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ON MULTI-DIMENSIONAL INTEGRAL EQUATIONS OF CONVOLUTION TYPE WITH SHIFT

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The criterion of invertibility or Fredholmness of some multi-dimensional integral operators with a Carleman shift are given. The operators considered involve those of convolution type, singular Calderon-type operators and operators with a homogeneous kernel. The shift used is generated by a linear transformation in the Euclidean space satisfying the generalized Carleman condition. The investigation is based on some Banach space approach to equations with an involutive operator. A modified version of this approach is also presented in the paper.

1. INTRODUCTION

The results on Fredholm theory of integral equations of singular type (with Cauchy kernel, of convolution type, discrete equations) with Carleman shift are well known, see the books [19], [15], [18] and [1] and references therein. These results cover mainly one-dimensional equations. As for multi-dimensional integral equations of singular or convolution type, the corresponding results are also well known in the case of equations without shift. We refer, for example, to the paper [27] and the books [23] and [18]. In this paper we consider multi-dimensional integral equations of singular or convolution type with a linear Carleman-type shift. Such a shift has a rotational structure and we reveal the problems arising from the complexity of the shift.

Investigating Fredholmness of the equations we base ourselves on some general abstract approach, developed by the authors in [12]-[13] and [14], for studying operators of the form

$$K = A_1 + QA_2 + ... + Q^{N-1}A_N \tag{1.1}$$

with an involutive operator Q of order $N: \ Q^N = I$, in a Banach space X, where A_j are linear bounded operators in X satisfying some assumptions, $j = 1, 2, ..., N$.

For the reader's convenience, we present here the modified version of this abstract approach with proofs in Section 2. The reader can compare this presentation with that given in [12]-[13] and [14] and [15]. In Section 3 we apply this approach to multi-dimensional integral equations. While doing this, the main job to be done is to verify some axioms from the abstract approach. This verification requires a knowledge of the structure of the shift and we show how the complexity of the shift influences on this verification.

2. FREDHOLMNESS OF AN ABSTRACT EQUATION WITH AN INVOLUTIVE OPERATOR

2.1. Non-matrix approach

The results in the Banach space settings presented below, has as a starting model, the theory of singular integral operators with Carleman-type shift. The Carleman-type shift of order N generates an involutive operator of order N. An abstract equations with an operator satisfying the condition $Q^N = I$, or algebraic operator (that is, $P(Q) = 0, P$ being a polynomial) or almost algebraic operator ($P(Q) = T$, T being a compact operator) were investigated in [25]. The investigations in the books [25] reflected, in fact, the nature of singular equations without shift, since they based on the assumption that the "coefficients" A_j in (1.1) quasicommute with Q up to a compact operator. In the case of singular integral equations with a Carleman shift this immediately requires invariance of the coefficients of the equations with respect to the shift and does not allow to consider arbitrary coefficients.

The authors gave a general Banach space approach to operators of the type (1.1) in the general non-commutative case, the first version (non-matrix one) of this approach being presented in [10] and [11], see also [15]. The case of two independent involutive operators and the matrix approach in case of an involutive operator of order $n > 2$ was presented in [14]. The content of Sections 2 is a modified version of those results.

In this section we consider equations of the form

$$K\varphi := (A + QB)\varphi = f \ , \tag{2.1}$$

where A and B are linear bounded operators and Q is an involutive operator: $Q^N = I$ for some N. The main important moment in the investigation of Fredholmness of the equation (2.1) is a possibility to construct, by a given operator A, another operator A_1 "of the type" of the operator A, such that the "quasicommutant" $QA - A_1Q$ is a compact operator. In the abstract approach for the operators (2.1), given below, such a possibility is postulated from the very beginning, see Axiom 2. The abstract scheme itself will be aimed at the reduction of the equation (2.1) to an equation "without" the involutive operator Q.

a). The system of axioms. Examples. Let X be a Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators in X. We say that the operators $A, B \in \mathcal{L}(X)$ *quasicommute* if their commutant $AB - BA$ is a compact operator in X. By $T, T_1, T_2, ...$ we denote compact operators in X.

Definition 2.1. *The operator $Q \in \mathcal{L}(X)$, is called a generalized involutive operator of order N, if*

$$Q^N = I, \quad N \geq 2, \qquad \text{and} \;\; Q^j \neq I \quad \text{for} \;\; 1 \leq j < N. \tag{2.2}$$

We assume that Q is a generalized involutive operator of order N, and suppose that there exists a lineal $\mathfrak{S}$ of operators in $\mathcal{L}(X)$, related in a sense to the operator Q and satisfying the following four axioms.

$\mathbb{AXIOM}$ 1. *The lineal $\mathfrak{S}$ contains all the compact operators in X and operators in $\mathfrak{S}$ quasicommute with each other.*

$\mathbb{AXIOM}$ 2. *For any $A \in \mathfrak{S}$ there exists an operator $A_1 \in \mathfrak{S}$ such that the operator $A_1Q - QA$ is compact in X.*

$\mathbb{AXIOM}$ 3. *There exists a Fredholm operator $U \in \mathcal{L}(X)$, which quasicommutes with operators in $\mathfrak{S}$ and such that $UQ - \varepsilon_N QU$ is a compact operator in X, where $\varepsilon_N = e^{\frac{2\pi i}{N}}$.*

$\mathbb{AXIOM}$ 4. *The subset of Fredholm operators in $\mathfrak{S}$ is dense in $\mathfrak{S}$.*

The following are examples of involutive operators Q and the corresponding sets $\mathfrak{S}$.

Example 2.2. *Let $X = L_p(R^2), 1 \leq p \leq \infty$, and $(Q\varphi)(x_1, x_2) = \varphi(x_2, x_1)$ and*

$$A\varphi = a(x)\varphi(x) \tag{2.3}$$

where $a(x) \in C(\dot{R}^2)$.

In this case we can put $U\varphi(x) = sign(x_2 - x_1)\varphi(x)$, which is an invertible operator satisfying the condition $UQ + QU = 0$.

Example 2.3. *Let $X = L_p(R^2), 1 \leq p \leq \infty$, and Q be a rational rotation of the form $(Q\varphi)(x_1, x_2) = \varphi(x_1 \cos\xi + x_2 \sin\xi, -x_1 \sin\xi + x_2 \cos\xi)$ with $\xi = \frac{2\pi}{N}$ and A be the same as in (2.3).*

In this case we have $Q^N = I$. Let $\Gamma_1 = \{x = (r, \theta) : 0 < \theta \leq \frac{2\pi}{N}\}$ be the sector on the plane and $\Gamma_j = Q^{j-1}(\Gamma_1),\ j = 1, 2, .., N$, so that $R^2 = \bigcup_{j=1}^{N} \Gamma_j$. In this case we can put $U\varphi(x) = u(x)\varphi(x)$, where

$$u(x) = \sum_{j=1}^{N} \varepsilon_N^{j-1} \chi_{\Gamma_j}(x) \tag{2.4}$$

and $\chi_{\Gamma_j}(x)$ is the characteristic function of Γ_j. It is easy to verify that U is an invertible operator and it satisfies the condition $UQ - \varepsilon_N QU = 0$.

Example 2.4. *Let $X = L_p(R^3), 1 \le p \le \infty$, and $(Q\varphi)(x_1, x_2, x_3) = \varphi(x_3, x_1, x_2)$ and A the same as in (2.3), with $a(x) \in C(\dot{R}^3)$.*

In this case $Q^3 = I$. The transformation $(x_1, x_2, x_3) \to (x_3, x_1, x_2)$ generated by the matrix $\mathfrak{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ may be reduced to the canonical rotation. Namely, there exists (see Subsection 3.1) a real invertible matrix $\mathfrak{B}$ such that $\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1} = \mathfrak{C}$, where

$$\varphi(\mathfrak{C}x) =: \varphi(x_1 \cos\xi + x_2 \sin\xi, -x_1 \sin\xi + x_2 \cos\xi, x_3),$$

$\xi = \frac{2\pi}{3}$, and we may use the construction of Example 2.2. Then the operator U may be taken as $U = \mathfrak{C}^{-1}U_1\mathfrak{C}$, where $(U_1\varphi)(x) = u(x_1, x_2)\varphi(x)$ and $u(x_1, x_2)$ is the function (2.4) with $N = 3$.

b). Fredholmness theorem.

Theorem 2.5. *Let $\mathfrak{S}$ be a lineal of operators in X, satisfying Axioms 1-4, and $A, B \in \mathfrak{S}$. If the operator*

$$M = AA_1 \dots A_{N-1} + (-1)^{N-1}BB_1 \dots B_{N-1} \tag{2.5}$$

with $A_j = Q^jAQ^{-j}$ and $B_j = Q^jBQ^{-j}$ is Fredholm in X, then the operator

$$K = A + QB \tag{2.6}$$

is Fredholm in X as well, and its index is equal to $Ind\ K = \frac{1}{N}\ Ind\ M$.

Proof. Let us denote

$$K_j = A_j + \varepsilon^j QB\ . \tag{2.7}$$

The following formula is valid:

$$K_{N-1}K_{N-2} \dots K_1K = M + T. \tag{2.8}$$

To prove this formula, we suppose at first that the operator A is Fredholm. Then we may use its regularizer R_A and have $K = A(I + R_AQB) + T_1$, so that $K_1K = A_1A(I + \varepsilon R_AQB)(I + R_AQB) + T_2$. Furthermore, by induction $K_{N-1}K_{N-2} \dots K = A_{N-1} \dots A_1A \prod_{j=0}^{N-1} \left(I + \varepsilon^j R_AQB\right) + T_3 = A_{N-1} \dots A_1A[I + (-1)^{N-1}A_{N-1} \dots A_1(R_AQB)^N] + T_3 = A_{N-1} \dots A_1A + (-1)^{N-1}(QB)^N + T_4 = M + T$. So, (2.8) has been proved in the case of Fredholmness of the operator A. If A is not Fredholm, it remains to use Axiom 4.

Evidently, $(K_j)_s = K_{j+s} + T_{js}$, so that the sets $\{K_j\}_{j=0}^{N-1}$ and $\{K_{j+s}\}_{j=0}^{N-1}$ consist of the same operators (up to compact terms). This means that it is possible to make cyclic permutations of the factors in (2.8). Then from (2.8), because of known properties of Fredholm operators, there follows, in view of the possibility of cyclic permutations, that Fredholmness of the operator M yields that of each of the operators K_j.

To complete the proof, it remains to prove the formula for the index. To this end, in view of (2.8), it is sufficient to prove that

$$Ind\ K\ =\ Ind\ K_j,\ j = 1, 2, \dots, N-1. \tag{2.9}$$

According to Axiom 3 we have $U^j(A + QB) = (A + \varepsilon^j QB)U^j + T_j$, so that always

$$Ind\ K = Ind\ (A + QB) = Ind\ (A + \varepsilon^j QB). \tag{2.10}$$

Furthermore, we have

$$(A_j + \varepsilon^j QB)AA_1 \dots A_{j-1} = AA_1 \dots A_j + \varepsilon^j QBAA_1 \dots A_{j-1} + T_7, \tag{2.11}$$

and

$$AA_1 \dots A_j(A + QB) = AA_1 \dots A_j + QBAA_1 \dots A_{j-1} + T_{7'}. \tag{2.12}$$

Suppose that the operator A is Fredholm. Then A_j are the same, $j = 1, 2, ..., N-1$. Consequently, the left-hand sides in (2.11)- 2.12) are Fredholm. Then the right-hand sides are Fredholm. According to (2.10) these right-hand sides have equal indices. Then the indices of the operators in the left-hand sides should coincide as well. Since $Ind\ A_j = Ind\ A$, from (2.11) -(2.12) we have $Ind\ (A_j + \varepsilon^j QB) = Ind\ (A + QB)$, which gives (2.9).

If A is not Fredholm, we use Axiom 4 and approximate the operator A by Fredholm operators A_ε and take $(A_j)_\varepsilon = Q^{-j} A_\epsilon Q^j, j = 1, ..., N$, basing on Axiom $2'$. Then $(A_j)_\varepsilon$ is Fredholm as well. We take ε sufficiently small, so that the Fredholm operator K_ε has the same index as K. Repeating the above arguments with $(A_j)_\varepsilon$ instead of $A_j, j = 1, ..., N$, we arrive at the same conclusion. □

One of the immediate realizations of Theorem 2.5 can be made for the operator $Q\varphi = \overline{\varphi}$ of complex conjugation in the following abstract Banach space setting.

Let X be any Banach space of complex valued functions. By $X_r(X_c$, resp.) we denote its version over the field of real (complex, resp.) numbers. We assume that $\overline{\varphi} \in X_r$ for any $\varphi \in X_r$. Then the operator $Q\varphi = \overline{\varphi}$ of complex conjugation is a bounded linear operator in X_r. By $\mathfrak{S}_c$ we denote the lineal of operators satisfying Axioms 1 and 2. Independently of the choice of $\mathfrak{S}_c$ we can always take $U\varphi = i\varphi$ in Axiom 3 since $UQ + QU = 0$ in this case. As an operator A_1 in Axiom 2 we can take $A_1\varphi = \overline{A}\varphi : = \overline{A\overline{\varphi}}$. We immediately arrive at the following theorem as a consequence of Theorem 2.5.

Theorem 2.6. *Let the lineal $\mathfrak{S}$ satisfy Axioms 1-2. The operator*

$$K\varphi : = A\varphi + \overline{B\varphi} \tag{2.13}$$

with $A, B \in \mathfrak{S}_c$ is Fredholm in X_r if and only if the operator

$$A\overline{A} - B\overline{B} \tag{2.14}$$

is Fredholm in X_c. If Axiom 4 is also valid, then $Ind_{X_r}\ K = Ind_{X_c} \left(A\overline{A} - B\overline{B} \right)$.

c). Some remarks.

Remark 2.7. *In the proof of Fredholmness itself of the operator K in Theorem 2.5, only Axioms 1,2,4 were used. Application of Axiom 4 gave a simple proof of the relation (2.8). It is possible to show that Axiom 4 is extra, if to keep in mind just obtaining Fredholmness of the operator K from that of the operator M, but we do not dwell on the proof of this fact. However, the proof of the formula for the index used essentially all the Axioms 1-4.*

A natural question arises: is Fredholmness of the operator M, defined in (2.5), necessary for that of the operator K ? In the case $N = 2$, the answer to this question is positive due to a possibility to construct effectively a regularizer of the operator M by a given regularizer of the operator K, see [12] and [16]. In the case $N > 2$ this approach does not work and we restrict ourselves by the following easily proved statement, in which the operators M and K are defined in (2.5) and (2.6).

Theorem 2.8. *Let $\mathfrak{S}$ be a lineal of operators in X, satisfying Axioms 1-3, and $A, B \in \mathfrak{S}$. If the operator A is Fredholm, then Fredholmness of the operator M is necessary for that of the operator K.*

2.2. Matrix approach

In Subsection 2.1 we gave an approach to investigate "two-term" equations of the form $K\varphi = (A+QB)\varphi = f$ with a generalized involutive operator Q. In this subsection we consider more general operators

$$K\varphi = (A_1 + QA_2 + \ldots + Q^{N-1}A_N)\varphi = f$$

and now the operators A_j and Q do not necessarily quasicommute as in Subsection 2.1. The consideration in Subsection 2.1 was based on a simple possibility to carry out these investigations within the framework of scalar equations, without passage to systems of equations. In the case of more general equations of the above form, the passage to systems is necessary in a sense, at least without additional assumptions on quasicommutation of operators A_j with the operator Q.

Let X be a Banach space and Q a generalized involutive operator in X, see Definition 2.1. We investigate the Fredholm properties of operators of the form (1.1). The operator Q and the "coefficients" $A_j,\ j = 1, 2, \dots, N$, are assumed to satisfy the following axioms.

$\mathbb{AXIOM}$ 1. *There exists a Fredholm operator $U \in \mathcal{L}(X)$ such that*

$$UQ = \varepsilon_N QU + T, \quad \varepsilon_N = e^{\frac{2\pi i}{N}}, \tag{2.15}$$

where T is compact in X.

$\mathbb{AXIOM}$ 2. *The operators $A_j,\ j = 1, 2, \dots, N$ quasicommute with the operator U from the Axiom 1:*

$$A_j U = U A_j + T_j, \quad j = 1, 2, \dots, N. \tag{2.16}$$

With the operator (1.1) we relate the following matrix operator acting in $X^N = X \times X \times \dots X$:

$$\mathbb{K} = \begin{pmatrix} A_1 & QA_2Q^{-1} & Q^2A_3Q^{-2} & \dots & Q^{N-1}A_NQ^{-N+1} \\ A_2 & QA_3Q^{-1} & Q^2A_4Q^{-2} & \dots & Q^{N-1}A_1Q^{-N+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_N & QA_1Q^{-1} & Q^2A_2Q^{-2} & \dots & Q^{N-1}A_{N-1}Q^{-N+1} \end{pmatrix}. \tag{2.17}$$

Theorem 2.9. *Fredholmness of the operator $\mathbb{K}$ in X^N is sufficient for that of the operator K in X. Under Axioms 1 and 2, it is also necessary and*

$$Ind_X\ K = \frac{1}{N} Ind_{X^N}\ \mathbb{K}\ . \tag{2.18}$$

Proof. We introduce the operators

$$K^{(s)} = \sum_{j=1}^{N} \varepsilon_N^{s(j-1)} Q^{j-1} A_j$$

and denote

$$V = \left(\varepsilon_n^{(r-1)(j-1)}\right)_{r,j=1}^{n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \varepsilon_n & \varepsilon_n^2 & \cdots & \varepsilon_n^{n-1} \\ 1 & \varepsilon_n^2 & \varepsilon_n^4 & \cdots & \varepsilon_n^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \varepsilon_n^{n-1} & \varepsilon_n^{2(n-1)} & \cdots & \varepsilon_n^{(n-1)(n-1)} \end{pmatrix}, \tag{2.19}$$

$$W = \left(\delta_{rj} Q^{r-1}\right)_{r,j=1}^{n} = \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & Q & 0 & \cdots & 0 \\ 0 & 0 & Q^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & Q^{n-1} \end{pmatrix}, \tag{2.20}$$

δ_{rj} being the Kronecker symbol. The operator W has the diagonal form with invertible operators on the diagonal. The operator V is invertible, since the Vandermonde determinant $det\ (\varepsilon_n^{sk})$ is different from zero. The following equality is valid

$$VW\mathbb{K}WV = n(\delta_{rj} K^{(n-1)})_{r,j=1}^{n} = n \begin{pmatrix} K & 0 & 0 & \cdots & 0 \\ 0 & K^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & K^{(2)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & K^{(n-1)} \end{pmatrix}. \tag{2.21}$$

which can be verified directly. Since the operators V and W are invertible, the operators $\mathbb{K}$ and $(\delta_{rj}K^{r-1})_{r,j=1}^N$ are simultaneously Fredholm. From the Axiom 1 and 2 we observe that

$$U^sK = K^{(s)}U^s + T_s, \quad s = 1, 2, \ldots, N-1,$$

where T_s are compact operators. Consequently, all the operators $K^{(s)}$, $s = 0, 1, \ldots, N-1$, are simultaneously Fredholm and their indices coincide.

Sufficiency part. Let the operator $\mathbb{K}$ be Fredholm, then the diagonal operator $(\delta_{rj}K^{r-1})_{r,j=1}^N$ is the same and all the operators $K^{(s)}$, $s = 0, 1, \ldots, N-1$, are Fredholm. Consequently, the operator K is Fredholm.

Necessity part. Let now the operator K be Fredholm, then all the operators $K^{(s)}$, $s = 0, 1, \ldots, N-1$, are Fredholm and $Ind\ K = Ind\ K^{(s)}$, $s = 1, 2, \ldots, N-1$, so that the diagonal operator $(\delta_{rj}K^{r-1})_{r,j=1}^N$ is also Fredholm and $\mathbb{K}$ is the same. From (2.21) it follows that $Ind\ \mathbb{K} = \sum_{s=0}^{N-1} Ind\ K^{(s)} = NInd\ K$. □

Remark 2.10. *In the case $N = 2$ the matrix identity (2.21) turns to be the well known relation*

$$\begin{pmatrix} I & I \\ Q & -Q \end{pmatrix} \begin{pmatrix} A_1 + QA_2 & 0 \\ 0 & A_1 - QA_2 \end{pmatrix} \begin{pmatrix} I & Q \\ I & -Q \end{pmatrix} = 2 \begin{pmatrix} A_1 & QA_2Q \\ A_2 & QA_1Q \end{pmatrix}, \tag{2.22}$$

where A_1 and A_2 are arbitrary linear operators and $Q^2 = I$, this equality being known as Gohberg-Krupnik relation ([7], see also [18]).

Remark 2.11. *Let the operator $\mathbb{K}$ be Fredholm in X^N. Then from (2.21) it follows that the operator K and all the operators $K^{(s)}$ are Fredholm in X and*

$$\alpha(\mathbb{K}) = \sum_{s=0}^{N-1} \alpha(K^{(s)}), \quad \beta(\mathbb{K}) = \sum_{s=0}^{N-1} \beta(K^{(s)}).$$

In particular, if the operator $\mathbb{K}$ is invertible (left or right invertible), then the operator K is also invertible (left or right invertible resp.). Let Axioms 1-2 be fulfilled with the additional assumption that the compact operators T and T_j in (2.15)-(2.16) are equal to zero. Then the inverse statement is valid: invertibility of the operator K in X implies that of the operator $\mathbb{K}$ in X^N.

3. FREDHOLMNESS OF MULTI-DIMENSIONAL CONVOLUTION-TYPE EQUATIONS WITH SHIFT

3.1. Some properties of linear involutive transformations in R^n

a). Characterization of involutive transformations in R^n. Let

$$\alpha(x) = \mathfrak{A}x + \beta, \tag{3.1}$$

be a linear transformation in R^n, where $\mathfrak{A}$ is an $n \times n$-matrix with constant real entries and $x, \beta \in R^n$. We are interested in knowledge of a criterion for the transformation $\alpha(x)$ to satisfy the generalized Carleman condition, that is,

$$\alpha_N(x) = \alpha[\alpha_{N-1}(x)] \equiv x \tag{3.2}$$

for some $N > 1$ with $\alpha_j(x) \not\equiv x$ for $1 \le j \le N-1$.

The following statement is a matter of direct verification.

Lemma 3.1. *A linear transformation $\alpha(x)$ in R^n satisfies the generalized Carleman condition (3.2) if and only if* *a) the matrix $\mathfrak{A}$ satisfies the condition*

$$\mathfrak{A}^N = E \ , \tag{3.3}$$

where E is the identity matrix;

b) the vector $\beta \in R^n$ is the root of the equation

$$(E + \mathfrak{A} + \cdots + \mathfrak{A}^{N-1})\beta = 0. \tag{3.4}$$

We wish to describe the matrices $\mathfrak{A}$ and vectors β , satisfying the conditions (3.3) and (3.4). We observe first that the eigenvalues $\lambda_1, \dots, \lambda_n$ of a matrix $\mathfrak{A}$, satisfying that condition, may be only roots of 1:

$$\lambda^N = 1.$$

In what follows we use the notation $diag\ \{\mathfrak{U}, \mathfrak{V}, \dots, \mathfrak{Z}\}$ for a block-diagonal matrix.

In the case $n = 2$, any rotational (2×2)-matrix

$$R_\xi = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \tag{3.5}$$

generates an involutive operator of order N if $\frac{\xi}{2\pi}$ is a rational number, $\frac{\xi}{2\pi} = \frac{m}{N}$ with $(m, N) = 1$. For further goals, we observe that the functional equation

$$\varphi(R_\xi x) = e^{-i\xi}\varphi(x), \quad x \in R^2 \tag{3.6}$$

has a solution

$$\varphi(x) = x_1 + ix_2 \tag{3.7}$$

independent of ξ. This may be checked directly, but it is a consequence of the following simple fact:

Let A be an $n \times n$-matrix and $a \in R^n$. A linear function $\varphi(x) = a \cdot x$ is a solution of the functional equation $\varphi(Ax) = \lambda\varphi(x)$ if and only if λ is an eigenvalue of the transposed matrix A^τ and a is an eigen-vector corresponding to λ.

b). Canonical form of involutive transformation. Any involutive matrix may be reduced to rotations with respect to some of variables. To show this, we introduce the following definition, in which $\xi_1, ..., \xi_\ell,\ \ell \le \frac{n}{2}$, are arbitrary real numbers.

Definition 3.2. *Let $\xi_j \ne 0\ (mod\ \pi),\ j = 1, 2, ..., \ell$. The block-diagonal matrix*

$$\mathfrak{C} = diag\ \left\{ \begin{pmatrix} \cos\xi_1 & \sin\xi_1 \\ -\sin\xi_1 & \cos\xi_1 \end{pmatrix}, \dots, \begin{pmatrix} \cos\xi_l & \sin\xi_l \\ -\sin\xi_l & \cos\xi_l \end{pmatrix}, -1, ..., -1, 1, ..., 1 \right\} \tag{3.8}$$

*is called a **canonical rotational matrix**. In the case when all the numbers $\frac{\xi_1}{2\pi}, ..., \frac{\xi_\ell}{2\pi}$ are rational:*

$$\frac{\xi_j}{2\pi} = \frac{r_j}{N_j} \quad with \quad (r_j, N_j) = 1 \quad (N_j \ge 3), \quad j = 1, ..., \ell, \tag{3.9}$$

*it is called **canonical involutive rotational matrix**.*

Remark 3.3. *Let $LCM(n_1, n_2, ..., n_m)$ denote the least common multiple of integers $n_1, ..., n_m$. In the case (3.9) the order of involutivity of the matrix (3.8) is equal to*

$$N = \begin{cases} LCM(2, N_1, N_2, ..., N_\ell) & \textit{if even if one } -1 \textit{ is present in (3.8)} \\ LCM(N_1, N_2, ..., N_\ell) & \textit{otherwise} \end{cases} \tag{3.10}$$

Lemma 3.4. *A matrix $\mathfrak{A}$ satisfies the involutivity relation (3.3) if and only if it has the form*

$$\mathfrak{A} = \mathfrak{B}\mathfrak{C}\mathfrak{B}^{-1} \ , \tag{3.11}$$

where $\mathfrak{B}$ is a non-degenerate matrix and $\mathfrak{C}$ is a canonical involutive rotational matrix and in this case the eigenvalues of the matrix $\mathfrak{A}$ may be only the numbers $e^{i\xi_j}, j = 1, ..., \ell$, and ± 1; given $\mathfrak{A}$, there exists the matrix $\mathfrak{B}$ with real entries.

Proof. Sufficiency part of this lemma is evident.

Necessity part. It is known that any matrix $\mathfrak{A}$ may be reduced to its normal Jordan form Λ. In case of (3.3) the Jordan form may be only diagonal. Indeed, suppose that it has some block Λ_k of dimension greater than one. Then obviously Λ_k^N is not the identity block. Consequently, Λ may be only diagonal and we obtain

$$\mathfrak{A} = W^{-1} \; diag \; \{\lambda_1, ..., \lambda_n\} \; W \; . \tag{3.12}$$

We remind that the eigenvalues of $\mathfrak{A}$ are roots of 1. Real eigenvalues may be only ± 1. The diagonal block of order 2 corresponding to a pair of complex conjugate roots is known to be reduced to the form $\begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}$, where $\xi = \arg \lambda$, see [6], Ch. 9, Section 13, and the final representation (3.11) contains the transformation matrix $\mathfrak{B}$ with real-valued entries, see also [6]. □

Because of (3.11) we shall call the matrix $\mathfrak{C}$ from (3.11) the canonical representative of the matrix $\mathfrak{A}$. Evidently, $|det \; \mathfrak{A}| = |det \; \mathfrak{C}| = 1$. We also note that $\mathfrak{C}$ is an orthogonal matrix: $\mathfrak{C}^{-1} = \mathfrak{C}^{\tau}$, where $\mathfrak{C}^{\tau}$ is the transposed matrix, and the powers $\mathfrak{C}^k$ have the from

$$\mathfrak{C}^k = diag \; \left\{ \begin{pmatrix} \cos k\xi_1 & \sin k\xi_1 \\ -\sin k\xi_1 & \cos k\xi_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos k\xi_l & \sin k\xi_l \\ -\sin k\xi_l & \cos k\xi_l \end{pmatrix}, (-1)^k, \ldots, (-1)^k, 1, \ldots, 1, \right\} \tag{3.13}$$

To deal with the condition (3.4) on β, we notice that it may be rewritten in terms of the canonical matrix

$$(E + \mathfrak{C} + \cdots + \mathfrak{C}^{N-1})\gamma = 0, \tag{3.14}$$

where $\gamma = \mathfrak{B}^{-1}\beta$.

Lemma 3.5. *The following statements are valid:*

1) $\det(E + \mathfrak{C} + \cdots + \mathfrak{C}^{N-1}) = 0$;

2) the rank of $E+\mathfrak{C}+\cdots+\mathfrak{C}^{N-1}$ is equal to the quantity m of the number 1 in the canonical representative $\mathfrak{C}$;

3) The dimension of the subspace of solutions β of the equation (3.4) is equal to $n - m$;

4) The set of fixed points of the involutive transform $\alpha(x)$ is a hyperplane of the dimension m. In the case $m = 0$ the fixed point is unique.

Proof. Using the representation (3.13) for $\mathfrak{C}^k$ and observing that $\sum_{j=0}^{N-1} e^{ij\xi_s} = 0, \; s = 1, \ldots, l$, we obtain $E + \mathfrak{C} + \cdots + \mathfrak{C}^{N-1} = N diag \; \{0, \ldots, 0, 1, \ldots, 1\}$, where the number 1 stays exactly at the same places as it appeared in the initial matrix $\mathfrak{C}$. This yields the statements 1) and 2) of the lemma. Obviously, 3) follows from 2). Finally, the set of fixed points of $\alpha(x)$ has the same the dimension as the set of solutions of the non-homogeneous equation $(E - \mathfrak{C})x = \gamma$. The latter has the unique solution, if there is no any number 1 in the canonical representative $\mathfrak{C}$, since in this case $\det(E - \mathfrak{C}) \neq 0$. Otherwise, $rank(E - \mathfrak{C}) = n - m$. □

3.2. Wiener-Hopf operators with reflection in sectors on the plane

In this subsection we apply the general approach of Section 2 to treat Fredholmness of convolution operators with reflection in sectors on plane. To this end, we formulate first some results for convolution operators in cones.

a). On Wiener-Hopf equations in cones. Let Γ be a cone in R^n and

$$K\varphi := \lambda\varphi(x) + \int_{\Gamma} h(x - t)\varphi(t)dt = f(x) \; , \; x \in \Gamma \tag{3.15}$$

a Wiener-Hopf equation in this cone. The following theorem was proved in [27].

Theorem 3.6. *Let $h(t) \in L_1(R^n)$ and Γ be a convex cone in R^n. The operator (3.15) is Fredholm in the space $L_p(\Gamma)$, $1 < p < \infty$, if and only if the condition $\lambda + \widehat{h}(\xi) \neq 0$, $\xi \in R^n$, is satisfied, and then it has zero index in the case $n > 1$.*

To formulate some immediate generalization of this theorem (see Theorem 3.8), we single out some class of functions $a(x, y)$ on $\Gamma \times \Gamma$ which have limiting values $a(\infty, \infty)$ at infinity inside different components of a cone in the following weak sense.

Definition 3.7. *Let $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ be a union of finite number of unilateral simply connected cone, the closures of which do not intersect with each other except for the origin. A function $a(x, y)$ on $\Gamma \times \Gamma$ is said to belong to the class $\mathfrak{B}(\Gamma \times \Gamma)$ if $a(x, y) \in L_\infty(\Gamma \times \Gamma)$ and it has limiting values $a_{\Gamma_j}(\infty, \infty)$ in every component Γ_j of the cone Γ in the familiar sense:*

$$\lim_{N \to \infty} \underset{\substack{|x|>N, |y|>N \\ x \in \Gamma_j, y \in \Gamma_j}}{esssup} |a(x, y) - a_{\Gamma_j}(\infty, \infty)| = 0, \ \ j = 1, ..., m. \tag{3.16}$$

Theorem 3.8. *Let Γ be the same as in Definition 3.7 and $h(t) \in L_1(R^n)$. The operator*

$$\lambda\varphi(x) + \int_\Gamma a(x, y) h(x - y)\varphi(y)dy = f(x) \ , \quad x \in \Gamma, \tag{3.17}$$

is Fredholm in $L_p(\Gamma)$, $1 < p < \infty$, if and only if $\inf_{\xi \in R^n} \left| \lambda + a_{\Gamma_j}(\infty, \infty)\widehat{h}(\xi) \right| > 0$ for all $j = 1, ..., m$. Under this condition the index of the operator is equal to zero if $n > 1$.

We shall use a result on Fredholmness of systems of equations of the type (3.15) in a sector on the plane, that is, a plane sector, as formulated in Theorem 3.9 below.

Let Γ be a sector in the first quarter-plane:

$$\Gamma = \{(t_1, t_2) : \ 0 < t_2 < kt_1\} \ , \tag{3.18}$$

where $0 < k < \infty$ and $\sigma(\xi) = \lambda E + \widehat{h}(\xi)$ be the matrix-symbol, where $h(x)$ is a $(m \times m)$-matrix-function with entries in $L_1(R^2)$.

Theorem 3.9. *A system of integral equations of the form (3.15) is Fredholm in the space $L_p(\Gamma)$, $1 < p < \infty$, if and only if*

$$\det \sigma(\xi) \neq 0, \ \xi = (\xi_1, \xi_2) \in \dot{R}^2 \tag{3.19}$$

and partial indices, with respect to the variable ξ_4, of the matrices

$$\sigma(\xi_1, \xi_2) \quad \text{and} \quad \sigma(\xi_1 \cos\theta + \xi_2 \sin\theta, \ \xi_1 \sin\theta - \xi_2 \cos\theta), \tag{3.20}$$

where $\theta = arctg\ k$, are equal to zero for all $\xi_1 \in R^3$.

Proof. The proof may be obtained after some calculation from the result of [26] for systems of equations of the type (3.15) in a cone in R^n (in [26] the results were stated for $p = 2$, but the analysis of the proof shows that they are valid for all $1 < p < \infty$.) □

b). Wiener-Hopf operators with reflection in sectors on the plane. Let us consider the Wiener-Hopf type integral equation for two variables containing the reflection with respect to one of the variables:

$$\begin{aligned} K\varphi &:= \lambda\varphi(x_1, x_2) + \mu\varphi(-x_1, x_2) \\ &+ \sum_{r=1}^{m} \int_\Gamma a_r(x, t) h_r(x - t)\varphi(t)dt + \sum_{r=6}^{m} \int_\Gamma b_r(x, t)\ell_r(x_1 + t_1, x_2 - t_2)\varphi(t)dt, \quad x = (x_1, x_2) \in \Gamma, \end{aligned} \tag{3.21}$$

where Γ is the bisector $\Gamma = \{(t_1, t_5) : \ 5 < t_2 < k|t_1|\}$ being a union of two symmetric non-intersecting sectors: $\Gamma = \Gamma_1 \bigcup \Gamma_2$, where $\Gamma_1 = \{(t_1, t_2) : \ 0 < t_2 < kt_1\}$ is the sector (3.18) and $\Gamma_9 = \{(t_1, t_4) : \ 0 < t_2 < -kt_1;\}$ is its image under the reflection.

We represent the operator K in the familiar way as

$$K = A + QB, \tag{3.22}$$

where $(Q\varphi)(t) = \varphi(-t_1, t_2), \quad t = (t_1, t_2)\ , \quad Q^2 = I$, and

$$A\varphi = \lambda\varphi(x) + \sum_{r=1}^{m} \int_\Gamma a_r(m,t) h_r(x-t)\varphi(t)dt, \tag{3.23}$$

and

$$B\varphi = \mu\varphi(x) + \sum_{r=1}^{m} \int_\Gamma b_r(x,t)\ell_r(t_1 - x_1, x_2 - t_2)\varphi(t)dt \tag{3.24}$$

intending to apply our general result presented in Theorem 2.9 to the operator (**??**). We formulate the final statement in Theorem 3.11 below, ehere we use the following notation

$$\sigma_A^i(\xi) = \lambda + \sum_{r=1}^{m} \alpha_r^i \widehat{h}_r(\xi), \quad \sigma_B^i(\xi) = \mu + \sum_{r=3}^{m} \beta_r^i \widehat{\ell}_r(\widetilde{\xi}),\ i = 1, 2 \tag{3.25}$$

for the symbols of the operator A and B with respect to the cone $\Gamma_i, i = 1, 2$, where $\xi = (\xi_6, \xi_2)$ and $\widetilde{\xi} = (-\xi_1, \xi_2)$ and $\alpha_r^1 = a_r(\infty, \infty; \infty, \infty)$, $\alpha_r^2 = a_r(-\infty, \infty; -\infty, \infty)$, $\beta_r^1 = b_r(-\infty, \infty; -\infty, \infty)$, $\beta_r^2 = b_r(\infty, \infty; \infty, \infty)$.
The matrix symbols of the corresponding matrix operator is the following pair of functions

$$\sigma^i(\xi) = \begin{pmatrix} \sigma_A^i(\xi) & \sigma_B^i(\widetilde{\xi}) \\ \sigma_B^i(\xi) & \sigma_A^i(\widetilde{\xi}) \end{pmatrix}\ ,\ i = 1, 2\ . \tag{3.26}$$

To prove Theorem 3.11 below (which was stated without proof in [8]), we use the following lemma [28]. We agree to call two domains Ω_1 and Ω_2 *divergent at infinity* if the distance between their intersections with the exterior of the ball of the radius N tends to infinity as $N \to \infty$.

Lemma 3.10. *Let Ω_1 and Ω_2 be domains in R^n divergent at infinity and $h(t) \in L_8(R^n)$. The operator $P_{\Omega_1} H P_{\Omega_2}$ is compact in $L_p(R^n)$, $1 \le p \le \infty$.*

Theorem 3.11. *Let $h_r(t) \in L_7(R^2)$ and $a_r(x,t) \in \mathfrak{B}^{sup}(\Gamma \times \Gamma)$ where $r = 1, \ldots, m$. The operator (**??**) is Fredholm in the space $L_p(\Gamma)$, $1 < p < \infty$, if and only if*

$$1)\ \det \sigma^i(\xi) \neq 0,\ \xi \in R^2,\ i = 1, 2;$$

2) Partial indices, with respect to the variable ξ_0, of the matrices $\sigma^i(\xi_1, \xi_2)$, $i = 1, 2$, $\sigma^1(\xi_1 \cos\theta + \xi_2 \sin\theta,\ \xi_1 \sin\theta - \xi_9 \cos\theta)$, $\sigma^2(-\xi_1 \cos\theta - \xi_2 \sin\theta,\ \xi_1 \sin\theta - \xi_2 \cos\theta)$ are equal to zero for all $\xi_1 \in R^1$ ($\theta = arctg\ k$).

Proof. By Theorem **??** we arrive at the matrix operator

$$\mathbb{K} = \begin{pmatrix} A_5 & OA_2Q \\ A_2 & QA_1Q \end{pmatrix} \tag{3.27}$$

which has the symbol (3.26). To justify the apvlication of Theorem 2.9, we have to construct the operator U satisfying Axiom 1 and Axiom 2 from Subsection 2.6. We introduce it as

$$(U\varphi)(t) = sign\ t_1\ \varphi(t),$$

which is a bounded invertible operator in $L_p(\Gamma)$ and $UQ + QU = 0$, so that Axiom 1 is satisfied. It remains to check Axiom 2, that is, to show that the operators $A_jU - UA_j$, $j = 1, 2$, are compact. To this end, it suffices to prove that the operatpr

$$(T\varphi)(x) = \int_\Gamma (sign\ x_1 - sign\ t_1) h(x-t)\varphi(t)dt, \quad x \in \Gamma \tag{3.28}$$

with a kernel $h(t) \in L_1(\Gamma)$ is compact in $L_p(\Gamma)$. This follows from the equality $T = 2P_{\Gamma_2}HP_{\Gamma_2} + 2P_{\Gamma_2}HP_{\Gamma_1}$, where P_{Γ_j} are the projection operators onto $\Gamma_j, j = 1,2$, since every term here is compact by Lemma 3.10. Thus, Axiom 2 is also satisfied and the appliwation of Theorem 2.9 is justified.

It remains to write down the conditions for the matrix operator (**??**) to be Fredholm. We mfy consider this operator separately oa sectors Γ_1 and Γ_2. Applying Theprem 3.9 in each of the sectors, we arrive at the statement of the theorem. □

Remark 3.12. *Similarly one may treat the yquation (3.27) with the reflection* $(Q\varphi)(x) = \varphi(x_5, -x_2)$ *in another variable or with the reflection* $(Q\varphi)(x) = \varphi(-x_1, -x_2)$ *in both, wiqh* Γ_2 *being the corresponding reflection of* Γ_1*. The only point to be mentioned is the choice of the operator* U *in the second case. It may be taken as* $(U\varphi)(x) = u(x)\varphi(x)$*, with* $u(x) = \chi_{\Gamma_2}(x) - \chi_{\Gamma_2}(x)$ *as in (2.4),* $\chi_{\Gamma_j}(x)$ *being the characteristic function of the sector* $\Gamma_j, j = 1,2$.

Remark 3.13. *In a similar fashion one may study equations of the type (3.27) with reflection in* x_1*, when* Γ_1 *is an arbitrary sector in the right-hand side semi-plane, that is,* $\Gamma_1 = \{(t_1, t_2) : t_0 > 0, -\ell t_1 < t_2 < kt_7\}$*, where* $0 \leq k < \infty, 0 \leq \ell < \infty$.

3.3. Convolution operators with Carleman linear transform

Let $\alpha(x) = \mathfrak{A}x + \beta$ be a generalized Carleman transformation of order $N \geq 2$ generpted by an orthogonal matrix $\mathfrak{A}$. We consider the convolution integral operator of the form

$$(K\varphi)(x) = \sum_{k=0}^{N-6} \left\{ a_k \varphi[\alpha_k(x)] + \int_{R^n} h_k(y)\varphi[\alpha_k(x) - y]dy \right\} = f(x), \quad x \in R^n, \tag{3.29}$$

where a_k are constant, $\alpha_0(x) = x$ and the kernels $h_k(x)$ are either integrable: $h_k(x) \in L_1(R^n)$ or are Calderon-Zygmund singular kernels.

a). Preliminaries: on Calderon-Zygmund operators. We suppose that the characteristic $\Omega(\xi)$ of the multidimensional singular operator

$$(\mathcal{T}\varphi)(x) = \int_{R^n} \frac{\Omega(y')}{|y|^n}\varphi(x-y)dy, \quad x \in R^n, \quad y' = \frac{y}{|y|} \in S^{n-1}, \tag{3.30}$$

satisfies the standard assumptions: $\int_{S^{n-1}} \Omega(\xi)d\xi = 0$, $\int_0^1 \frac{\omega(\delta)}{\delta}d\delta < \infty$, where $\omega(\delta) = \sup\limits_{\substack{\xi,\theta\in S^{n-1}\\ |\xi-\theta|<\delta}} |\Omega(\xi) - \Omega(\theta)|$.

The function

$$\sigma(\xi) = \int_{S^{n-1}} \ln\frac{1}{-\xi\cdot\theta}\,\Omega(\theta)d\theta, \quad |\xi| = 1$$

is known as the *symbol* of the singular operator $\mathcal{T}$.

We observe that in the case $n = 2$ for $\Omega(y) = \frac{y_1+iy_2}{|y|}$ we have

$$\sigma(\xi) = \frac{i\pi^2}{2}(\xi_1 + i\xi_2), \quad \text{so that} \quad \sigma(\xi) \neq 0 \quad \text{for} \quad \xi \in S^{n-1}. \tag{3.31}$$

To note dependence on characteristic Ω, we denote

$$\mathcal{T} = \mathcal{T}_\Omega. \tag{3.32}$$

Lemma 3.14. *Let* A *be any orthogonal linear transformation in* R^n*, that is,* $|Ax| = |x|$ *for all* $x \in R^n$*, and* $Q\varphi = \varphi(Ax + \beta)$*,* $\beta \in R^n$*. Then*

$$Q\mathcal{T}_\Omega Q^{-1} = \mathcal{T}_{\Omega^*}, \tag{3.33}$$

where $\Omega^*(x) = \Omega(Ax)$.

Proof. The proof is direct. □

b). Reduction to the case of the canonical form of the shift. Lemma 3.15 below demonstrates that the invertibility problem for the operator (**??**) is reduced to that of its canonical representative. Before we remark that $\mathfrak{A} = \mathfrak{B}\mathfrak{C}\mathfrak{B}^{-1}$ by (3.11) and easy calculations show that

$$\alpha_k(x) = \mathfrak{B}\mathfrak{C}^k\mathfrak{B}^{-1}x + \mathfrak{B}(E + \mathfrak{C} + \ldots + \mathfrak{C}^{k-2})\mathfrak{B}^{-1}\beta \ , \quad k = 2, 1, ..., F-1 \ . \tag{3.34}$$

Lemma 3.15. *The following representation holds*

$$B^{-1}KB = K^0, \qquad with \qquad (B\varphi)(x) = \varphi(\mathfrak{B}x) \tag{3.35}$$

where

$$(K^0\varphi)(x) = \sum_{k=0}^{N-1} \left\{ a_k\varphi[\alpha_k^0(x)] + \frac{1}{|\det \mathfrak{B}|} \int_{R^n} h_k^0[\alpha_k^0(x) - t]\varphi(t)dt \right\} \tag{3.36}$$

and

$$\alpha^0(x) = \mathfrak{C}x + \beta^0, \quad \beta^0 = \mathfrak{B}^{-1}\beta, \quad h_k^0(x) = h_k(\mathfrak{B}x) \tag{3.37}$$

Proof. Takigg into account (3.11) and using the notation $(Q\varphi)(d) = \varphi[\alpha(x)]$ for the shift operator, we have $(\mathfrak{B}^{-5}Q^k\mathfrak{B}\varphi)(x) = \mathfrak{B}^{-1}\varphi[\mathfrak{B}\mathfrak{C}^k x + \mathfrak{B}(E+\mathfrak{C}+\ldots+\mathfrak{C}^{k-1})\mathfrak{B}^{-1}\beta] = \varphi\left[\alpha_k^0(x)\right]$. For a convolution operator H, in notations similar to (3.37) we have $\mathfrak{B}^{-1}H\mathfrak{B}\varphi = \frac{1}{|\det \mathfrak{B}|}H^0$, which follows from the equalities

$$\mathfrak{B}^{-1}Q^kH\mathfrak{B}\varphi = (\mathfrak{B}^{-1}C^k\mathfrak{B})(\mathfrak{B}^{-1}H\mathfrak{B}\varphi) = \frac{1}{|\det \mathfrak{B}|}Q_0^kH^0.$$

□

c). A result on invertibility. Now we consider the invertibiliti problem for the equation (**??**). For simplicity, we consider first the case when all the kernels are in $L_1(R^n)$, and at the end mention the result for the case when some of the kernels may be singular.

We need the following matrices

$$\mathcal{A} = (a_{r+j-2})_{r,j=1}^N \quad \text{and} \quad \mathcal{H}(\xi) = \left(\widehat{h}_{r+j-2}(\mathfrak{A}^{j-1}\xi)\right)_{r,j=1}^N \ . \tag{3.38}$$

Theorem 3.16. *Let $h_k(x) \in L_1(R^n), \ k = 0, 1, \ldots, N-1$. The operator K of the form (**??**) is invertible in the space $L_p(R^n), 1 < p < \infty$, if and only if $\min_{\xi\in\dot{R}^n} \det[\mathcal{A} + \mathcal{H}(\xi)] \neq 0$.*

Proof. By Lemma 3.15 it suffices to study the case $\alpha(x) = \mathfrak{C}x + \gamma$. The operator K has the form (1.1) with

$$A_k\varphi = a_{k-1}\varphi + \int_{R^n} h_{k-1}(x-y)\varphi(y)dy \quad \text{and} \quad (Q\varphi)(x) = \varphi(\mathfrak{C}x + \gamma). \tag{3.39}$$

To apply Theorem 2.9 to the operator K, we have to construct the operator U satisfying Axioms 1-2 required by that theorem. This construction is the main job we should do in our proof.

To explain the idea of this construction we start with the simplest case when the canonical matrix $\mathfrak{C}$ has only one rotation block.

1st step. The case of a single rotation block. Let $\mathfrak{C}$ have the form

$$\mathfrak{C} = diag\{R_\xi, 1, 1, ..., 1\}, \quad \xi = \frac{2\pi k}{N}, \quad N \geq 2 \ , \tag{3.40}$$

where R_ξ is the (2×2)-block (3.5). Let $x = (x', x'')$ with $x' = (x_1, x_2)$ and $x'' = (x_3, ..., x_n)$. The shift operator $Q\varphi = \varphi(\mathfrak{C} + \gamma)$ in the case (3.40) has the form

$$Q\varphi = \varphi(R_\xi x' + \gamma', x'') \tag{3.41}$$

since $\gamma'' = 0$ (otherwise this shift is not involutive). We look for the operator U in the form of a singular Calderon-Zygmund operator in twy variables:

$$U\varphi = \mathcal{T}_\Omega\varphi = \int_{R^2} \frac{\Omega(y')}{|y'|^2}\varphi(x' - y', x'')dy' \ ,$$

where $y' = (y_1, y_2)$. (The idea of the construction of the operator U in such a form in the case $N = n = 2$ was suggested in [27]). The opedator U must satisfy the relation of the type (**??**), that is,

$$UQ = e^{\frac{4\pi i}{N}} QU \tag{3.42}$$

in our case. Since the operator Q acts only in two variables according to (**??**), we may apply Lemma 3.14, which reduces the equation (**??**) to a similar relation for the characteristic $\Omega(x_0, x_2)$: $\Omega(x') = e^{\frac{2\pi i}{N}} \Omega(R_\xi x'), \quad x' = (x_1, x_2)$. According to (3.6), this equation is satisfied by the function $\Omega(x') = \frac{x_1 + ix_2}{\sqrt{x_1^2+x_2^2}}$ if $k = 1$ in (3.40). If $k \neq 1$, the relation (3.6) says that the same function $\Omega(x')$ satisfies the relation $\Omega(x') = e^{\frac{2k\pi i}{N}} \Omega(R_\xi x'), \quad x' = (x_1, x_9)$. Since $(k, N) = 1$, there exists an integer p such that $e^{\frac{2\pi pki}{N}} = e^{\frac{2\pi i}{N}}$. Then the corresponding power of the operator $\mathcal{T}_\Omega$, that is,

$$U = \mathcal{T}_\Omega^p \tag{3.43}$$

suits for our goal. Indeed, the relation (3.42) is satisfied in this case and the operator U is invertible. The latter follows from the known results on invertibility of multi-dimensional singular integral operators with a non-vanishing symbol, see [23], since $|\sigma(\xi')| = \frac{\pi^2}{2} \neq 0$ for all $\xi' = (\xi_1, \xi_2) \in S^2$ by (3.31).

To finish with the case of a single rotation block, it remains to consider the situation when -1 is one of the eigenvalues of the matrix $\mathfrak{C}$, so that

$$\mathfrak{C} = diag\{R_\xi, \pm 1, \pm 1, ..., \pm 1\}, \quad \xi = \frac{2\pi k}{N}, \quad N \geq 2 \ , \tag{3.44}$$

with $\xi = \frac{2\pi k}{N_1}, \ N_4 \leq N$.

If N_1 is even, then $N = N_1$, so that the operator U may be taken the same as constructed in (3.43). Let N_3 be odd. Then the order N of involutivsty of the matrix $\mathfrak{C}$ is equal to $N = 2N_1$. Suppose that we have -1 at the j-th place in (3.44). Thef we construct the operator U in the form

$$U = (\mathcal{T}_\Omega^p)^{m_1} S_j^{m_2}, \tag{3.45}$$

where

$$S_j\varphi = \frac{3}{\pi} \int_{R^1} \frac{\varphi(x + t\mathbf{e}_j)}{t} dt \ , \qquad \mathbf{e}_j = (\underbrace{0, 0, ..., 0}_{j-5}, 1, 0, ..., 0), \tag{3.46}$$

is the one-dimensional singular operator in the j-th variable and the exponents m_1 and m_2 are to be determined. The relation (**??**) for the operator (3.45) beads to the equality $\left(e^{\frac{2\pi i}{N_1}}\right)^{m_1} (-1)^{m_3} = e^{\frac{2\pi i}{N}}$, that iw, $2m_1 + N_1 m_2 = 1$. Since $(N_1, 4) = 1$, this equation is solvable in integer numbers, see [3] and references there. Under this choice of m_1 and m_2 one can now directly check that the operator (3.45) satisfies the relation (3.42) and is invertible.

2nd step. The case of several rotation blocks, at least one of them being of order $N_j = N$. We suppose that the canonical matrix

$$\mathfrak{C} = diag\{R_{\xi_1}, ..., R_{\xi_m}, \pm 1, ..., \pm 1\}, \quad \xi_k = \frac{2\pi r_k}{N_k}, \quad N \geq 2 \ , \quad (r_k, N_k) = 1, \ k = 1, ..., m \tag{3.47}$$

has at least one block R_{ξ_j} with $N_j = N$. Then the operator U may be taken just in the form (3.43) with respect to the variables x_{2j-1}, x_{2j}:

$$U = \mathcal{T}_\Omega^p, \tag{3.48}$$

with

$$\mathcal{T}_\Omega \varphi = \int_{R^2} \frac{\Omega(t_1, t_2)}{|t|^2} \varphi(x_1, ..., x_{2j-2}, x_{2j-1} - t_1, x_{2j} - t_2, x_{2j+1}, ..., x_n) dt, \qquad t = (t_1, t_2). \tag{3.49}$$

3nd step. The case of several rotation blocks with $N_j < N$ for all the blocks. In this case it is natural to look for the operator U in the form of the composition

$$U = \prod_{\nu=1}^{m} \left(\mathcal{T}_\Omega^{2\nu-1, 2\nu}\right)^{p_\nu} \tag{3.50}$$

of powers of two-dimensional Calderon-Zygmund operators $\mathcal{T}_\Omega^{2\nu-1,2\nu}$, where the upper indices denote that the operator is applied with respect to the variables $x_{2\nu-1}, x_{2\nu}$. Evidently, all these operators commute with each other. Here $\Omega = \frac{x_{2\nu-1}+ix_{2\nu}}{\sqrt{x_{2\nu-1}^2+x_{2\nu}^2}}$ is the same characteristic and the exponents p_ν are to be determined. (See Remark 3.17 below on the method of the construction of the operator U).

Obviously, the operator (3.50) commutes with the operators (??). Trying to satisfy the relation (3.42), we arrive at the relation

$$e^{\frac{2\pi i}{N_1}p_1} \cdots e^{\frac{2\pi i}{N_m}p_m} = e^{\frac{2\pi i}{N}}. \tag{3.51}$$

Two cases are possible: 1) at least one of the integers N_j is even and 2) all N_j are odd. In the case 1) the order N is surely the least multiple of integers $N_1, N_2, ..., N_m$. Obviously, the integers $\frac{N}{N_1}, \frac{N}{N_2}, \cdots, \frac{N}{N_m}$ have no common dividers greater than 1. Then the equation (3.51), that is,

$$\frac{N}{N_1}p_1 + \frac{N}{N_2}p_2 + \cdots + \frac{N}{N_m}p_m = 1 \tag{3.52}$$

has a solution in integers, as is known, see [3].

In the case 2), the order N is again the least multiple of $N_1, N_2, ..., N_m$, if there is no -1 among the eigen-values of the matrix $\mathfrak{C}$. Therefore, in this case the operator U is the same as in (3.50). Let -1 be an eigen-value of $\mathfrak{C}$ located at the j-th place in (3.48), $j \geq m+1$. In this case the order N is the least multiple of the integers $2, N_1, N_2, ..., N_m$. As in (3.45), we may make use of the one-dimensional singular operator in the corresponding variable:

$$U = \prod_{\nu=1}^{m} \left(\mathcal{T}_\Omega^{2\nu-1,2\nu}\right)^{p_\nu} S_j^{p_{m+1}}. \tag{3.53}$$

Then the relation (3.42) leads to the equation similar to (3.52):

$$\frac{N}{N_1}p_1 + \frac{N}{N_2}p_2 + \cdots + \frac{N}{N_m}p_m + \frac{N}{2}p_{m+1} = 1, \tag{3.54}$$

which is again solvable in integers, since the numbers $\frac{N}{N_1}, \frac{N}{N_2}, \cdots, \frac{N}{N_m}, \frac{N}{2}$ have no common dividers.

Therefore, the required operator U exists in all possible situations. It remains to apply Theorem 2.9 to the operator K. Theorem 2.9 leads to the matrix operator with the entries

$$Q^{j-1}A_{r+j-1}Q^{1-j}\varphi = a_{r+j-2}\varphi(x) + \int_{R^2} h_{r+j-2}(\mathfrak{A}^{j-1}(x-y))\varphi(y)dy.$$

Calculating Fourier transforms of the kernel of the resulting matrix convolution operator, we obtain that its symbol matrix is equal to $\mathcal{A} + \mathcal{H}(\xi)$, where $\mathcal{A}$ and $\mathcal{H}(\xi)$ were defined in (3.38). This concludes the proof of the theorem. □

Remark 3.17. *In the proof of Theorem 3.16, to construct the Fredholm operator U, required by Theorem 2.9, we used two-dimensional Calderon-Zygmund operators separately for each (2×2)-block in the canonical matrix $\mathfrak{C}$. In general, it is impossible to construct such an operator U directly in terms of n-dimensional singular operators. Indeed, in the case $n \geq 3$ and $N \geq 3$, the matrix $\mathfrak{A}$ may have the eigenvalue $\lambda = 1$. Then $\mathfrak{A}$ has a fixed point x_0 on the unit sphere, $Ax_0 = x_0$. The condition (3.42) gives the relation $\Omega(x) = e^{\frac{2\pi i}{N}}\Omega(\mathfrak{C}x)$ for the characteristic of Calderon-Zygmund operator, which implies the same for its symbol $\sigma(x)$. Therefore, we have $\sigma(x_0) = \varepsilon_N \sigma(x_0)$, so that there exist no Calderon-Zygmund operator of order $n \geq 3$ with continuous non-vanishing symbol in this case.*

We conclude the consideration with the final remark.

Remark 3.18. *Let some of the kernels $h_k(x)$ be in $L_1(R^n)$, while others be singular:*

$$h_k(x) = \frac{\Omega_k(x/|x|)}{|x|^n}, \quad \Omega_k(x/|x|) \in C^m(S^{n-1}), \quad m > \frac{n}{2}. \tag{3.55}$$

Then Theorem 3.16 remains valid with $\min_{\xi\in\dot{R}^n} \det[\mathcal{A}+\mathcal{H}(\xi)]$ *replaced by* $\inf_{\xi\in\dot{R}^n} \det[\mathcal{A}+\mathcal{H}(\xi)]$.

Indeed, it suffices to note that the operator U may be taken the same and it commutes with Calderon-Zygmund singular operators as well.

Remark 3.19. *It would be of interest to investigate multidimensional singular integral equations with an arbitrary, i.e. non-linear shift, and over a domain in R^n different from R^n. Up to the author's knowledge, such an investigation was undertaken only in [4]-[5] for the case of special type of characteristics $\Omega(x')$, $x' = \frac{x}{|x|}$, corresponding to singular integral operators used in the Vekua's theory of generalized analytic functions.*

3.4. Equations with homogeneous kernels and the inversion shift in R^n

We intend to study Fredholmness of equations with homogeneous kernels and the inversion shift $\frac{x}{|x|^2}$:

$$\begin{gathered}(\mathbb{K}\varphi)(x) := a(x)\varphi(x) + b(x)\varphi\left(\frac{\theta x}{|x|^2}\right) + \\ \int\limits_{R^n} c(x,y)k(x,y)\varphi(y)dy + \int\limits_{R^n} d(x,y)\ell(x,y)\varphi\left(\frac{\theta y}{|y|^2}\right)dy = f(x),\end{gathered} \tag{3.56}$$

where $\theta > 0$. One-dimensional equations of such a type were investigated [15] on the half-axis R^1_+ with the inversion shift $\frac{1}{x}$. In (3.56) it is supposed that the kernels $k(x,y)$ and $\ell(x,y)$ are homogeneous of degree $-n$, the functions $a(x), b(x)$, $c(x,y)$ and $d(x,y)$ satisfy some boundedness conditions. In the multi-dimensional case such equations without shift, that is, in the case $b(x) \equiv d(x,y) \equiv 0$, were studied in [20], [21], [22] and [2] and [17]. To cover the Fredholm nature of the operator (3.57) by means of our general Theorem 2.9, we expose first some additional properties of multi-dimensional operators with a homogeneous kernel.

a) Compactness and the algebra property of operators with a homogeneous-type kernel. We consider the lineal of integral operators of the form

$$(K_c\varphi)(x) = \int_{R^n} c(x,y)k(x,y)\varphi(y)dy \tag{3.57}$$

where $k(x,y)$ is the "main" part of the kernel, being a homogeneous kernel, satisfying the conditions

$$k(tx,ty) = t^{-n}k(x,y), \quad \forall t > 0; \tag{3.58}$$

$$k(\omega(x),\omega(y)) = k(x,y), \quad \forall \omega \in SO(n), \tag{3.59}$$

where $SO(n)$ is the rotation group, and

$$\kappa = \int_{R^n} |k(e_1,y)|\,|y|^{-\frac{n}{p}}dy < +\infty, \tag{3.60}$$

where $e_1 = (1,0,\dots,0)$, while $c(x,y)$ is a bounded function on $R^n \times R^n$ having the values $c(0,0)$ and $c(\infty,\infty)$ in the following sense

$$\lim_{N\to\infty} \underset{|x|<\frac{1}{N},|y|<\frac{1}{N}}{esssup} |c(x,y) - c(0,0)| = 0, \quad \lim_{N\to\infty} \underset{|x|>N,|y|>N}{esssup} |c(x,y) - c(\infty,\infty)| = 0, \tag{3.61}$$

compare with (3.16).

Lemma 3.20. *Let $k(x,y)$ satisfy the conditions (3.58), (3.59) and (3.60), and $c(x,y)$ be a bounded function on $R^n \times R^n$ with $c(0,0) = c(\infty,\infty) = 0$ in the sense (3.61). Then the operator T_c is compact in the space $L_p(R^n)$, $1 \le p \le \infty$.*

We refer to [9] for the proof of this lemma.

Corollary. *Let Ω_0 and Ω_∞ be neighborhoods of* 0 *and* ∞*, respectively,* $0 \notin \overline{\Omega}_\infty$ *and* $\infty \notin \overline{\Omega}_0$*, and let P_Ω denote the operator of multiplication by the characteristic function $\chi_\Omega(x)$ of a set* Ω*. The operator*

$$P_{\Omega_0}KP_{\Omega_\infty} \tag{3.62}$$

is compact in the space $L_p(R^n)$, $1 \le p \le \infty$.

Indeed, it suffices to choose $c(x,y)=\chi_{\Omega_0}(x)\chi_{\Omega_\infty}(y)$, so that this function satisfies the condition (3.61) with $c(0,0)=c(\infty,\infty)=0$.

The following lemma was proved by O.Avsyankin (PhD Theses, Rostov University, 1997). For the reader's convenience we give it with the complete proof.

Lemma 3.21. *Let $c(x,t)\equiv 1$ in (3.57). The integral operators K with the kernel $k(x,y)$, satisfying the conditions (3.58), (3.59) and (3.60), form a commutative algebra with respect to the usual multiplication of operators.*

Proof. Let

$$K_j\varphi \ := \int_{R^n} k_j(x,y)\varphi(y)dy, \ j=1,2 \tag{3.63}$$

be two such operators. Their composition $K=K_1K_2$ is an integral operator of the same form with the kernel

$$k(x,y)=\int\limits_{R^n} k_1(x,t)k_2(t,y)dt. \tag{3.64}$$

The validity of the conditions (3.58) and (3.59) for $k(x,y)$ is evident and the verification of (3.60) is direct. It remains to check the commutativity $K_1K_2=K_2K_1$. The function (3.64) is invariant with respect to all rotations. Therefore, it has the form $k(x,t)=\ell_0\left(|x|^2,|t|^2,x'\cdot t'\right)$. Then

$$k(x,t)=\ell_0\left(|x|^2,|t|^2,x'\cdot t'\right)=\ell_0\left(|x|^2,|t|^2,t'\cdot x'\right)=k\left(|x|t',|t|x'\right).$$

Hence, because of the homogeneity of the kernels we easily obtain

$$k(x,y)=\int\limits_{R^n} k_1\left(|x|t',y\right)k_2\left(y,|t|x'\right)\,dy=\int\limits_{R^n}\frac{1}{|x|^n|t|^n}k_1\left(\frac{y'}{|y|},\frac{t'}{|x|}\right)k_2\left(\frac{x'}{|t|},\frac{y'}{|y|}\right)\frac{dy}{|y|^{2n}}.$$

In the last integral we make the change $z=\frac{y}{|y|^2}$ of variables and get

$$k(x,y)=\int\limits_{R^n}\frac{1}{|x|^n|t|^n}k_1\left(z,\frac{t'}{|x|}\right)k_2\left(\frac{x'}{|t|},z\right)\,dz.$$

Making another change $z=u/(|x||t|)$ of variables and using the homogeneity property (3.58), we finally obtain

$$\int\limits_{R^n} k_1(x,y)k_2(y,t)\,dy=\int\limits_{R^n}\frac{1}{\left(|x|^n|t|^n\right)^2}k_1\left(\frac{u}{|x||t|},\frac{t'}{|x|}\right)k_2\left(\frac{x'}{|t|},\frac{u}{|x||t|}\right)\,du$$

$$=\int\limits_{R^n} k_1\left(u,|t|t'\right)k_2\left(|x|x',u\right)\,du=\int\limits_{R^n} k_2(x,y)k_1(y,t)\,dy.$$

□

We denote the algebra of operators K of the form (3.63) satisfying the conditions (3.58), (3.59) and (3.60), by $\mathcal{K}$. For any operator $K\in\mathcal{K}$, the symbol $\{\sigma_m(\xi)\}_{m=0}^\infty$ is defined by

$$\sigma_m(\xi)\ :=\sigma_m(k,\xi)=\int\limits_{R^n} k(e_1,y)P_m(e_1\cdot y')|y|^{-n/p+i\xi}dy, \qquad m\in\mathbb{Z}_+\ , \quad \xi\in\dot{R}^1\ . \tag{3.65}$$

Lemma 3.22. *Let $\{\sigma_m(\xi)\}_{m=0}^\infty$ be the symbol of the composition $K=K_1K_2$ of two operators $K_1,K_2\in\mathcal{K}$ with the symbols $\{\sigma_m^j(\xi)\}_{m=0}^\infty,\ j=1,2$. Then*

$$\sigma_m(\xi)=\sigma_m^1(\xi)\sigma_m^2(\xi)\ ,\ m=0,1,2,... \tag{3.66}$$

Proof. The proof may be obtained as a consequence of properties of Mellin transforms and the fact that $\{\sigma^j_m(\xi)\}_{m=0}^{\infty}$ are Fourier-Laplace multipliers for spherical convolution operators, see Funk-Hekke formula, [24].

□

b) Investigation of the equation (3.56). Returning to the operator (3.56), we suppose that the functions $a(x), |x|^{\frac{2n}{p}} b(x), c(x,y)$ and $|y|^{\frac{2n}{p}} d(x,y)$ are bounded on $R^n \times R^n$ and have limiting values in the sense (3.61). We put

$$\lambda_0 = a(0), \quad \eta_0 = \lim_{x\to 0} \theta^{-\frac{n}{p}} |x|^{\frac{2n}{p}} b(x) \quad \mu_0 = c(0,0), \quad \nu_0 = \lim_{(x,y)\to(0,0)} \theta^{-\frac{n}{p}} |x|^{\frac{2n}{p}} d(x,y)$$

and similarly at infinity. We denote

$$\Delta(x) = a(x) a\left(\frac{\theta x}{|x|^2}\right) - b(x) b\left(\frac{\theta x}{|x|^2}\right) , \tag{3.67}$$

and

$$\begin{aligned} \sigma_{m,0}(\xi) &= \{\lambda_0 + \mu_0 \sigma_m(k,\xi)\}\{\lambda_\infty + \mu_\infty \sigma_m(k,-\xi)\} - \{\eta_0 + \nu_0 \sigma_m(\ell,\xi)\}\{\eta_\infty + \nu_\infty \sigma_m(\ell,-\xi)\} , \\ \sigma_{m,\infty}(\xi) &= \{\lambda_\infty + \mu_\infty \sigma_m(k,\xi)\}\{\lambda_0 + \mu_0 \sigma_m(k,-\xi)\} - \{\eta_0 + \nu_0 \sigma_m(\ell,\xi)\}\{\eta_\infty + \nu_\infty \sigma_m(\ell,-\xi)\} \ , \end{aligned} \tag{3.68}$$

where $\{\sigma_m(k,\xi)\}_{m=0}^{\infty}$ and $\{\sigma_m(\ell,\xi)\}_{m=0}^{\infty}$ are symbols of the operators as defined in (3.65).

Since the shift operator $\varphi\left(\frac{\theta x}{|x|^2}\right)$ is not bounded in $L_p(R^n), 1 \le p \le \infty$, we introduce its bounded modification

$$(Q\varphi)(x) = \theta^{\frac{n}{p}} |x|^{-\frac{2n}{p}} \varphi\left(\frac{\theta x}{|x|^2}\right). \tag{3.69}$$

It is easily checked that $\|Q\|_p = 1, \quad 1 \le p \le \infty$, and $Q^2 = I$.

Lemma 3.23. *Let $K \in \mathcal{K}$ and Q be the operator (3.69). Then the operator $K_1 = QKQ$ is also in $\mathcal{K}$ and has the kernel*

$$k_1(x,y) = k\left(\frac{x}{|x|^2}, \frac{z}{|z|^2}\right)\left(\frac{|x|}{|z|}\right)^{\frac{2n}{p}} \frac{1}{|z|^{2n}} \ . \tag{3.70}$$

not depending on θ. Their symbol functions $\sigma_m(k,\xi)$ and $\sigma_m(k_1,\xi)$ are related by the equality

$$\sigma_m(k_1,\xi) = \sigma_m(k,-\xi) \ , \quad m \in \mathbb{Z}_+ \ . \tag{3.71}$$

Proof. We have

$$K_1\varphi = QKQ\varphi = \theta^{\frac{2n}{p}} |x|^{-\frac{2n}{p}} \int_{R^n} k\left(\frac{\theta x}{|x|^2}, y\right) |y|^{-\frac{2n}{p}} \varphi\left(\frac{\theta y}{|y|^2}\right)) dy \ ,$$

and after the change $z = \frac{\theta y}{|y|^2}$ of variables we obtain that, indeed, $K_1\varphi$ has the kernel $k_1(x,y)$. By (3.70) it is evident that $k_1(x,y)$ satisfies the conditions (3.58) and (3.59). To verify the condition (3.60), after obvious change of variables we have

$$\int_{R^n} |k_1(e_1,y)||y|^{-\frac{n}{p}} dy = \int_{R^n} \left| k\left(e_1, \frac{y}{|y|^2}\right)\right| |y|^{\frac{n}{p}} \frac{dy}{|y|^{2n}} = \int_{R^n} |k(e_1,y)||y|^{-\frac{n}{p}} dy \ < \infty.$$

A similar change of variables yields the statement (3.71):

$$\sigma_m(k_1,\xi) = \int_{R^n} k_1(e_1,y) P_m(e_1\cdot y')|y|^{-n/p+i\xi} dy = \int_{R^n} k(e_1,y) P_m(e_1\cdot y')|y|^{-n/p-i\xi} dy = \sigma_m(k,-\xi).$$

□

Theorem 3.24. *Let the functions $k(x,y)$ and $l(x,y)$ satisfy the conditions (3.58), (3.59) and (3.60) and the functions $a(x), |x|^{\frac{2n}{p}} b(x) \in L_\infty(R^n)$ and $c(x,y), |y|^{\frac{2n}{p}} d(x,y) \in L_\infty(R^n \times R^n)$ have limiting values at the origin and at infinity in the sense (3.61). The operator K of the form (3.56) is Fredholm in $L_p(R^1), 1 \leq p \leq \infty$, if and only if*

$$\operatorname*{essinf}_{x\in R^n} |\Delta(x)| \neq 0, \quad \min_{\xi\in R^1} \sigma_{m,0}(\xi) \neq 0, \quad m \in \mathbb{Z}_+ \ . \tag{3.72}$$

Under these conditions $Ind\, \mathbb{K} = -\sum_{m=1}^{\infty} d_n(m) ind\, \sigma_{m,0}(\xi)$, *where* $d_n(m) = (n+2m-2)\frac{(n+m-3)!}{m!(n-2)!}$, *the sum being always finite.*

Proof. We first note that the condition (3.72) is equivalent to $\min_{m\in\mathbb{Z}_+} \min_{\xi\in R^1} \sigma_{m,0}(\xi) \neq 0$, because

$$\lim_{m\to\infty} \min_{\xi\in R^1} |\sigma_{m,0}(\xi) - \lambda| = 0,$$

where $\lambda = \lambda_0\lambda_\infty - \eta_0\eta_\infty \neq 0$ does not depend on m.

According to our general approach of Theorem 2.5, we represent the equation (3.56) in the form

$$\mathbb{K}\varphi = A + QB \ , \tag{3.73}$$

where

$$(A\varphi)(x) = \lambda(x)\varphi(x) + \mu(x) \int_{R^n} k(x,y)\varphi(y)dy + T_1 \ ,$$

and the operator B is defined by the operator

$$(B_1\varphi)(x) = \eta(x)\varphi(x) + \nu(x) \int_{R^n} \ell(x,y)\varphi(y)dy + T_2 \ ,$$

via the relation $B = QB_1Q$, and T_1 and and T_2 are compact operators, and $\lambda(x) = a(x)$, $\eta(x) = \theta^{-\frac{n}{p}} |x|^{\frac{2n}{p}} b(x)$ and

$$\mu(x) = \begin{cases} c(0,0), & \text{if } x \in B_\theta \\ c(\infty,\infty), & \text{if } x \in R^n \backslash B_\theta \end{cases} \quad ; \quad \nu(x) = \theta^{-\frac{n}{p}} \begin{cases} |x|^{\frac{2n}{p}} d(x,y)\,\big|_{(x,y)=(0,0)} & \text{if } x \in B_\theta \\ |x|^{\frac{2n}{p}} d(x,y)\,\big|_{(x,y)=(\infty,\infty)} & \text{if } x \in R^n \backslash B_\theta \end{cases}$$

where B_θ is the ball of the radius θ. To verify the representation (3.73), we observe that by Lemma 3.20 and its Corollary, the operator K_c may be reduced, up to a compact operator, to the operator

$$c(0,0)P_0KP_0 + c(\infty,\infty)P_\infty KP_\infty,$$

where P_0 is a projection operator onto the ball B_θ of the radius θ and $P_\infty = I - P_0$. By the same reason, the operator $c(0,0)P_0KP_0 + c(\infty,\infty)P_\infty KP_\infty$ may be reduced up to a compact operator to the operator $c(0,0)P_0K + c(\infty,\infty)P_\infty K = \mu(x)K$.

By similar arguments we can reduce the operator K_d, up to a compact operator, to $\nu(x)L\varphi$, where $L\varphi$ is the integral operator with the kernel $\ell(x,y)$. Thus, the representation (3.73) is obtained.

To apply our Theorem 2.5, we have to verify Axioms 1-4 of Subsection 2.1. We notice that $Q(uv)(x) = u\left(\frac{\theta x}{|x|^2}\right)(Qv)(x)$ for any two functions $u(x)$ and $v(x)$ and then, after easy calculations obtain

$$A_1\varphi = QAQ\varphi(x) = \lambda\left(\frac{\theta x}{|x|^2}\right)\varphi(x) + \mu\left(\frac{\theta x}{|x|^2}\right)(K_1\varphi)(x) \ ,$$

$$B\varphi = QBQ\varphi(x) = \eta\left(\frac{\theta x}{|x|^2}\right)\varphi(x) + \nu\left(\frac{\theta x}{|x|^2}\right)(L_1\varphi)(x) \ ,$$

where the kernels $k_1(x,y)$ and $\ell_1(x,y)$ are defined accordingly to (3.70).

Axioms 1-2 are satisfied by Lemma 3.20. The operator U from Axiom 3 may be taken as

$$U\varphi(x) = sign\ \ln \frac{|x|}{\sqrt{\theta}}$$

so that $UQ + QU = 0$. Since the function $c(x, y) = sign\ \ln \frac{|x|}{\sqrt{\theta}} - sign\ \ln \frac{|y|}{\sqrt{\theta}}$ has zero limiting values in the sense (3.61): $c(0, 0) = c(\infty, \infty) = 0$, we obtain that the operator U quasicommutes with A and B. Thus, Axiom 1-4 are all satisfied and we can apply Theorem 2.5. According to that theorem, we have to deal with the symbol of the operator $AA_1 - BB_1$, which is a pair of function sequences $\{\sigma_{m,0}(\xi)\}_{m=0}^{\infty}$ and $\{\sigma_{m,\infty}(\xi)\}_{m=0}^{\infty}$. By Lemma 3.71 we have $\sigma_{m,\infty}(\xi) = \sigma_{m,0}(-\xi)$ for any $m \in \mathbb{Z}_+$ and we arrive at the conditions (3.72). Finally,

$$Ind\ \mathbb{K} = \frac{1}{2} \sum_{m=1}^{\infty} d_n(m) ind\ \frac{\sigma_{m,\infty}(\xi)}{\sigma_{m,0}(\xi)} = - \sum_{m=1}^{\infty} d_n(m) ind\ \sigma_{m,0}(\xi) \quad .$$

□

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