $k^1_{ij}(x,y)$ calculated by the entries $k_{ij}(x,y)$ via the formula (2.18). We denote

$$\begin{aligned} M_1 &= A_1 C = \tfrac{1}{2}\left[(A+B)^{-1} + (A-B)^{-1}\right]C, \\ M_2 &= B_1 C = \tfrac{1}{2}\left[(A+B)^{-1} - (A-B)^{-1}\right]C. \end{aligned} \tag{2.30}$$

The matrix-symbol of the obtained system may be written in terms of the matrices M_1 and M_2, according to (2.14), as

$$\sigma_0(z) = \begin{pmatrix} I + M_1(0)\mathcal{K}_{++}(z) + M_2(0)\mathcal{K}^1_{++}(z) & M_1(0)\mathcal{K}_{+-}(z) + M_2(0)\mathcal{K}^1_{+-}(z) \\ M_1(0)\mathcal{K}_{-+}(z) + M_2(0)\mathcal{K}^1_{-+}(z) & I + M_1(0)\mathcal{K}_{--}(z) + M_2(0)\mathcal{K}^1_{--}(z) \end{pmatrix},$$

$$\sigma_\infty(z) = \begin{pmatrix} I + M_1(\infty)\mathcal{K}_{++}(z) + M_2(\infty)\mathcal{K}^1_{++}(z) & M_1(\infty)\mathcal{K}_{+-}(z) + M_2(\infty)\mathcal{K}^1_{+-}(z) \\ M_1(\infty)\mathcal{K}_{-+}(z) + M_2(\infty)\mathcal{K}^1_{-+}(z) & I + M_1(\infty)\mathcal{K}_{--}(z) + M_2(\infty)\mathcal{K}^1_{--}(z) \end{pmatrix},$$

representing a pair of $(2m \times 2m)$-matrices. The $(m \times m)$-blocs $\mathcal{K}_{\pm\pm}(z)$ and $\mathcal{K}^1_{\pm\pm}(z)$ here are the matrix symbols

$$\{\mathcal{K}_{rj,\pm\pm}(z)\}_{r,j=1}^m \quad \text{and} \quad \left\{\mathcal{K}^1_{rj,\pm\pm}(z)\right\}_{r,j=1}^m$$

corresponding to the matrices $K(x,y) = \{k_{rj}(x,y)\}_{r,j=1}^m$ and $K^1(x,y) = \left\{k^1_{rj}(x,y)\right\}_{r,j=1}^m$ where the entries $k^1_{rj}(x,y)$ are calculated by the entries $k_{rj}(x,y)$ via the formula (2.18). It is easy to see that the connections (2.21)-(2.24) remain valid when $\mathcal{K}_{\pm\pm}(z)$ and $\mathcal{K}^1_{\pm\pm}(z)$ are matrices. Making use of those connections, we calculate the matrices (2.13) and obtain that the $(m \times m)$-blocs $\sigma_0^{kj}(z)$ and $\sigma_\infty^{kj}(z)$ have the form:

$$\sigma_0^{11}(z) = I + [M_1(0) + ictg\ z\pi M_2(0)]\mathcal{K}_{++}(z) + \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{-+}(z)\ ,$$

$$\sigma_0^{12}(z) = [M_1(0) + ictg\ z\pi M_2(0)]\mathcal{K}_{+-}(z) + \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{--}(z)\ ,$$

$$\sigma_0^{21}(z) = [M_1(0) - ictg\ z\pi M_2(0)]\mathcal{K}_{-+}(z) - \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{++}(z)\ ,$$

$$\sigma_0^{22}(z) = I + [M_1(0) - ictg\ z\pi M_2(0)]\mathcal{K}_{--}(z) - \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{+-}(z)$$

and similarly for $\sigma_\infty^{kj}(z),\ k,j = 1,2$, with $M_1(0)$ and $M_2(0)$ replaced by $M_1(\infty)$ and $M_2(\infty)$, respectively.

Similarly to Theorem 2.7 we obtain the following result.

Theorem 2.8. *Let the entries of the matrices $A(x)$, $B(x)$, $C(x)$ be in $C(\dot{R}^1)$ and the entries of the matrix $K(x,y)$ satisfy the conditions (2.3)-(2.4). The operator of the form (2.27) is Fredholm in the space $L_p^\gamma(R^1)$, $1 < p < \infty, -1 < \gamma < p-1$, if and only if* $\det[A(x) \pm B(x)] \neq 0$, $x \in \dot{R}^1$ *and*

$$\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0\ , \quad \det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0 \tag{2.31}$$

for all $\xi \in \dot{R}^1$. *Under these conditions*

$$Ind_{L^p_\gamma}\ N \ = \ ind\ \frac{\det[A(x)-B(x)]}{\det[A(x)+B(x)]} \ + \ ind\ \frac{\det\sigma_\infty\left(i\xi+1-\frac{1+\gamma}{p}\right)}{\det\sigma_0\left(i\xi+1-\frac{1+\gamma}{p}\right)}\ . \tag{2.32}$$

We shall also need a result similar to Theorem 2.8 in the situation when the matrix singular operator is perturbed by several non-compact operators with homogeneous kernels with different variable coefficients

$$N\varphi :\equiv A(x)\varphi(x)+B(x)(S\varphi)(x)+\sum_{j=1}^{\ell}C_j(x)\int_{-\infty}^{\infty}K_j(x,y)\varphi(y)\,dy=f(x), \tag{2.33}$$

where $A(x)$, $B(x)$, $C_j(x)$ are $(m\times m)-$ matrices with entries continuous on $\dot{R}^1$, and $K_j(x,y)$ are matrix kernels with entries satisfying the conditions (2.3) and (2.4). The corresponding theorem given below is not derived from Theorem 2.8, but its proof may be obtained in the same way as that of Theorem 2.8.

As in (2.30), we introduce the matrices $M_{j,1}=A_1C_j$ and $M_{j,2}=B_1C_j, j=1,...,\ell$, where A_1 and B_1 are the matrices (2.28). Let $\sigma_0(z)$ and $\sigma_\infty(z)$ be the matrices

$$\sigma_0(z)=\begin{pmatrix}\sigma_0^{11}(z) & \sigma_0^{12}(z)\\ \\ \sigma_0^{21}(z) & \sigma_0^{22}(z)\end{pmatrix}\ ,\qquad \sigma_\infty(z)=\begin{pmatrix}\sigma_\infty^{11}(z) & \sigma_\infty^{12}(z)\\ \\ \sigma_\infty^{21}(z) & \sigma_\infty^{22}(z)\end{pmatrix}, \tag{2.34}$$

with the $(m\times m)$-matrix entries

$$\sigma_0^{11}(z)=I+\sum_{j=1}^{\ell}\left\{[M_{j,1}(0)+ictg\ z\pi M_{j,2}(0)]\mathcal{K}_{j,++}(z)+\frac{i}{\sin z\pi}M_{j,2}(0)\mathcal{K}_{j,-+}(z)\right\}\ , \tag{2.35}$$

$$\sigma_0^{12}(z)=\sum_{j=1}^{\ell}\left\{[M_{j,1}(0)+ictg\ z\pi M_{j,2}(0)]\mathcal{K}_{j,+-}(z)+\frac{i}{\sin z\pi}M_{j,2}(0)\mathcal{K}_{j,--}(z)\right\}\ , \tag{2.36}$$

$$\sigma_0^{21}(z)=\sum_{j=1}^{\ell}\left\{[M_{j,1}(0)-ictg\ z\pi M_{j,2}(0)]\mathcal{K}_{j,-+}(z)-\frac{i}{\sin z\pi}M_{j,2}(0)\mathcal{K}_{j,++}(z)\right\}\ , \tag{2.37}$$

$$\sigma_0^{22}(z)=I+\sum_{j=1}^{\ell}\left\{[M_{j,1}(0)-ictg\ z\pi M_{j,2}(0)]\mathcal{K}_{j,--}(z)-\frac{i}{\sin z\pi}M_{j,2}(0)\mathcal{K}_{j,+-}(z)\right\} \tag{2.38}$$

and similarly for $\sigma_\infty^{kr}(z),\ k,r=1,2$, with $M_{j,1}(0)$ and $M_{j,2}(0)$ replaced by $M_{j,1}(\infty)$ and $M_{j,2}(\infty)$, respectively.

Theorem 2.9. *Let the entries of the matrices* $A(x)$, $B(x)$, $C_j(x), j=1,2,...,\ell$, *be in* $C(\dot{R}^1)$ *and the entries of the matrices* $K_j(x,y), j=1,...,\ell$, *satisfy the conditions (2.3),(2.4). The operator of the form (2.33) is Fredholm in the space* $L_p^\gamma(R^1)$, $1<p<\infty$, $-1<\gamma<p-1$, *if and only if* $\det[A(x)\pm B(x)]\neq 0,\ x\in\dot{R}^1$, *and the conditions (2.31) are satisfied in which the entries of the matrices* $\sigma_0(z)$ *and* $\sigma_\infty(z)$ *are given by the formulas (2.35)-(2.38). Under these conditions the formula (2.32) for the index is valid.*

3 Equations with an involutive operator

We shall base ourselves on the following Theorem 3.10 on Fredholmness of equations with an involutive operators in Banach spaces, see this theorem in the book [4] or its reformulation in [6]. Let X be an abstract Banach space. We suppose that the following Axioms 1 and 2 are valid.

$\mathbb{AXIOM}$ 1. *There exists a Fredholm operator* $U \in \mathcal{L}(X)$ *which anti-quasicommutes with* Q, that is,

$$UQ + QU = T \tag{3.39}$$

where T *is compact in* X.

$\mathbb{AXIOM}$ 2. *The operators* A *and* B *quasicommute with the operator* U *from the Axiom* 1: $AU = UA + T_1,\ BU = UB + T_2$. *where* T_1 *and* T_2 *are compact in* X.

Theorem 3.10. *Let* $A, B, Q \in \mathcal{L}(X)$ *and* $Q^2 = I, Q \neq \pm I$. *The operator* $K = A + QB$ *is Fredholm in* X *if the operator*

$$\mathbb{K} = \begin{pmatrix} A & QBQ \\ B & QAQ \end{pmatrix} \tag{3.40}$$

is Fredholm in $X^2 = X \times X$. *Under the additional assumption that Axioms* 1 *and* 2 *are satisfied, Fredholmness of the operator* $\mathbb{K}$ *is also necessary for that of the operator* K *and*

$$Ind_X\ K = \frac{1}{2} Ind_{X^2}\ \mathbb{K}\ .$$

4 Investigation of the equation (1.1)

We consider operator (1.1) in the space $L_p^\gamma(R^1), 1 < p < \infty,\ -1 < \gamma < p - 1$. Let $\nu = \frac{2}{p}(\gamma + 1)$. We suppose that

$$a(x),\ b(x),\ c(x),\ |x|^\nu sign\ x\ \mathfrak{a}(x),\ |x|^\nu sign\ x\ \mathfrak{b}(x),\ |x|^\nu sign\ x\ \mathfrak{c}(x)\ \in C(\dot{R}^1) \tag{4.1}$$

and the kernels $k(x, y)$ and $\ell(x, y)$ satisfy the integrability conditions (2.3), (2.4):

$$\int_{-\infty}^{\infty} |k(\pm 1, y)| \frac{dy}{|y|^{\frac{1+\gamma}{p}}} < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{dy}{|y|^{\frac{1+\gamma}{p}}} \left| \int_{-\infty}^{\infty} \frac{k(t, y)}{t \pm 1} dt \right| < \infty \tag{4.2}$$

and similarly for $\ell(x, y)$.

To formulate Theorem 4.1 below, we use the following notation:

$$\widetilde{a}(x) = a\left(\frac{1}{x}\right), \quad \widetilde{b}(x) = b\left(\frac{1}{x}\right) \quad \widetilde{c}(x) = c\left(\frac{1}{x}\right), \tag{4.3}$$

$$\mathfrak{a}_1(x) = |x|^{-\nu} sign\ x\ \mathfrak{a}\left(\frac{1}{x}\right),\ \mathfrak{b}_1(x) = |x|^{-\nu} sign\ x\ \mathfrak{b}\left(\frac{1}{x}\right),\ \mathfrak{c}_1(x) = |x|^{-\nu} sign\ x\ \mathfrak{c}\left(\frac{1}{x}\right), \tag{4.4}$$

$$\Delta(x) := [a(x)+b(x)][\widetilde{a}(x)-\widetilde{b}(x)] - [\mathfrak{a}(x)-\mathfrak{b}(x)][\widetilde{\mathfrak{a}}(x)+\widetilde{\mathfrak{b}}(x)] \tag{4.5}$$

and

$$A(x) = \begin{pmatrix} a(x) & \widetilde{\mathfrak{a}}_1(x) \\ \\ \mathfrak{a}_1(x) & \widetilde{a}(x) \end{pmatrix}, \qquad B(x) = \begin{pmatrix} b(x) & -\widetilde{\mathfrak{b}}_1(x) \\ \\ \mathfrak{b}_1(x) & -\widetilde{b}(x) \end{pmatrix}, \tag{4.6}$$

$$C(x) = \begin{pmatrix} 0 & -\widetilde{\mathfrak{b}}_1(x) \\ \\ 0 & -\widetilde{b}(x) \end{pmatrix}, \qquad D(x) = \begin{pmatrix} c(x) & \widetilde{\mathfrak{c}}_1(x) \\ \\ \mathfrak{c}_1(x) & \widetilde{c}(x) \end{pmatrix}, \tag{4.7}$$

and

$$A_1(x) = [A(x)+B(x)]^{-1}A(x)[A(x)-B(x)]^{-1}, \quad B_1(x) = -[A(x)+B(x)]^{-1}B(x)[A(x)-B(x)]^{-1},$$

under the assumption that the matrices $A(x) \pm B(x)$ are invertible. We observe that

$$\Delta(x) = det\ [A(x)+B(x)], \quad \widetilde{\Delta}(x) = det\ [A(x)-B(x)]. \tag{4.8}$$

We also introduce the matrix

$$K(x,y) = \begin{pmatrix} k_{11}(x,y) & k_{12}(x,y) \\ \\ k_{21}(x,y) & k_{22}(x,y) \end{pmatrix}, \tag{4.9}$$

where $k_{11}(x,y) = k(x,y)$ and $k_{21}(x,y) = \ell(x,y)$ and

$$k_{22}(x,y) = \frac{1}{y^2}\left(\frac{|y|}{|x|}\right)^{\nu} k\left(\frac{1}{x},\frac{1}{y}\right), \qquad k_{12}(x,y) = \frac{1}{y^2}\left(\frac{|y|}{|x|}\right)^{\nu} \ell\left(\frac{1}{x},\frac{1}{y}\right), \tag{4.10}$$

and the corresponding (4×4)-matrix symbol

$$\begin{pmatrix} \mathcal{M}_{++}(z) & \mathcal{M}_{+-}(z) \\ \\ \mathcal{M}_{-+}(z) & \mathcal{M}_{--}(z) \end{pmatrix} \tag{4.11}$$

with the (2×2)-blocks

$$\mathcal{M}_{\pm\pm}(z) = \begin{pmatrix} \mathcal{K}_{\pm\pm}(z) & \mathcal{L}_{\pm\pm}(z) \\ \\ \mathcal{K}_{\pm\pm}(2-\nu-z) & \mathcal{L}_{\pm\pm}(2-\nu-z) \end{pmatrix}, \tag{4.12}$$

where $\mathcal{K}_{\pm\pm}(z)$ and $\mathcal{L}_{\pm\pm}(z)$ are the Mellin transforms:

$$\mathcal{K}_{\pm\pm}(z) = \int_0^\infty y^{z-1}k(\pm 1, \pm y)dy, \quad \mathcal{L}_{\pm\pm}(z) = \int_0^\infty y^{z-1}\ell(\pm 1, \pm y)dy,$$

and $k(x,y)$ and $\ell(x,y)$ being the initial kernels from (1.1).

For the posterior calculation of the blocks $\mathcal{M}_{\pm\pm}(z)$ we observe that for $z = i\xi + 1 - \frac{1+\gamma}{p}$ we have

$$2-\nu-z = -i\xi + 1 - \frac{1+\gamma}{p}. \tag{4.13}$$

Let $\sigma_0(z)$ and $\sigma_\infty(z)$ be the matrices (2.34) with the (2×2)-entries calculated by formulas (2.35)-(2.38) with $\ell = 2$. In our case we have

$$\sigma_0^{11}(z) = I - [\alpha(z)B_1(0) + i\beta(z)A_1(0)]\, C(0)$$
$$+A_1(0)D(0)\mathcal{M}_{++}(z) + iB_1(0)D(0)\,[ctg\ \pi z\ \mathcal{M}_{++}(z) + cosec\ \pi z\ \mathcal{M}_{-+}(z)]\,, \qquad (4.14)$$

$$\sigma_0^{12}(z) = [u(z)B_1(0) - iv(z)A_1(0)]\, C(0)$$
$$+A_1(0)D(0)\mathcal{M}_{+-}(z) + iB_1(0)D(0)\,[ctg\ \pi z\ \mathcal{M}_{+-}(z) + cosec\ \pi z\ \mathcal{M}_{--}(z)]\,, \qquad (4.15)$$

$$\sigma_0^{21}(z) = [u(z)B_1(0) + iv(z)A_1(0)]\, C(0)$$
$$+A_1(0)D(0)\mathcal{M}_{-+}(z) - iB_1(0)D(0)\,[ctg\ \pi z\ \mathcal{M}_{-+}(z) + cosec\ \pi z\ \mathcal{M}_{++}(z)]\,, \qquad (4.16)$$

$$\sigma_0^{22}(z) = I - [\alpha(z)B_1(0) - i\beta(z)A_1(0)]\, C(0)$$
$$+A_1(0)D(0)\mathcal{M}_{--}(z) - iB_1(0)D(0)\,[ctg\ \pi z\ \mathcal{M}_{--}(z) + cosec\ \pi z\ \mathcal{M}_{+-}(z)]\,, \qquad (4.17)$$

where $\alpha(z), \beta(z), u(z)$ and $v(z)$ are the functions

$$\alpha(z) = \frac{1 + \cos \nu\pi}{\sin z\pi \sin(z+\nu)\pi}, \quad \beta(z) = \frac{\sin \nu\pi}{\sin z\pi \sin(z+\nu)\pi},$$

and

$$u(z) = \frac{\cos(z+\nu)\pi + \cos \nu\pi}{\sin z\pi \sin(z+\nu)\pi}, \quad v(z) = \frac{\sin(z+\nu)\pi + \sin \nu\pi}{\sin z\pi \sin(z+\nu)\pi}.$$

and similarly for $\sigma_\infty(z)$ with $A_1(0)$ replaced by $A_1(\infty)$ etc.

Theorem 4.1. *Let the kernels $k(x,y)$ and $\ell(x,y)$ satisfy the conditions (4.2). Under assumptions (4.1) the operator K is Fredholm in the space $L_p^\gamma(R^1), 1 < p < \infty, -1 < \gamma < p-1$, if and only if*

$$\Delta(x) \neq 0, \quad x \in \dot{R}^1 \quad \text{and} \quad \det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0, \qquad \xi \in \dot{R}^1. \qquad (4.18)$$

Under these conditions

$$Ind_{L_p^\gamma}\ K = \ ind\ \Delta(x)\ -\ ind\ \det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right). \qquad (4.19)$$

Proof. Since the shift operator in the form $\varphi\left(\frac{1}{x}\right)$ is unbounded (in the space $L_p^\gamma(R^1)$, for example), we introduce the involutive operator Q by the formula

$$(Q\varphi)(x) = \frac{sign\ x}{|x|^\nu} \varphi\left(\frac{1}{x}\right) \qquad (4.20)$$

which gives a bounded and isometric operator: $\|Q\varphi\|_{L_p^\gamma} = \|\varphi\|_{L_p^\gamma}$.

We represent the operator (1.1) in the form $K = A + QB$, where

$$A\varphi = a\varphi + bS\varphi + cH\varphi, \qquad B\varphi = \mathfrak{a}_1\varphi + \mathfrak{b}_1 S\varphi + \mathfrak{c}_1 L\varphi \tag{4.21}$$

and

$$H\varphi = \int_{-\infty}^{\infty} k(x,y)\varphi(y)dy, \qquad L\varphi = \int_{-\infty}^{\infty} \ell(x,y)\varphi(y)dy \tag{4.22}$$

and the notation (4.4) is used.

To apply the general Theorem 3.10, we need to verify Axioms 1-2 . The most important requirement of those axioms is the existence of a Fredholm operator U anti-commuting with the involutive operator Q and commuting with the operators of the type A, B. In the case when the operators (1.1) did not contain the singular operator S, we could take U as an operator of multiplication by a piece-wise constant function with a jump at the point $x = 1$ (as it was done in [6]). This is now impossible because of the presence of the singular operator. This problem is now overcome by introduction of a special kind of the operator U. Namely, let

$$S^{\alpha}\varphi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \left(\frac{|t|}{|x|}\right)^{\alpha} \frac{\varphi(t)dt}{t-x} \tag{4.23}$$

be the weighted singular operator. We introduce the operator U in terms of the operator S^{α} with $\alpha = \frac{\nu-1}{2}$:

$$U\varphi = \frac{x^2-1}{x^2+1}\varphi(x) \ + \ ie^{-x^2-\frac{1}{x^2}}(S^{\frac{\nu-1}{2}}\varphi)(x). \tag{4.24}$$

The operator (4.24) is Fredholm in the space $L_p^{\gamma}(R^1)$ and satisfies the relation $UQ + QU = 0$, so that Axiom 1 is fulfilled.

Now we have to verify Axiom 2. To this end, we notice that all the coefficients involved in (4.21) and (4.24) are continuous on $C(\dot{R}^1)$, so that they commute, up to a compact term, with the singular operator and with operators with a homogeneous kernel.

Then it remains to verify only the quasicommutation of the operator $v_0 S^{\frac{\nu-1}{2}}$ with the operators S and H and L, where $v_0(x) = exp\left(-x^2 - \frac{1}{x^2}\right)$. For the operator S, the corresponding commutant $v_0 S^{\frac{\nu-1}{2}} S - S v_0 S^{\frac{\nu-1}{2}}$ may be calculated, up to compact terms, by means of the formulas (24.9′) and-(24.10′) from [4], p. 148):

$$\left(v_0 S^{\frac{\nu-1}{2}} S - S v_0 S^{\frac{\nu-1}{2}}\right)\varphi = v_0\left(S^{\frac{\nu-1}{2}} S - S S^{\frac{\nu-1}{2}}\right) + T\varphi$$

$$= \frac{v_0(x)}{\pi} tg\ \frac{(\nu-1)\pi}{4} \int_{-\infty}^{\infty} \frac{sign\ x \ - \ sign\ t}{t-x} \left[\frac{sign\ t}{sign\ x}\left|\frac{t}{x}\right|^{\frac{\nu-1}{2}} - 1\right]\varphi(t)dt + T\varphi \tag{4.25}$$

where T is a compact operator. The same is true for the first term in (4.25), since it is an operator with a kernel homogeneous of degree -1 and the coefficient $v_0(x)$ vanishes at the points $x = 0$ and $x = \infty$, which provides the compactness of the operator, see for example Theorem 2.9 from [5] on compactness of operators with homogeneous kernels in the case of continuous coefficients vanishing at infinity and at the origin.

As regards the quasicommutativity of the operator $v_0 S^{\frac{\nu-1}{2}}$ with an operator H with a homogeneous kernel, in the commutant $v_0 S^{\frac{\nu-1}{2}} H - H v_0 S^{\frac{\nu-1}{2}}$ even every term proves to be a compact operator. Indeed,

$$v_0 S^{\frac{\nu-1}{2}} H = S^{\frac{\nu-1}{2}} v_0 H + T = T_1$$

with a compact operator U, since $v_0 H$ is compact.

Axioms 1-2 being satisfied, we may apply Theorem 3.10 which leads to the matrix operator $\mathbb{K} = \begin{pmatrix} A & QBQ \\ B & QAQ \end{pmatrix}$. To calculate the operators $A_1 = QAQ$ and $B_1 = QBQ$, we observe that $QSQ^{-1} = -S^{\nu-1}$, so that

$$A_1\varphi = \widetilde{a}\varphi - \widetilde{b}S^{\nu-1} + \widetilde{c}H_*\varphi, \quad B_1\varphi = \widetilde{\mathfrak{a}_1}\varphi - \widetilde{\mathfrak{b}_1}S^{\nu-1} + \widetilde{\mathfrak{c}_1}L_*\varphi,$$

where the notation (4.3) is used and

$$H_*\varphi = \int_{-\infty}^{\infty} k_{12}(x,y)\varphi(y)dy, \quad L_*\varphi = \int_{-\infty}^{\infty} k_{22}(x,y)\varphi(y)dy,$$

where $k_{12}(x,y)$ and $k_{22}(x,y)$ are kernels (4.10).

Thus, we arrive at the matrix operator

$$\mathbb{K} = \begin{pmatrix} aI + bS + cH & \widetilde{\mathfrak{a}_1}I - \widetilde{\mathfrak{b}_1}S^{\nu-1} + \widetilde{\mathfrak{c}_1}L_* \\ \\ \mathfrak{a}_1 I + \mathfrak{b}_1 S + \mathfrak{c}_1 L & \widetilde{a}I - \widetilde{b}S^{\nu-1} + \widetilde{c}H_* \end{pmatrix} \tag{4.26}$$

which may be rewritten as a matrix singular integral operator perturbed by two matrix operators with kernels homogeneous of degree -1 in the form:

$$\mathbb{K}\phi = A(x)\phi(x) + \frac{1}{\pi i}B(x)\int_{-\infty}^{\infty}\frac{\phi(y)\,dy}{y-x} + C(x)\int_{-\infty}^{\infty}K_0(x,y)\phi(y)\,dy + D(x)\int_{-\infty}^{\infty}K(x,y)\phi(y)\,dy \tag{4.27}$$

where $\phi(x) = \{\varphi_1(x), \varphi_2(x)\}$. The matrix kernel $K(x,y)$ was defined in (4.9) while

$$K_0(x,y) = \begin{pmatrix} k_0(x,y) & 0 \\ \\ 0 & k_0(x,y) \end{pmatrix}, \tag{4.28}$$

where $k_0(x,y)$ is the kernel

$$k_0(x,y) = \frac{1}{\pi i}\frac{\left(\frac{|y|}{|x|}\right)^{\nu-1} - 1}{y-x}. \tag{4.29}$$

Matrix singular operators of the type (4.27) were investigated in Section 1, see the equation (2.35). We intend to apply Theorem 2.9 for which we have to verify the assumptions of that theorem. The conditions of Theorem 2.9 on the matrix coefficients $A(x), B(x)$ and $C_1(x) = C(x)$ and $C_2(x) = D(x)$ are satisfied because of the assumptions (4.1). It remains to show that the entries of the matrices $K_0(x,y)$ and $K(x,y)$ satisfy the conditions (4.2). In the case of $K_0(x,y)$, its entry $k_0(x,y)$ is of special type and the conditions (4.2) are easily seen to be satisfied. In the case of the matrix $K(x, y)$, the fulfillment of the conditions (4.2) for the entries $k(x, y) = k_{12}(x, y)$ and $\ell(x, y) = k_{21}(x, y)$ was postulated in the assumptions of the theorem. To check the conditions (4.2), for example, for the entry $k_{12}(x, y)$, we remind that the second of the assumptions in (4.2) is a condition sufficient for the composition SH to have the kernel satisfying the first of conditions in (4.2). Thus, to verify (4.2), in the case under the consideration we have to check the first of the conditions in (4.2) for the composition SH_* where H_* is the operator with the kernel $k_{12}(x, y)$. Since $SQ = -QS^{\nu-1}$, we have

$$SQHQ = -QS^{\nu-1}HQ = Q(S - S^{\nu-1})HQ - QSHQ.$$

On the right-hand side of this formula, for the term $QSHQ$ the conditions from (4.2) is valid by the assumption, while for the first term this condition is satisfied automatically, because operators with homogeneous kernels satisfying the first of the conditions in (4.2) form an algebra.

Therefore, all the assumptions of Theorem 2.9 are satisfied. Applying this theorem, we calculate the entries $\sigma_0^{rj}(z)$ and $\sigma_\infty^{rj}(z)$ by formulas (2.35)-(2.38) and note that in those formulas $\ell = 2$ and the matrix symbols $\mathcal{K}_{1,\pm\pm}(z)$ are given by (2.21), while $\mathcal{K}_{2,\pm\pm}(z)$ are generated by matrix (4.11). The matrix symbol $\mathcal{K}_{1,\pm\pm}(z)$ corresponds to the case when the homogeneous kernels are absent in (1.1) and this symbol is known, see [4], p. 145, formula (23.17). Taking this calculation into account, after easy transformations we arrive at the matrix symbols $\sigma_0(z)$ and $\sigma_\infty(z)$ with the blocks defined in (4.14)-(4.17). The application of Theorem 2.9 gives (4.18)-(4.19) if we take the relation

$$\det \sigma_\infty \left(-i\xi + 1 - \frac{1}{p} \right) = \det \sigma_0 \left(i\xi + 1 - \frac{1}{p} \right) \tag{4.30}$$

into account. The latter may be verified directly. □

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