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Fractional Weyl-Riesz Integrodifferentiation of Periodic Functions of Two Variables via the Periodization of the Riesz Kernel

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Abstract

We consider the periodization of the Riesz fractional integrals (Riesz potentials) of two variables and show that already in this case we come across different effects, depending on whether we use the repeated periodization, first in one variable, and afterwards in another one, or the so called double periodization. We show that the naturally introduced doubly-periodic Weyl-Riesz kernel of order $0 < \alpha < 2$ in general coincides with the periodization of the Riesz kernel, the repeated periodization being possible for all $0 < \alpha < 2$, while the double one is applicable only for $0 < \alpha < 1$. This is obtained as a realization of a certain general scheme of periodization, both repeated and double versions. We prove statements on coincidence of the corresponding periodic and non-periodic convolutions and give an application to the case of the Riesz kernel.

Key Words and Phrases: fractional integration, Weyl integration, periodic convolutions, periodization
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1. Introduction

In the theory of one-dimensional fractional integration it is well known that the periodic fractional integral (Weyl integral) of a 2π-periodic function $f(x)$, generally speaking, coincides with the properly interpreted Liouville fractional integral of f, see [18], Lemma 19.3. In fact, this is nothing else but the statement that the periodic Weyl fractional kernel $\sum_{n=-\infty}^{\infty}{}' \frac{e^{inx}}{(in)^\alpha}$ is the periodization of the Liouville kernel $\frac{x_+^{\alpha-1}}{\Gamma\alpha}$.

The notion of periodization, at least for fractional integration, appears in the paper H. Weyl [19]. A general idea of the periodization of a function given on a real line, is well exposed in the book A.Zygmund [21]. The periodization of functions is well known in harmonic analysis (in particular, in application to sampling of signals), its central point being the Poisson summation formula, see e.g. the books [4] or [2], p. 248-257, on the

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Poisson summation formula, or the papers [3], [5] on relations between this formula and sampling theorems. We mention also the paper [1] devoted to periodization of singular integrals.

In the case of many variables, it is easy to write down the periodization of the mixed (repeated) fractional integration in each variable, when we easily separate variables. For completeness we dwell briefly on this easy case in Subsection 4.6, item **a)**. More difficult is the case of a "real" multidimensional fractional integration, when we cannot separate variables.

There exist many forms of multidimensional fractional integro-differentiation, the reader may be referred to [17], Ch. 9 and [18], Ch.5. In this paper we dwell on the case of the Riesz fractional integrals (Riesz potentials) of two variables and show that already in this case we come across different effects, depending on whether we use the repeated periodization, first in one variable, and afterwards in another one, or the so called double periodization. We show that the naturally introduced doubly-periodic Weyl-Riesz kernel of order $0 < \alpha < 2$ in general coincides with the periodization of the Riesz kernel, the repeated periodization being possible for all $0 < \alpha < 2$, while the double one is applicable only for $0 < \alpha < 1$, see Theorems 4.20 and 4.21. This is obtained as a realization of a certain general scheme of periodization, both repeated or double which is developed in Section 4.

Our interest to the periodization of fractional integrals is stirred up, in particular, by the growing number of applications of fractional calculus, see for example the recent book [8], the survey [11], the papers [6], [9] and [15] and references therein. We mention also the paper [10] in which the Feller semigroups generated by periodic fractional Weyl derivatives were studied.

The presentation is as follows. In Section 2 we give some one-dimensional background on fractional integrals of periodic functions. Section 3 contains a general approach to the periodization of functions of one variable, mainly based on [21], but with some modifications and specifications, and show how it works in case of the one-dimensional Riesz kernel.

The main Section 4 is purely two-dimensional. In Subsections 4.2-4.4 we develop a general approach to the periodic and double periodization itself, keeping in mind applications of this approach to the fractional integration operators. In Subsection 4.5 we prove the main statements on coincidence of the corresponding periodic and non-periodic convolutions. Section 4.6 contains an application of those results to the case of the Riesz kernel.

Section 5 contains some final remarks on possible generalizations.

2. The one-dimensional background: Weyl and Weyl-Riesz periodic fractional integration.

For a 2π-periodic function $f(x), x \in \mathbb{R}^1$ we write

$$f(x) \sim \sum_{n=-\infty}^{\infty} f_n e^{inx}, \qquad f_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\, dx.$$

The Riemann-Liouville fractional integrodifferentiation does not preserve periodicity of functions, as is well known. To keep a function periodic, one should introduce the fractional integration as a periodic convolution

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi(x-t)f(t)dt \tag{2.1}$$

with a suitable periodic kernel $\Psi(x)$ playing the same role as the power function $\frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$ does in the non-periodic case. This was an original idea of Weyl [19] who introduced the fractional integration keeping periodicity via the Fourier series representation

$$W_\pm^\alpha f(x) \sim {\sum_{n=-\infty}^{\infty}}' \frac{f_n}{(\pm in)^\alpha}\, e^{inx} \tag{2.2}$$

the dash indicating that the term $n=0$ is omitted. This definition leads to convolution (2.1) of the form

$$W_\pm^\alpha f(x) = \frac{1}{2\pi}\int_0^{2\pi} \Psi_\pm^\alpha(x-t)f(t)dt, \qquad \alpha>0 \tag{2.3}$$

where

$$\Psi_\pm^\alpha(x) = {\sum_{n=-\infty}^{\infty}}' \frac{e^{inx}}{(\pm in)^\alpha} = 2\sum_{n=1}^{\infty}\frac{\cos\left(nx \mp \frac{\alpha\pi}{2}\right)}{n^\alpha} \tag{2.4}$$

the signs $\pm$ corresponding to the left- and right-hand side forms of fractional integration, see details on both forms in [18], Section 19. Starting from (2.2)-(2.3), Weyl showed that in the case of "nice" functions $f(t)$ this definition coincides with

$$W_\pm^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}f(x\mp t)dt \tag{2.5}$$

which was introduced by J. Liouville [12], p. 8, see also [13] and [14]. However, the integral in (2.5) is not absolutely convergent in case of periodic functions and it is in reality treated as conventionally convergent in a special way:

$$W_\pm^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\lim_{\substack{n\to\infty\\ n\in\mathbb{Z}_+}}\int_0^{2\pi n} t^{\alpha-1}f(x\mp t)dt \tag{2.6}$$

and under the condition that $\int_0^{2\pi} f(x)\,dx = 0$, see details in [18], Subsection 19.2.

It is worth noticing that the non-absolutely convergent Weyl integral (2.5) of a periodic function may be transformed to the following absolutely convergent form

$$W_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty f(x-t)\left\{t^{\alpha-1} - \left(2\pi\left[\frac{t}{2\pi}\right]\right)^{\alpha-1}\right\}dt \tag{2.7}$$

where $\left[\frac{t}{2\pi}\right]$ stands for the entire part of $\frac{t}{2\pi}$, which was observed by M.Mikolas [16], see [18], p. 353.

By the Hurwitz formula [20] for the generalized Riemann function $\zeta(s,a)$, kernel (2.4) may be written in terms of the function $\zeta(s,a)$:

$$\Psi^{\alpha}_{\pm}(t) = \frac{(2\pi)^{\alpha}}{\Gamma(\alpha)}\zeta\left(1-\alpha, \pm\frac{t}{2\pi}\right), \qquad 0 < t < 2\pi. \tag{2.8}$$

It is known that

$$\frac{1}{2\pi}\Psi^{\alpha}_{\pm}(t) = \frac{t_{+}^{\alpha-1}}{\Gamma(\alpha)} + r_{\alpha}(t) \tag{2.9}$$

where the function

$$r_{\alpha}(t) = \frac{1}{\Gamma(\alpha)}\lim_{n\to\infty}\left[\sum_{m=1}^{n}(t+2\pi m)_{+}^{\alpha-1} - \frac{(2\pi)^{\alpha-1}}{\alpha}\, n^{\alpha}\right] \tag{2.10}$$

is infinitely differentiable for $t \in (-2\pi, 2\pi]$ (see [18], p. 349).

The one-dimensional Weyl-Riesz fractional integration of periodic functions is introduced via

$$W^{\alpha}f(x) \sim \sum_{n=-\infty}^{\infty}{}' \frac{f_n}{|n|^{\alpha}}e^{inx} \tag{2.11}$$

so that

$$W^{\alpha}f(x) = \frac{1}{2\cos\frac{\alpha\pi}{a}}\left[W^{\alpha}_{+}f + W^{\alpha}_{-}f\right] = \frac{1}{2\pi}\int_0^{2\pi}\Psi^{\alpha}(t)f(x-t)\,dt \tag{2.12}$$

with

$$\Psi^{\alpha}(t) = \frac{\Psi^{\alpha}_{+}(t) + \Psi^{\alpha}_{-}(t)}{2\cos\frac{\alpha\pi}{2}} = 2\sum_{n=1}^{\infty}\frac{\cos nt}{n^{\alpha}}, \tag{2.13}$$

see [18], Subsection 19.3. By relation (2.8), the function $\Psi(t)$ may be also written in the form

$$\Psi^{\alpha}(t) = \frac{\pi^{\alpha}}{2^{1-\alpha}\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}\left[\zeta\left(1-\alpha, \frac{t}{2\pi}\right) + \zeta\left(1-\alpha, -\frac{t}{2\pi}\right)\right], \qquad 0 < t < 2\pi.$$

3. Periodization in the one-dimensional case.

For a function $k(x)$ defined on $x \in \mathbb{R}^1$ by $\widehat{k}(\xi)$ and $\widetilde{k}(\xi)$ we denote the direct and inverse Fourier transforms:

$$\hat{k}(\xi) = \int_{-\infty}^{\infty} k(x)e^{i\xi x}\,dx, \qquad \tilde{k}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)e^{-i\xi x}\,dx;$$

$\chi_m(x)$ will stand for the characteristic function of the interval $[2\pi m, 2\pi(m+1)]$:

$$\chi_m(x) = \begin{cases} 1, & x \in [2m\pi, 2(m+1)]\pi, \\ 0, & x \notin [2m\pi, 2(m+1)\pi] \end{cases};$$

the means of a function $k(x), x \in \mathbb{R}^1$ over the interval $[2\pi m, 2\pi(m+1)]$ will be denoted as

$$M_m = M_m(k) = \frac{1}{2\pi}\int_{2\pi m}^{2\pi(m+1)} k(x)\,dx, \qquad m = 0, \pm1, \pm2, ... \tag{3.1}$$

3.1 The general scheme.

a). Definitions. Given a function $k(x)$ locally integrable on $\mathbb{R}^1$, by its periodization usually one means a construction of a periodic function $\mathcal{K}(x):\ \mathcal{K}(x+2\pi)=\mathcal{K}(x),\quad x\in\mathbb{R}^1$, related to the function $k(x)$ by the following properties:

A) its Fourier coefficients $\mathcal{K}_m$ coincide with the values of the Fourier transform of the function k at the points $\xi=m=0,\pm 1,\pm 2,...$:

$$\mathcal{K}_m=\widetilde{k}(m), \tag{3.2}$$

(with the convergence of the Fourier integral at infinity specially discussed),
B) there holds the following coincidence of convolutions:

$$\int_0^{2\pi}\mathcal{K}(t)f(x-t)\,dt=\int_{-\infty}^{\infty}k(t)f(x-t)\,dt \tag{3.3}$$

for all 2π-periodic functions $f(x)$.

To this end, we need to say more about the function $k(x)$ than just that it is locally integrable.

Definition 3.1. We say that a locally integrable function $k(x)$ is *admissible* if the function $k_*(x)$ defined by $k_*(x)=k(x)-M_m,\quad x\in[2\pi m,2\pi(m+1)]$, that is, the function

$$k_*(x)=k(x)-\sum_{m=-\infty}^{\infty}M_m\ \chi_m(x),\qquad x\in\mathbb{R}^1 \tag{3.4}$$

belongs to $L^1(\mathbb{R}^1)$. The periodic function

$$\mathcal{K}(x)=\sum_{m=-\infty}^{\infty}k_*(x+2\pi m)=\sum_{m=-\infty}^{\infty}[k(x+2\pi m)-M_m] \tag{3.5}$$

will be referred to as the *periodization* of the function $k_*(x)$ (or of the function $k(x)$.)

In the case of symmetric convergence of the series in (3.5), formula (3.5) is equivalent to

$$\mathcal{K}(x)=\lim_{n\to\infty}\left\{\sum_{m=-n}^{n}k(x+2\pi m)-A_n\right\} \tag{3.6}$$

where

$$A_n=\frac{1}{2\pi}\int_{-2\pi n}^{2\pi(n+1)}k(t)\,dt=\frac{K(2\pi n+2\pi)-K(-2\pi n)}{2\pi}, \tag{3.7}$$

$K(x)$ being a primitive of the function $k(x)$ (compare with (2.9)-(2.10)).

a) Convergence of series (3.5).

Lemma 3.2. *If the kernel $k(x)$ is admissible, then the series defining its periodization $\mathcal{K}(x)$ converges absolutely for almost all x and in the norm of $L^1(0,2\pi)$, and*

$$\int_0^{2\pi}|\mathcal{K}(x)|\,dx\le\sum_{m=-\infty}^{\infty}\int_0^{2\pi}|k_*(x+2\pi m)|\,dx=\int_{-\infty}^{\infty}|k_*(x)|\,dx. \tag{3.8}$$

Proof. The equality in (3.8) is obvious. It is known that convergence of the series $\sum \int_a^b |f_m(t)|\ dt$ implies convergence of the series $\sum |f_m(t)|\ dt$ for almost all $t \in [a, b]$, see [21]. Therefore, from equality in (3.8) there follows the absolute convergence of the series defining the function $\mathcal{K}(x)$. As for the inequality in (3.8), it follows from the Fatou theorem for integrals. From this inequality convergence in L^1-norm is also derived. □

In the following lemma we give a sufficient condition for a function $k(x)$ to be admissible. To this end, we assume that $k(x)$ is differentiable in every interval $2\pi m \le x \le 2\pi(m+1)$ for large $|m| \ge N$, with possible jumps at the points $x = 2\pi m$. Let

$$\beta_m = \max_{2\pi m \le x \le 2\pi(m+1)} |k'(x)|, \qquad |m| \ge N. \tag{3.9}$$

Lemma 3.3. *Let the series* $\sum\limits_{|m|\ge N} \beta_m$ *converge. Then the function* $k_*(x)$ *is admiisible and*

$$\|k_*\|_{L^1} \le 2\int_{-2\pi N}^{2\pi N} |k(x)|\ dx + \frac{4\pi^2}{3} \sum_{|m|\ge N} \beta_m. \tag{3.10}$$

Proof. We have

$$\int_{-\infty}^{\infty} |k_*(x)|\ dx = \sum_{m=-\infty}^{\infty} \int_{2\pi m}^{2\pi(m+1)} |k_*(x)|\ dx = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{2\pi m}^{2\pi(m+1)} \left| \int_{2\pi m}^{2\pi(m+1)} [k(x) - k(t)]\ dt \right| dx.$$

Representing the difference $k(x) - k(t)$ as $k(x) - k(t) = \int_t^x k'(s)\ ds$, we arrive at (3.10) after easy evaluation. □

The following lemma provides an exactification of the convergence statement of Lemma 3.2.

Lemma 3.4. *Under the assumptions of Lemma 3.3*

$$\sum_{m=-\infty}^{\infty} |k_*(x_0+2\pi m)| \le \frac{1}{2\pi} \sum_{|m|\le N-1} |k(x_0+2\pi m)| + \frac{1}{2\pi} \int_{-2\pi N}^{2\pi N} |k(x)|\ dx + 2\pi \sum_{|m|\ge N} \beta_m \tag{3.11}$$

so that series (3.5) defining the periodization of $k(x)$ *converges absolutely at any point* $x_0 \in [0, 2\pi]$ *such that the values* $k(x_0 + 2\pi m)$, $m = 0, \pm 1, \pm 2, .., \pm(N-1)$, *are finite. Besides this,*

$$\int_0^{2\pi} |\mathcal{K}(x)|\ dx \le \int_0^{\infty} |k_*(x)|\ dx. \tag{3.12}$$

Proof. For $x_0 \in [0, 2\pi]$ we have

$$\sum_{m=-\infty}^{\infty} |k_*(x_0 + 2\pi m)| \le \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left| \int_{2\pi m}^{2\pi(m+1)} [k(x_0 + 2\pi m) - k(t)]\ dt \right|$$

from which the inequality (3.11) easily follows. The proof of (3.12) is also direct. □

c). Achieving goals A) and B)

Theorem 3.5. *Let $k(x)$ be an admissible kernel. The Fourier coefficients $\mathcal{K}_m$ coincide with the values of the Fourier transform $\tilde{k}_*$ at integer points:*

$$\mathcal{K}_m = \tilde{k}_*(m), \qquad m = 0, \pm 1, \pm 2, ... \tag{3.13}$$

and $\mathcal{K}_0 = \tilde{k}_*(0) = 0$.

Proof. Indeed,

$$\mathcal{K}_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-imt} \sum_{j=-\infty}^{\infty} k_*(t+2\pi j)\, dt.$$

By Lemma 3.2, the series converges absolutely and we may integrate it term by term after which we easily obtain (3.13). The equality $\mathcal{K}_0 = \tilde{k}_*(0) = 0$ is obvious. □

Theorem 3.6. *If $k_*(x) \in L_1(\mathbb{R}^1)$, then the Fourier transform $\widetilde{k}(\xi)$ of the kernel $k(x)$ exists at the least at integer points $\xi = \pm 1, \pm 2, ...$ in the following sense:*

$$\tilde{k}(m) = \frac{1}{2\pi} \lim_{\substack{n_1\to\infty, n_2\to\infty \\ n_1,n_2\in\mathbb{Z}_+}} \int_{-2\pi n_1}^{2\pi n_2} e^{-imt} k(t)\, dt \tag{3.14}$$

and

$$\tilde{k}(m) = \tilde{k}_*(m), \qquad m = \pm 1, \pm 2, \pm 3, ... \tag{3.15}$$

Proof. Indeed, for $m \neq 0$ we have

$$\int_{-2\pi n_1}^{2\pi n_2} k(t) e^{-imt}\, dt \;=\; \sum_{j=-n_1}^{n_2-1} \int_{2\pi j}^{2\pi j+2\pi} [k(t) - M_j] e^{imt}\, dt \;=\; \int_{-2\pi n_1}^{2\pi n_2} k_*(t) e^{-imt}\, dt.$$

It suffices to refer to the fact that the Fourier transform $\tilde{k}_*(\xi)$ exists in the usual sense. □

Corollary. From Theorems 3.5 and 3.6 it follows that the periodization $\mathcal{K}(x)$ of the kernel $k(x)$ may be represented as

$$\mathcal{K}(x) = \sum_{m=-\infty}^{\infty}{}' \tilde{k}(m) e^{imx}$$

at least in the case when $k_*(x) \in L_1(\mathbb{R}^1)$, the Fourier integrals $\tilde{k}(m)$ being treated in the sense of (3.14).

The following theorem shows that convolution on real line with the kernel $k(x)$ coincides with the periodic convolution with the kernel $\mathcal{K}(x)$, but the former must be treated as a conventionally convergent at infinity in a special way.

Theorem 3.7. *Let $k(x), x \in \mathbb{R}^1$ be an admissible kernel. Then for almost all x*

$$\int_0^{2\pi} \mathcal{K}(t) f(x-t)\, dt = \int_{-\infty}^{\infty} k(t) f(x-t)\, dt \tag{3.16}$$

for any 2π-periodic function $f(x) \in L^1(0,2\pi)$ with $f_0 = 0$, provided that the integral on the right-hand side is interpreted as conventionally convergent at infinity in the following sense:

$$\int_{-\infty}^{\infty} k(t)f(x-t)\,dt = \lim_{\substack{\min(N_1,N_2)\to\infty \\ N_1,N_2\in\mathbb{Z}_+}} \int_{-2\pi N_1}^{2\pi N_2} k(t)f(x-t)\,dt\ . \tag{3.17}$$

The representation by an absolutely convergent integral

$$\int_0^{2\pi} \mathcal{K}(t)f(x-t)\,dt = \int_{-\infty}^{\infty} [k(t) - \mathcal{M}(t)]\,f(x-t)\,dt \tag{3.18}$$

is also valid, where $\mathcal{M}(t)$ is a piece-wise constant function: $\mathcal{M}(t) = \frac{1}{2\pi}\int_{2\pi\left[\frac{t}{2\pi}\right]}^{2\pi\left[\frac{t}{2\pi}\right]+1} k(s)\,ds$.

Proof. The series defining $\mathcal{K}(x)$ converges in L^1-norm, by Lemma 3.2. Therefore, when substituting (3.5) into the left-hand side of (3.16), we may interchange the integration and summation, by Young theorem for convolutions. As a result, we have

$$\int_0^{2\pi} \mathcal{K}(t)f(x-t)\,dt = \lim_{\min(N_1,N_2)\to\infty} \int_0^{2\pi} f(x-t) \sum_{m=-N_1}^{N_2} [k(t+2\pi m) - M_m]\,dt. \tag{3.19}$$

Since $\int_0^{2\pi} f(x)dx = 0$, we obtain

$$\int_0^{2\pi} \mathcal{K}(t)f(x-t)\,dt = \lim_{\min(N_1,N_2)\to\infty} \int_{-2\pi N_1}^{2\pi N_2} f(x-t)k(t)\,dt$$

from which (3.16) follows with the interpretation (3.17) of the integral. To obtain (3.18) from (3.19), it suffices to choose $N_1 = N_2 = n$ and note that $M_m\Big|_{m=\left[\frac{t}{2\pi}\right]} = \mathcal{M}(t)$. □

3.2 The case of the Riesz kernel $k(x) = \frac{|x|^{\alpha-1}}{\gamma_1(\alpha)}$.

As is well known (see, for instance, [17], p.37), the kernel of the multidimensional Riesz potential operator is given by

$$k_\alpha(x) = \frac{|x|^{\alpha-n}}{\gamma_n(\alpha)}, \qquad 0<\alpha<n, \qquad x=(x_1,...,x_n)\in\mathbb{R}^n, \tag{3.20}$$

with the normalizing constant $\gamma_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$.

Let $n=1$ and let

$$\mathcal{K}^\alpha(x) = \frac{1}{\gamma_1(\alpha)} \sum_{m=-\infty}^{\infty} \left(|x+2m\pi|^{\alpha-1} - c_m\right), \qquad \gamma_1(\alpha) = 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}, \tag{3.21}$$

be the periodization of the kernel $k_\alpha(x) = \frac{|x|^{\alpha-1}}{\gamma_1(\alpha)}$ with $0<\alpha<1$, where

$$c_m = \frac{1}{2\pi}\int_{2\pi m}^{2\pi m+2\pi} |t|^{\alpha-1}\,dt = \frac{(2\pi)^{\alpha-1}}{\alpha} \begin{cases} (m+1)^\alpha - m^\alpha, & m\geq 0 \\ |m|^\alpha - (|m|-1)^\alpha, & m<0. \end{cases} \tag{3.22}$$

Periodization (3.21) coincides with the function $\Psi^\alpha(x)$ defined in (2.13), see Theorem 3.9 below.

Lemma 3.8. *The kernel $k_\alpha(x)$, $0 < \alpha < 1$, is admissible (in the sense of Definition 3.1).*

Proof. Indeed, since $\frac{d}{dx}k_\alpha(x) = \frac{\alpha-1}{\gamma_1(\alpha)}|x|^{\alpha-2}$ *sign* x, we have $\beta_m \leq \frac{c}{m^{2-\alpha}}$ with $c > 0$ not depending on m. Therefore, the series $\sum\limits_{|m|\leq 1} \beta_m$ converges and $k_\alpha(x)$ is admissible by Lemma 3.3. □

Theorem 3.9. *Let $0 < \alpha < 1$. The Riesz-Weyl kernel (2.13) coincides with periodization (3.21) of the Riesz kernel up to the constant factor 2π:*

$$\Psi^\alpha(x) \equiv 2\pi\mathcal{K}^\alpha(x) \tag{3.23}$$

and it may be also represented as

$$\Psi^\alpha(x) = \frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}} \lim_{n\to\infty}\left[2\pi\sum_{m=-n}^{n}|x+2m\pi|^{\alpha-1} - \frac{2}{\alpha}(2\pi n)^\alpha\right]. \tag{3.24}$$

Proof. From (3.22) we have $\sum\limits_{m=-n}^{n} c_m = (2\pi)^{\alpha-1}\ \frac{n^\alpha+(n+1)^\alpha}{\alpha}$ so that formula (3.24) follows directly from (3.21), if (3.23) is proved.

To obtain (3.23), we notice that the Fourier coefficients $\mathcal{K}^\alpha_m$ of the function $\mathcal{K}^\alpha_m$ coincide with

$$\widetilde{k}_\alpha(m) = \frac{1}{2\pi}\frac{1}{\gamma_1(\alpha)}\int_{-\infty}^{\infty}\frac{e^{-ixm}}{|x|^{1-\alpha}}\,dx = \frac{1}{2\pi}\frac{1}{|m|^\alpha}, \qquad m = \pm1, \pm2, ...$$

in view of Theorems 3.5 and 3.6, these theorems being applicable since the Riesz kernel $k_\alpha(x)$ is admissible by Lemma 3.8. Then (3.23) follows directly from the definition given in (2.11).

□

Corollary. The Riesz-Weyl kernel $\Psi^\alpha(x)$ and the Riesz kernel $k^\alpha(x)$ differ in $(-2\pi, 2\pi)$ by an infinitely differentiable term.

The following theorem is a corollary to Theorem 3.7 for the case of the Riesz kernel, see an analogous version for the Liouville kernel in [18], p. 353.

Theorem 3.10. *Let $f(x)$ be a 2π-periodic function, $f(x) \in L^1(0, 2\pi)$ and $\int_0^{2\pi} f(x)\,dx = 0$. Then the Weyl-Riesz fractional integral (2.12) of the function f coincides with the Riesz fractional integral:*

$$\frac{1}{2\pi}\int_0^{2\pi}\Psi^\alpha(t)f(x-t)\,dt = \frac{1}{\gamma_1(\alpha)}\int_{-\infty}^{\infty}\frac{f(x-t)\,dt}{|t|^{1-\alpha}}, \qquad 0<\alpha<1, \tag{3.25}$$

provided that the integral on the right-hand side is interpreted as conventionally convergent at infinity as in (3.17). The representation by an absolutely convergent integral

$$\frac{1}{2\pi}\int_0^{2\pi}\Psi^\alpha(t)f(x-t)\,dt=\frac{1}{\gamma_1(\alpha)}\int_{-\infty}^{\infty}f(x-t)\left\{|t|^{\alpha-1}-\left(2\pi\left[\frac{|t|}{2\pi}\right]\right)^{\alpha-1}\right\}\,dt \tag{3.26}$$

is also valid.

P r o o f. It suffices to choose $k(t)=\frac{1}{\gamma_1(\alpha)}|t|^{\alpha-1}$ in the statements of Theorem 3.7. To prove (3.26), the easiest way is to make use of the relation

$$\frac{1}{2\pi}\int_0^{2\pi}\Psi^\alpha(t)f(x-t)\,dt=W_+^\alpha f+W_-^\alpha f,$$

(see (2.12)), take into account that $W_-^\alpha f=QW_+^\alpha Qf$, where $Qf(x)=f(-x)$ and make use of representation (2.7) for $W_+^\alpha f$. □

4. Periodization of functions of two variables.

Let $f(x,y)$ be a doubly 2π-periodic function on $\mathbb{R}^2$: $f(x+2\pi,y)=f(x,y+2\pi)=f(x,y)$, $(x,y)\in\mathbb{R}^2$. Let $S=\{(x,y):0<x<2\pi,0<y<2\pi\}$ and

$$f(x,y)\sim\sum_{m,n}f_{mn}e^{i(mx+ny)}, \tag{4.1}$$

where $\sum\limits_{m,n}=\sum\limits_{(m,n)\in\mathbb{Z}^2}$ and $f_{mn}=\frac{1}{(2\pi)^2}\iint_S f(x,y)e^{i(mx+ny)}\,dxdy$.

For a function $k(x,y)$ defined on $\mathbb{R}^2$ we use the notation

$$\hat{k}(\xi,\eta)=\iint_{\mathbb{R}^2}k(x,y)e^{i(\xi x+\eta y)}\,dxy,\qquad \tilde{k}(\xi,\eta)=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}k(x,y)e^{-i(\xi x+\eta y)}\,dxy;$$

by S_{mn} we denote the square

$$S_{mn}=\{(x,y):2\pi m<x<2\pi(m+1),2\pi n<y<2\pi(n+1)\},\qquad m,n=0,\pm1,\pm2,...$$

so that $S=S_{00}$; by $\chi_{mn}(x,y)=\chi_m(x)\chi_n(y)$ we designate the characteristic function of the square S_{mn}; we shall also need the notation

$$M_{mn}^{12}(k)=\frac{1}{(2\pi)^2}\iint_{S_{mn}}k(x,y)\,dxdy \tag{4.2}$$

for the means of the function $k(x,y)$ over S_{mn}, and

$$M_m^1(k,y)=\frac{1}{2\pi}\int\limits_{2\pi m}^{2\pi(m+1)}k(s,y)\,ds,\qquad M_n^2(k,x)=\frac{1}{2\pi}\int\limits_{2\pi n}^{2\pi(n+1)}k(x,t)\,dt \tag{4.3}$$

for the one-dimensional means over the corresponding intervals.

4.1 Weyl-Riesz fractional integration of periodic functions of two variables.

The Weyl-Riesz fractional integration of periodic functions of two variables may be introduced in a natural way as

$$I^\alpha f(x,y) = \sum_{|m|+|n|\neq 0} \frac{f_{mn}}{(m^2+n^2)^{\frac{\alpha}{2}}}\; e^{i(mx+ny)}. \tag{4.4}$$

This operator has the form

$$I^\alpha f(x,y) = \frac{1}{(2\pi)^2} \iint_S \Psi^\alpha(\xi,\eta) f(x-\xi, y-\eta)\; d\xi d\eta \tag{4.5}$$

where

$$\Psi^\alpha(\xi,\eta) = \sum_{|m|+|n|\neq 0} \frac{e^{i(mx+ny)}}{(m^2+n^2)^{\frac{\alpha}{2}}}. \tag{4.6}$$

Obviously,

$$\Psi^\alpha(\xi,\eta) = \Psi^\alpha(\xi) + \Psi^\alpha(\eta) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\cos m\xi \cos n\eta}{(m^2+n^2)^{\frac{\alpha}{2}}}$$

in notation (2.13).

We will show that operator (4.5) may be obtained as a result of the periodization of the Riesz potential over $\mathbb{R}^2$ of order α. Therefore, we have to study the periodization of the two-dimensional Riesz kernel as defined in (3.20), that is,

$$k_\alpha(x,y) = \frac{(x^2+y^2)^{\frac{\alpha-2}{2}}}{\gamma_2(\alpha)}, \qquad 0<\alpha<2, \qquad (x,y)\in\mathbb{R}^2 \tag{4.7}$$

with $\gamma_2(\alpha) = \frac{2^\alpha \pi \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)} = 2^\alpha \Gamma^2\left(\frac{\alpha}{2}\right) \sin\frac{\alpha\pi}{2}$. As is well known (see e.g. [17], p. 38),

$$\tilde{k}_\alpha(\xi,\eta) = \frac{1}{(2\pi)^2}\frac{1}{(\xi^2+\eta^2)^{\frac{\alpha}{2}}}. \tag{4.8}$$

4.2 On double and repeated periodization of functions of two variables.

Similarly to the case of one variable, by a given function $k(x,y)$ one can organize a doubly periodic function in the form

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k(x+2\pi m, y+2\pi n) \tag{4.9}$$

but before one has to modify $k(x,y)$ in such a way that it would have zero mean value over every square S_{mn}. There are two natural ways to realize this procedure:

i) to subtract directly the mean over S_{mn} from $k(x,y)$ when considered on S_{mn},
ii) to arrange a similar process first with respect to x and afterwards with respect to y.

The approach based on the former way will be referred to as the *double periodization* and in the latter case we shall speak about the *repeated periodization.* Both the approaches have their advantages and disadvantages:

1) In the case of the double periodization, the obtained construction will have mean value zero over the square S_{00}, while in the case of the repeated periodization partial mean values in each variable over $[0,2\pi]$ will be identically equal to zero.

2) In applications to kernels $k(x,y)$ with singularities, in particular to the Riesz kernel (4.7), the double periodization proves to be more restrictive; thus the repeated periodization allows us to consider all the orders $0<\alpha<2$, while the double periodization is possible only for $0<\alpha<1$: when $1\le\alpha<2$, the corresponding series (4.9) diverges for the Riesz kernel in the case of the double periodization.

3) In the case when the kernel $k(x,y)$ has singularities at the lines $x=0$ and $y=0$ (for example, in the case of the mixed fractional order integration of order α in x and of order β in y), the double periodization is not applicable at all.

4) The above arguments are in favor of the repeated periodization. However, there appear arguments in favor of the double periodization when we wish to show that the periodic convolution whose kernel is the corresponding periodization of a kernel $k(x,y)$, is the same as the non-periodic convolution on $\mathbb{R}^2$ with the kernel $k(x,y)$ itself. This coincidence is valid for all periodic functions $f(x,y)$ with $f_{00}=0$ in the case of the double periodization, while in the case of the repeated factorization such a coincidence takes place on functions $f(x,y)$ with a stronger restriction on the Fourier coefficients:

$$f_{m0}=f_{0n}=0, \qquad m=0,\pm1,\pm2,\dots\,, \quad n=0,\pm1,\pm2,\dots \tag{4.10}$$

see Theorem 4.9.

Obviously, to single out the subspace of functions $f(x,y)\in L^1(S_{00})$ which satisfy condition (4.10), means to organize the factor space $L^1(S_{00})/A$ modulo the class A of functions of the form $a(x)+b(y)$.

Lemma 4.1. *Let $f\in L_1(S_{00})$. The condition $f_{m0}=0$ for all $m\in\mathbb{Z}$ is equivalent to the condition*

$$\int_0^{2\pi} f(x,y)\,dy=0 \quad \text{for almost all} \quad x\in[0,2\pi].$$

Similarly,

$$f_{0n}=0,\ n\in\mathbb{Z} \qquad \Leftrightarrow \qquad \int_0^{2\pi} f(x,y)\,dx=0 \quad \text{for almost all}\ \ y\in[0,2\pi].$$

Proof. It suffices to observe that $f_{m0}=\frac{1}{(2\pi)^2}\int_0^{2\pi} e^{imx}g(x)\ dx$ with $g(x)=\int_0^{2\pi} f(x,y)\,dy$ and similarly for f_{0n}. □

4.3 The double periodization of functions of two variables.

a) The function $k_{**}(x,y)$ and its integrability on $\mathbb{R}^2$. Let $k(x,y)$ be a locally integrable function on $\mathbb{R}^2$. Similarly to (3.4) we introduce the function $k_{**}(x,y)$ on the plane $\mathbb{R}^2$ by the formula

$$k_{**}(x,y) = k(x,y) - \sum_{mn} M^{12}_{mn}(k)\ \chi_{mn}(x,y), \qquad (x,y) \in \mathbb{R}^2, \tag{4.11}$$

where $\chi_{mn}(x,y)$ is the characteristic function of the square S_{mn}. Evidently,

$$k_{**}(x,y) = \frac{1}{(2\pi)^2} \iint_{S_{mn}} [k(x,y) - k(s,t)]\, dsdt, \qquad (x,y) \in S_{mn}. \tag{4.12}$$

The obvious formula is valid:

$$\sum_{|j|\le m} \sum_{|\ell|\le n} k_{**}(x+2j\pi, y+2\ell\pi) = \sum_{|j|\le m} \sum_{|\ell|\le n} k(x+2j\pi, y+2\ell\pi) - C_{mn}, \tag{4.13}$$

where $(x,y) \in S_{00}$ and

$$C_{mn} \ = \ \frac{1}{(2\pi)^2} \int_{-2\pi m}^{2\pi(m+1)} \int_{-2\pi n}^{2\pi(n+1)} k(x,y)\, dxdy.$$

It is clear that $k_{**}(x,y) \in L_1(\mathbb{R}^2)$ if $k_(x,y) \in L_1(\mathbb{R}^2)$ and

$$\|k_{**}\|_{L_1(\mathbb{R}^2)} \le 2\|k\|_{L_1(\mathbb{R}^2)}, \tag{4.14}$$

which can be checked directly. However, we are interested in the cases where $k_{**}(x,y) \in L_1(\mathbb{R}^2)$, but a locally integrable function $k(x,y)$ is not necessarily in $L_1(\mathbb{R}^2)$. The possibility for a function $k_{**}(x,y)$ to be integrable over $\mathbb{R}^2$ can be obtained due to local smoothness of the function $k(x,y)$, see Lemma 4.2.

Below we give some conditions on the function $k(x,y)$ sufficient for the function $k_{**}(x,y)$ to be in $L^1(\mathbb{R}^2)$. In those conditions it will be assumed that the function $k(x,y)$ satisfies the following conditions:

1) $k(x,y)$ is integrable on S_{00};

2) $k(x,y)$ is bounded on every square S_{mn} with $|m|+|n| \ne 0$.

Let

$$\beta_{mn} = \sup_{\substack{(x,y)\in S_{mn}\\ (s,t)\in S_{mn}}} |k(x,y) - k(s,t)| \qquad \text{for} \qquad |m|+|n| \ne 0. \tag{4.15}$$

In the case where the function $k(x,y)$ is differentiable in every square S_{mn}, $|m|+|n| \ne 0$, one may take

$$\beta_{mn} = 2\pi \left(\sup_{S_{mn}} |k'_x(x,y)| + \sup_{S_{mn}} |k'_y(x,y)| \right). \tag{4.16}$$

We do not assume the function $k(x,y)$ to be bounded on the square S_{00} in order to be able to admit functions with singularity at the origin (like the Riesz kernel).

Lemma 4.2. *Suppose that the series*

$$\beta : = \sum_{|m|+|n|\ne 0} \beta_{mn} \tag{4.17}$$

converges. Then $k_{**}(x,y) \in L^1(\mathbb{R}^2)$ *and*

$$\|k_{**}\|_{L_1(\mathbb{R}^2)} \le \|k\|_{L_1(S_{00})} + (2\pi)^2\beta. \tag{4.18}$$

Proof. We have

$$\|k_{**}\|_{L_1(\mathbb{R}^2)} = \sum_{mn} \iint_{S_{mn}} |k(x,y) - M_{mn}|\, dxdy$$

$$= \frac{1}{(2\pi)^2} \sum_{mn} \iint_{S_{mn}} \left| \iint_{S_{mn}} |k(x,y) - k(s,t)|\, dsdt \right| dxdy.$$

Passing to constants (4.15) in every term with $|m| + |n| \neq 0$, after easy calculations we arrive at (4.18). □

b) Convergence of the series defining the double periodization. Now by means of the function $k_{**}(x,y)$ we construct the following doubly 2π-periodic function

$$\mathcal{K}(x,y) = \sum_{mn} k_{**}(x + 2\pi m, y + 2\pi n). \tag{4.19}$$

According to (4.13) we may rewrite this series as the limit

$$\mathcal{K}(x,y) = \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\le m} \sum_{|\ell|\le n} k(x + 2\pi j, y + 2\pi \ell) - C_{mn} \right]. \tag{4.20}$$

Lemma 4.3. *Suppose that* $k_{**}(x,y) \in L^1(\mathbb{R}^2)$*. Then series (4.19) converges absolutely for almost all* $(x,y) \in \mathbb{R}^2$ *and may be also represented by (4.20). In the case where* $k(x,y)$ *satisfies the assumptions 1) and 2) and numerical series (4.17) converges, series (4.19) converges absolutely for any point* $(x_0,y_0) \in S_{00}$ *at which the function* $k(x,y)$ *is finite and*

$$\sum_{mn} |k_{**}(x_0 + 2\pi m, y_0 + 2\pi n)| \le |k(x_0,y_0) - M_{00}| + \beta. \tag{4.21}$$

Proof. The almost everywhere convergence is derived by the same arguments as in the proof of Lemma 3.2. Let $0 \le x_0 \le 2\pi,\ 0 \le y_0 \le 2\pi$. Denote $|k(x_0,y_0) - M_{00}| = a$ for brevity. We have

$$\sum_{mn} |k_{**}(x_0 + 2\pi m, y_0 + 2\pi n)| = a + \sum_{|m|+|n|\neq 0} |k(x + 2\pi m, y + 2\pi n) - M_{mn}|$$

$$\le a + \frac{1}{(2\pi)^2} \sum_{|m|+|n|\neq 0} \left| \iint_{S_{mn}} [k(x + 2\pi m, y + 2\pi n) - k(s,t)]\ dsdt \right| \le a + \sum_{mn} \beta_{mn}.$$

□

4.4 The repeated periodization of functions of two variables.

a) The function $k_*^*(x,y)$. Now for a that function $k(x,y)$ locally integrable on $\mathbb{R}^2$, we construct a function $k_*^*(x,y)$ with mean value zero in each variable, first by introducing the function

$$k_*(x,y) = k(x,y) - M_m^1(k,y), \qquad 2\pi m < x < 2\pi(m+1), \quad y \in \mathbb{R}^1, \tag{4.22}$$

and then the function

$$k_*^*(x,y) = k_*(x,y) - M_n^2(k_*,x), \qquad 2\pi n < y < 2\pi(n+1), \quad x \in \mathbb{R}^1 \tag{4.23}$$

where $M_m^1(k,y)$ and $M_n^2(k_*,x)$ are the one-dimensional means, see (4.3). From (4.22) and (4.23) it follows that the function $k_*^*(x,y)$ has the following representation on the whole plane :

$$k_*^*(x,y) = k(x,y) - \sum_{m=-\infty}^{\infty} \chi_m(x) M_m^1(k,y) \tag{4.24}$$

$$- \sum_{n=-\infty}^{\infty} \chi_n(y) M_n^2(k,x) + \sum_{mn} \chi_m(x)\chi_n(y) M_{mn}^{12}(k), \qquad (x,y) \in \mathbb{R}^2,$$

compare with (3.4) and (4.11). It is not hard to see that

$$k_*^*(x,y) = \frac{1}{(2\pi)^2} \iint_{S_{mn}} [k(x,y) - k(s,y) - k(x,t) + k(s,t)]\, dsdt, \quad (x,y) \in S_{mn}. \tag{4.25}$$

The mixed difference obtained in (4.25) has the representation

$$k(x,y) - k(s,y) - k(x,t) + k(s,t) = \int_s^x \int_t^y \frac{\partial^2 k}{\partial \xi \partial \eta}\, d\xi d\eta \tag{4.26}$$

under the assumption that the mixed derivative of $k(x,y)$ exists. Therefore,

$$|k_*^*(x,y)| \le \max_{(x,y)\in S_{mn}} \left| \frac{\partial^2 k}{\partial x \partial y} \right| (2\pi)^2, \qquad (x,y) \in S_{mn}. \tag{4.27}$$

Similarly to (4.14) we have the following statement.

Lemma 4.4. *Let $k(x,y) \in L_1(\mathbb{R}^2)$. Then $k_*^*(x,y) \in L_1(\mathbb{R}^2)$ and $\|k_*^*\|_{L_1(\mathbb{R}^2)} \le 4\|k\|_{L_1(\mathbb{R}^2)}$.*

Proof. We have

$$\|k_*^*\|_{L^1(\mathbb{R}^2)} = \sum_{mn} \iint_{S_{mn}} k_*^*(x,y)\, dxdy.$$

Making use of representation (4.25), we obtain

$$\|k_*^*\|_{L_1(\mathbb{R}^2)} \le \frac{1}{(2\pi)^2} \sum_{mn} \iint_{S_{mn}} dxdy \iint_{S_{mn}} (|k(x,y)| + |k(s,y)| + |k(x,t)| + |k(s,t)|)\, dsdt$$

from which lemma's statement easily follows. □

However, of more importance is derivation of integrability of $k_*^*(x,y)$ from local smoothness of the function $k(x,y)$ in the situation when $k(x,y)$ may be not integrable at infinity. To this end, we introduce the series

$$\mu = \sum_{|m|+|n|\neq 0} \mu_{mn} \quad \text{with} \quad \mu_{mn} = \max_{(x,y)\in S_{mn}} \left| \frac{\partial^2 k}{\partial x \partial y} \right| . \tag{4.28}$$

Lemma 4.5. *Let $k(x,y) \in L^1(S_{00})$ and suppose that series (4.28) converges. Then $k_*^*(x,y) \in L^1(\mathbb{R}^2)$ and*

$$\|k_*^*\|_{L^1(\mathbb{R}^2)} \leq 4\|k\|_{L^1(S_{00})} + (2\pi)^4\mu. \tag{4.29}$$

Proof. We have

$$\|k_*^*\|_{L^1(\mathbb{R}^2)} = \iint_{S_{00}} |k_*^*(x,y)|\, dxdy + \sum_{|m|+|n|\neq 0} \iint_{S_{mn}} |k_*^*(x,y)|\, dxdy.$$

By (4.27) we arrive at (4.29). □

Finally we introduce the repeated periodization of the function $k(x,y)$ as

$$\mathbb{K}(x,y) = \sum_{mn} k_*^*(x + 2\pi m, y + 2\pi n). \tag{4.30}$$

As it follows from (4.24), a relation of type (4.13) in this case has the form

$$\sum_{|j|\leq m} \sum_{|\ell|\leq n} k_*^*(x + 2\pi j, y + 2\pi \ell)$$

$$= \sum_{|j|\leq m} \sum_{|\ell|\leq n} \left[k(x+2\pi j, y+2\pi\ell) - M_j^1(k, y+2\pi\ell) - M_\ell^2(k, x+2\pi j) + M_{j\ell}^{12}(k) \right]$$

for $(x,y) \in S_{00}$, so that the corresponding analogue of (4.20) is

$$\mathbb{K}(x,y) = \lim_{\min(m,n)\to 0} \left[\sum_{|j|\leq m} \sum_{|\ell|\leq n} k(x+2\pi j, y+2\pi\ell) - A_m(y) - B_n(x) + C_{mn} \right] \tag{4.31}$$

for $(x,y) \in S_{00}$, where

$$A_m(y) = \frac{1}{2\pi} \int_{-2\pi m}^{2\pi(m+1)} k(s,y)\, ds \qquad B_n(x) = \frac{1}{2\pi} \int_{-2\pi n}^{2\pi(n+1)} k(x,t)\, dt$$

and C_{mn} is the same as in (4.13).

b) Convergence of the series defining the repeated periodization.

Lemma 4.6. *Let $k(x,y)$ satisfy the assumptions of Lemma 4.5. Then series (4.30) converges absolutely at any point $(x_0,y_0)\in S_{00}$ for which the following values are finite:*

$$k(x_0,y_0),\quad \int_0^{2\pi} k(x_0,t)dt \quad \textit{and} \quad \int_0^{2\pi} k(s,y_0)ds$$

and

$$\sum_{mn}|k_*^*(x_0+2\pi m, y_0+2\pi n)|\le |A(x_0,y_0)|+(2\pi)^2\mu, \tag{4.32}$$

where $A(x_0,y_0)=k(x_0,y_0)-\int_0^{2\pi}k(x_0,t)dt-\int_0^{2\pi}k(s,y_0)ds+M_{00}$.

Proof. Indeed,

$$\sum_{mn}|k_*^*(x+2\pi m,y+2\pi n)|=|k_*^*(x,y)|+\sum_{|m|+|n|\neq 0}|k_*^*(x+2\pi m,y+2\pi n)|$$

$$=|A(x_0,y_0)|+\frac{1}{(2\pi)^2}\sum_{|m|+|n|\neq 0}\iint_{S_{mn}}[k(x,y)-k(s,y)-k(x,t)+k(s,t)]\,dsdt$$

in view of (4.25), whence (4.32) follows according to (4.27). □

4.5 Fourier coefficients of the periodizations $\mathcal{K}(x,y)$ and $\mathbb{K}(x,y)$ and coincidence between the corresponding periodic and non-periodic convolutions.

For a doubly 2π-periodic function $f(x,y)$ we consider the periodic convolutions

$$\mathcal{K}f(x,y)=\iint_{S_{00}}\mathcal{K}(s,t)f(x-s,y-t)\,dsdt\sim\sum_{mn}\mathcal{K}_{mn}f_{mn}e^{i(mx+ny)} \tag{4.33}$$

and

$$\mathbb{K}f(x,y)=\iint_{S_{00}}\mathbb{K}(s,t)f(x-s,y-t)\,dsdt\sim\sum_{mn}\mathbb{K}_{mn}f_{mn}e^{i(mx+ny)} \tag{4.34}$$

whose kernels are the double and repeated periodizations of a given locally integrable kernel $k(x,y)$, as defined in (4.19), (4.30). Theorem 4.8 below shows that they coincide, generally speaking, with the non-periodic convolution on $\mathbb{R}^2$ with the kernel $k(x,y)$.

Lemma 4.7. *Let $k_{**}\in L_1(\mathbb{R}^2)$. The double periodization $\mathcal{K}(x,y)$ has mean value zero:*

$$\iint_{S_{00}}\mathcal{K}(x,y)\,dxdy=0. \tag{4.35}$$

Similarly, when $k_*^*\in L_1(\mathbb{R}^2)$, the repeated periodization $\mathbb{K}(x,y)$ has partial mean values equal to zero:

$$\int_0^{2\pi}\mathbb{K}(s,y)\,ds=\int_0^{2\pi}\mathbb{K}(x,t)\,dt=0 \tag{4.36}$$

for almost all $x, y \in [0, 2\pi]$.

P r o o f. Statement (4.35) follows directly from the definition of the function $\mathcal{K}(x, y)$. In fact, (4.36) also is a consequence of the definition of $\mathbb{K}(x, y)$, but may be also checked directly via representation (4.25). □

Theorem 4.8. *Suppose that $k_{**}(x, y) \in L^1(\mathbb{R}^2)$. The Fourier coefficients $\mathcal{K}_{mn}$ of the periodization (4.19) coincide with the values of the Fourier transforms of the functions k_{**} at the points (m, n):*

$$\mathcal{K}_{mn} = \widetilde{k_{**}}(m, n), \qquad m, n \in \mathbb{Z}\ , \quad |m| + |n| \neq 0, \tag{4.37}$$

and similarly

$$\mathbb{K}_{mn} = \widetilde{k^*_*}(m, n), \qquad m, n \in \mathbb{Z}\ , \qquad m \neq 0, \quad n \neq 0, \tag{4.38}$$

in the case where $k^*_*(x, y) \in L^1(\mathbb{R}^2)$.

Also

$$\mathcal{K}_{mn} = \widetilde{k}(m, n), \qquad |m| + |n| \neq 0 \tag{4.39}$$

and

$$\mathbb{K}_{mn} = \widetilde{k}(m, n), \qquad m \neq 0, \quad n \neq 0, \tag{4.40}$$

where the Fourier transform of the locally integrable function $k(x, y)$ on the right-hand sides of (4.39) and (4.40) exists at the least at integer points (m, n) in the following sense:

$$\tilde{k}(m, n) = \lim_{\substack{\min(m_1, m_2) \to \infty \\ m_1, m_2 \in \mathbb{Z}_+}} \ \lim_{\substack{\min(n_1, n_2) \to \infty \\ n_1, n_2 \in \mathbb{Z}_+}} \int_{-2\pi m_1}^{2\pi m_2} \int_{-2\pi n_1}^{2\pi n_2} k(x, y) e^{-i(mx+ny)}\, dxdy.$$

In the excluded cases we have

$$\mathcal{K}_{00} = 0, \qquad \mathbb{K}_{m0} = \mathbb{K}_{0n} = 0. \tag{4.41}$$

Theorem 4.8 is similar to statements of Theorem 3.5 and Lemma 3.6 and is proved in the same way. Relations (4.40) follow also from (4.35) and (4.36).

Corollary. The double and repeated periodization of the kernel $k(x, y)$ are nothing else but

$$\mathcal{K}(x, y) = \sum_{mn}{}' \tilde{k}(m, n) e^{i(mx+ny)} \quad \text{and} \quad \mathbb{K}(x, y) = \sum_{mn}{}'' \tilde{k}(m, n) e^{i(mx+ny)},$$

where the dash $'$ as usual means omission of the term with $m = n = 0$, while the double dash $''$ means that all the terms with $m = 0$ or $n = 0$ are omitted.

The next theorem generalizes Theorem 3.7.

Theorem 4.9. *I. Suppose that $k_{**}(x, y) \in L^1(\mathbb{R}^2)$. Then a.e. on $\mathbb{R}^2$*

$$\iint_{S_{00}} \mathcal{K}(s, t) f(x - s, y - t)\, dsdt = \iint_{\mathbb{R}^2} k(s, t) f(x - s, y - t)\, dsdt \tag{4.42}$$

for any (2π)-periodic function $f(x,y) \in L^1(S_{00})$ with $f_{00} = 0$ provided that the integral on the right hand side of (4.42) is interpreted as

$$\lim_{\substack{\min(m_1,m_2)\to\infty \\ m_1,m_2\in\mathbb{Z}_+}} \lim_{\substack{\min(n_1,n_2)\to\infty \\ n_1,n_2\in\mathbb{Z}_+}} \int_{-2\pi m_1}^{2\pi m_2} \int_{-2\pi n_1}^{2\pi n_2} k(s,t) f(x-s, y-t)\, dsdt. \tag{4.43}$$

II. Let $k_*^*(x,y) \in L^1(\mathbb{R}^2)$. Then a.e. on $\mathbb{R}^2$

$$\iint_{S_{00}} \mathbb{K}(s,t) f(x-s,y-t)\, dsdt = \iint_{\mathbb{R}^2} k(s,t) f(x-s,y-t)\, dsdt \tag{4.44}$$

for any (2π)-periodic function $f(x,y) \in L^1(S_{00})$ satisfying conditions (4.10) under the same interpretation (4.43) of the integral on the right-hand side.

Proof. The proof is similar to that of Theorem 3.7. For example, representation (4.44) is obtained from the relation

$$\iint_{S_{00}} \mathbb{K}(s,t) f(x-s,y-t)\, dsdt \tag{4.45}$$

$$= \lim_{\min(m,n)\to\infty} \sum_{|j|\le m} \sum_{|\ell|\le n} \iint_{S_{j\ell}} \left[k(s,t) - M_j^1(k,t) - M_\ell^2(k,s) + M_{j\ell}^{12}(k)\right] f(x-s,y-t)\, dsdt$$

if one notices that the terms with $M_j^1(k,t)$, $M_\ell^2(k,t)$ and $M_{j\ell}^{12}(k)$ disappear since the function $f(x,y)$ satisfies conditions (4.10) and, therefore, the corresponding repeated integrals are equal to zero by Lemma 4.1. □

4.6 Periodization of kernels of two-dimensional fractional integration operators.

a) Periodization of the kernel of the mixed fractional integration. This case is not in fact two-dimensional being easily reduced to repeated one-dimensional application of operations. Of much more interest is the periodization of the Riesz kernel to which we pass in the next item, after we dwell briefly on the main points for the mixed fractional integration. A natural definition of the mixed fractional integration of order α in x and of order β in y of doubly periodic functions (4.1) is

$$W^{\alpha,\beta} f(x,y) = \sum_{\substack{mn \\ m\neq 0, n\neq 0}} \frac{f_{mn}}{(im)^\alpha (in)^\beta} e^{i(mx+ny)}, \qquad \alpha > 0,\ \beta > 0. \tag{4.46}$$

It may be written as a periodic convolution

$$W_{++}^{\alpha,\beta} f(x,y) = \frac{1}{(2\pi)^2} \iint_{S_{00}} \Psi_{++}^{\alpha,\beta}(s,t) f(x-s,y-t)\, dsdt \tag{4.47}$$

with the kernel

$$\Psi_{++}^{\alpha,\beta}(x,y) = \sum_{\substack{mn \\ m\neq 0, n\neq 0}} \frac{e^{i(mx+ny)}}{(im)^\alpha (in)^\beta} = 4 \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\cos mx \cos ny}{m^\alpha n^\beta} = \Psi_+^\alpha(x) \Psi_+^\beta(y) \tag{4.48}$$

where $\Psi_+^\alpha(x)$ was defined in (2.4). The kernel $\Psi_{++}^{\alpha,\beta}(x,y)$ may be obtained as a periodization of the kernel $k(x,y) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}\,\frac{y_+^{\beta-1}}{\Gamma(\beta)}$. The double periodization is not applicable in this case (one cannot obtain the convergent series (4.19) just by subtracting only the means over squares as in (4.11)). Under the repeated periodization (4.30), the corresponding one-dimensional means are equal to

$$M_m^1(k,y) = a_m(\alpha)\frac{y^{\beta-1}}{\Gamma(\beta)} \qquad \text{and} \qquad M_n^2(k,x) = a_n(\beta)\frac{x^{\alpha-1}}{\Gamma(\alpha)}$$

with $a_m(\alpha) = \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[(m+1)_+^\alpha - m_+^\alpha\right]$ and $M_{mn}^{12}(k) = a_m(\alpha)a_n(\beta)$.

The terms $A_m(y), B_n(x)$, and C_{mn} from (4.31) are equal to

$$A_m(y) = \frac{(2\pi)^{\alpha-1}}{\Gamma(\alpha+1)}(m+1)^\alpha\,\frac{y_+^{\beta-1}}{\Gamma(\beta)}, \qquad B_n(x) = \frac{(2\pi)^{\beta-1}}{\Gamma(\beta+1)}(n+1)^\beta\,\frac{x_+^{\alpha-1}}{\Gamma(\alpha)} \tag{4.49}$$

and

$$C_{mn} = \frac{(2\pi)^{\alpha+\beta-2}}{\Gamma(\beta+1)\Gamma(\beta+1)}\,(m+1)^\alpha(n+1)^\beta \tag{4.50}$$

for all $m \geq 0, n \geq 0$, being each equal to zero if $m < 0$ or $n < 0$.

Theorem 4.10. *The series defining the repeated periodization $\mathbb{K}(x,y)$ of the kernel $k(x,y) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}\,\frac{y_+^{\beta-1}}{\Gamma(\beta)}$ in the case $0 < \alpha < 1$ and $\beta > 1$ converges for all $(x,y) \in \mathbb{R}^2$ except for the lines $x = 2\pi m$ and $y = 2\pi n$, $m,n \in \mathbb{Z}$ (and uniformly in any square $\{2\pi m + \varepsilon \leq x \leq 2\pi(m+1) - \varepsilon, 2\pi n + \varepsilon \leq y \leq 2\pi(n+1) - \varepsilon\}$) and*

$$\mathbb{K}(x,y) = \frac{1}{(2\pi)^2}\Psi_{++}^{\alpha,\beta}(x,y). \tag{4.51}$$

The formula

$$\frac{1}{(2\pi)^2}\Psi_{++}^{\alpha,\beta}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\lim_{\min(m,n)\to\infty}\left[\sum_{j=0}^{m-1}\sum_{\ell=0}^{n-1}(x+2\pi j)_+^{\alpha-1}(y+2\pi\ell)_+^{\beta-1}\right. \tag{4.52}$$

$$\left. -\ \frac{(2\pi)^{\alpha-1}}{\alpha}m^\alpha y_+^{\beta-1}\ -\ \frac{(2\pi)^{\beta-1}}{\beta}n^\beta x_+^{\alpha-1}\ +\ \frac{(2\pi)^{\alpha+\beta-2}}{\alpha\beta}m^\alpha n^\beta\right], \qquad (x,y) \in S_{00},$$

also holds, and for all doubly 2π-periodic functions $f(x,y) \in L^1(S_{00})$ satisfying condition (4.10), the coincidence

$$W_{++}^{\alpha,\beta}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty \frac{f(x-s,y-t)}{s^{\alpha-1}t^{\beta-1}}\,dsdt \tag{4.53}$$

is valid (under the corresponding interpretation of the integral on the right-hand side).

P r o o f. The convergence of series (4.30) for the kernel $k(x,y) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}\,\frac{y_+^{\beta-1}}{\Gamma(\beta)}$ follows from Lemma 4.6 since $\mu_{mn} = \frac{const}{m^{2-\alpha}n^{2-\beta}}$ and series (4.28) converges. The coincidence (4.51) is a consequence of Theorem 4.8, while (4.53) follows from Theorem 4.9. To obtain (4.52), it suffices to refer to (4.31) and (4.49)-(4.50).

□

b) Periodization of the Riesz kernel $k_\alpha(x,y)$; convergence of the series. First we observe that the means $M^{12}_{mn}(k_\alpha)$ of the Riesz kernel (4.7) over the square S_{mn} have the following asymptotics

$$M^{12}_{mn}(k_\alpha) = k_\alpha(2\pi m, 2\pi n) + o\left(\frac{1}{(|m|+|n|)^{2-\alpha}}\right) = \frac{1}{(m^2+n^2)^{1-\frac{\alpha}{2}}}\left[\frac{(2\pi)^{\alpha-2}}{\gamma_2(\alpha)} + o(1)\right] \tag{4.54}$$

as $|m|+|n| \to \infty$.

Indeed, we have

$$M^{12}_{mn}(k) = \frac{1}{(2\pi)^2}\iint_{S_{mn}} k_\alpha(x,y)\,dxdy = \frac{1}{(2\pi)^{2-\alpha}\gamma_2(\alpha)}\int\limits_m^{m+1} ds \int\limits_n^{n+1} \frac{dt}{(s^2+t^2)^{1-\frac{\alpha}{2}}}$$

and taking into account that $m \le s \le m+1, n \le t \le n+1$, obtain $[(m+1)^2+(n+1)^2]^{\frac{\alpha}{2}-1} \le (2\pi)^{2-\alpha}\gamma_2(\alpha)M^{12}_{mn}(k) \le (m^2+n^2)^{\frac{\alpha}{2}-1}$, from which (4.54) follows.

Let

$$S^\varepsilon_{00} = \{(x,y) \in S_{00}, x^2+y^2 \ge \varepsilon^2\}, \qquad 0<\varepsilon<2\pi.$$

In Lemmas 4.11 and 4.12, by V_ε, $\varepsilon > 0$ we denote the union of arbitrarily small neighborhoods of the vertices $(0,0),(0,2\pi),(2\pi,0),(2\pi,2\pi)$ lying inside the square S_{00}.

Lemma 4.11. *Let $0<\alpha<1$. Series (4.19) defining the double periodization of the Riesz kernel $k_\alpha(x,y)$ converges absolutely for all $(x,y) \in S_{00}$ and uniformly on any truncated square $S_{00}\backslash V_\varepsilon$.*

Proof. The Riesz kernel is integrable on S_{00} and infinitely differentiable outside the origin. Therefore, by Lemma 4.3 it suffices to check that $\beta < \infty$, that is, series (4.17) converges. To make use of (4.16), we observe that $\frac{\partial}{\partial x}k_\alpha(x,y) = \frac{\alpha-2}{\gamma_2(\alpha)}x(x^2+y^2)^{\frac{\alpha}{2}-2}$ and similarly for $\frac{\partial}{\partial y}k_\alpha(x,y)$. Consequently,

$$\beta_{mn} \le const \cdot \frac{\max(m,n)}{(m^2+n^2)^{2-\frac{\alpha}{2}}} \le \frac{c}{(m^2+n^2)^{\frac{3-\alpha}{2}}}.$$

Hence

$$\sum_{|m|+|n|\ne 0} \beta_{mn} \le c \sum_{|m|+|n|\ne 0} \frac{1}{(m+n)^{3-\alpha}}$$

$$= 4c\left(\sum_{n=1}^\infty \frac{1}{n^{3-\alpha}} + \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{(m+n)^{3-\alpha}}\right) \le c_1 + 4c\sum_{m=1}^\infty\sum_{n=m+1}^\infty \frac{1}{n^{3-\alpha}}$$

and it remains to note that $\sum_{n=m+1}^\infty \frac{1}{n^{3-\alpha}} \le \frac{c}{m^{2-\alpha}}$. □

Lemma 4.12. *Let $0<\alpha<2$. Series (4.30) defining the repeated periodization of the Riesz kernel $k_\alpha(x,y)$ converges absolutely for all $(x,y) \in S_{00}$ and uniformly on any* *subsquare* $\{(x,y) : \varepsilon < x < 2\pi - \varepsilon, \varepsilon < y < 2\pi - \varepsilon\}$*; in the case* $\alpha > 1$ *it converges uniformly on the larger set* S_{00}/V_ε.

Proof. The absolute convergence in S_{00} follows from Lemma 4.6. The uniform convergence is easily derived from estimate (4.32) if one shows that $\mu < \infty$. To prove this, we note that $\frac{\partial^2 k_\alpha(x,y)}{\partial x \partial y} = const \frac{xy}{(x^2+y^2)^{3-\frac{\alpha}{2}}}$, so that for series (4.28) we have

$$\mu \leq c \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn}{(m^2+n^2)^{3-\frac{\alpha}{2}}}$$

which converges for all $0 < \alpha < 2$. Indeed,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn}{(m^2+n^2)^{3-\frac{\alpha}{2}}} \leq 2^{3-\frac{\alpha}{2}} \sum_{m=1}^{\infty} m \sum_{n=1}^{\infty} \frac{n}{(m+n)^{6-\alpha}}$$

$$\leq 2^{3-\frac{\alpha}{2}} \sum_{m=1}^{\infty} m \sum_{n=m+1}^{\infty} \frac{n}{n^{5-\alpha}} \leq c_1 \sum_{m=1}^{\infty} \frac{1}{n^{3-\alpha}} < \infty.$$

In the case $\alpha > 1$ it suffices to observe that the function $A(x,y)$ from (4.32) is uniformly bounded on $S_{00} \backslash V_\varepsilon$. □

Let $\mathcal{K}_\alpha(x,y)$ and $\mathbb{K}_\alpha(x,y)$ be the double and repeated periodizations of the Riesz kernel $k_\alpha(x,y)$. According to (4.19) and (4.31), for $(x,y) \in S_{00}$ we have

$$\mathcal{K}_\alpha(x,y) = \frac{1}{\gamma_2(\alpha)} \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\leq m} \sum_{|\ell|\leq n} \frac{1}{[(x+2\pi j)^2 + (y+2\pi \ell)^2]^{1-\frac{\alpha}{2}}} - c_{mn} \right] \quad (4.55)$$

in the case $0 < \alpha < 1$ and

$$\mathbb{K}_\alpha(x,y) = \quad (4.56)$$

$$\frac{1}{\gamma_2(\alpha)} \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\leq m} \sum_{|\ell|\leq n} \frac{1}{[(x+2\pi j)^2 + (y+2\pi \ell)^2]^{1-\frac{\alpha}{2}}} - a_m(y) - a_n(x) + c_{mn} \right]$$

in the case $0 < \alpha < 2$, with

$$a_n(y) = \frac{1}{2\pi} \int_{-2\pi m}^{2\pi(m+1)} \frac{ds}{(s^2+y^2)^{1-\frac{\alpha}{2}}}, \qquad c_{mn} = \frac{1}{(2\pi)^2} \int_{-2\pi m}^{2\pi(m+1)} \int_{-2\pi n}^{2\pi(n+1)} \frac{dsdt}{(s^2+t^2)^{1-\frac{\alpha}{2}}}. \quad (4.57)$$

Below we study the behavior of the terms $a_m(y)$ and c_{mn} when $m,n \to \infty$.

c) Periodization of the Riesz kernel $k_\alpha(x,y)$; asymptotics of the terms c_{mn} and $a_m(y)$ as $m,n \to \infty$.

For the term c_{mn} from (4.57)we start from the following relation

$$c_{mn} = \frac{2^\alpha}{\pi^{2-\alpha}} g_\alpha(m,n) + M^{12}_{mn}(k_\alpha) \quad (4.58)$$

where
$$g_\alpha(m,n) = \int_0^m \int_0^n \frac{dsdt}{(s^2+t^2)^{1-\frac{\alpha}{2}}} \tag{4.59}$$
and $\lim_{|m|+|n|\to\infty} M^{12}_{mn}(k_\alpha) = 0$. The relation (4.58) is obtained directly with $M^{12}_{m,n}(k_\alpha)$ equal to the mean (4.2) corresponding to the kernel $\gamma_2(\alpha)k_\alpha(x,y)$. Its tendency to zero follows from (4.54).

So we have to study the behavior of $g_\alpha(m,n)$ as $|m|+|n|\to\infty$. Obviously, $g_\alpha(m,n)$ is a homogeneous function of m and n of degree α:
$$g_\alpha(m,n) = (m^2+n^2)^{\frac{\alpha}{2}} g_\alpha(\mu,\nu), \qquad \mu = \frac{m}{\sqrt{m^2+n^2}}, \qquad \nu = \frac{n}{\sqrt{m^2+n^2}}, \qquad \mu^2+\nu^2 = 1. \tag{4.60}$$

Lemma 4.13. *$g_\alpha(\mu,\nu)$ satisfies the estimates*
$$\frac{\pi}{2\alpha}[\min(\mu,\nu)]^\alpha \le g_\alpha(\mu,\nu) \le \frac{\pi}{2\alpha} \tag{4.61}$$
and takes its maximal value when $\mu=\nu=\frac{\sqrt{2}}{2}$.

Proof. Obviously,
$$\iint_{D_r} \frac{dxdy}{(x^2+y^2)^{1-\frac{\alpha}{2}}} \le g_\alpha(m,n) \le \iint_{D_R} \frac{dxdy}{(x^2+y^2)^{1-\frac{\alpha}{2}}}$$
where D_r and D_R are the quarters of the circles:
$$D_r = \{(x,y):\ x>0, y>0,\ x^2+y^2<r\}$$
of the radii $r=\min(m,n)$ and $R=\sqrt{m^2+n^2}$. Passing to polar coordinates we easily obtain (4.61).

To find the maximum value, we represent $g_\alpha(\mu,\nu)$ as the function of $\lambda=\mu^2$:
$$g_\alpha(\mu,\nu) = \int_0^\lambda \int_0^{1-\lambda} f(x,y)dxdy := h(\lambda)$$
with $f(x,y) = \frac{1}{4\ \sqrt{xy}(x+y)^{\frac{2-\alpha}{2}}}$. Then easy calculations yield
$$h'(\lambda) = \int_0^{1-\lambda} f(\lambda,x)\, dx + \int_0^\lambda f(1-\lambda,x)\, dx.$$
We have $h'(\frac{1}{2})=0$ and
$$h'(\lambda) = \int_0^\lambda [f(\lambda,x)-f(1-\lambda,x)]\, dx + \int_\lambda^{1-\lambda} f(\lambda,x)\, dx, \qquad 0<\lambda<\frac{1}{2},$$
$$h'(\lambda) = \int_0^{1-\lambda} [f(\lambda,x)-f(1-\lambda,x)]\, dx + \int_{1-\lambda}^\lambda f(\lambda,x)\, dx, \qquad \frac{1}{2}<\lambda<1,$$

so that $h'(\lambda) > 0$ for $0 < \lambda < \frac{1}{2}$ and $h'(\lambda) < 0$ for $\frac{1}{2} < \lambda < 1$, which ends the proof. □

The following lemma gives a precise expression for $g_\alpha(\mu, \nu)$ in terms of the Gauss hypergeometric function

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}(1-sz)^{-a} ds. \quad (4.62)$$

Lemma 4.14. *$g_\alpha(\mu, \nu)$ may be calculated by the formula*

$$g_\alpha(\mu, \nu) = \frac{\nu}{\alpha\mu} F\left(\frac{1+\alpha}{2}, 1; \frac{3}{2}; -\frac{\nu^2}{\mu^2}\right) \; + \; \frac{\mu}{\alpha\nu} F\left(\frac{1+\alpha}{2}, 1; \frac{3}{2}; -\frac{\mu^2}{\nu^2}\right). \quad (4.63)$$

In particular,

$$g_1(\mu, \nu) = \mu \; ln\left(1 + \frac{1}{\mu}\right) + \nu \; ln\left(1 + \frac{1}{\nu}\right) \quad (4.64)$$

in the case $\alpha = 1$.

Proof. Passing to polar coordinates we have

$$g_\alpha(\mu, \nu) = \int_0^\theta d\varphi \int_0^{\frac{\mu}{\cos\varphi}} r^{\alpha-1}\, dr + \int_\theta^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\nu}{\sin\varphi}} r^{\alpha-1}\, dr$$

with $\theta = arctg\frac{\nu}{\mu}$. Hence

$$g_\alpha(\mu, \nu) = \frac{1}{\alpha}\left(\mu^\alpha \int_0^\theta \frac{d\varphi}{\cos^\alpha\varphi} + \nu^\alpha \int_\theta^{\frac{\pi}{2}} \frac{d\varphi}{\sin^\alpha\varphi}\right).$$

The changes $\cos\varphi = \sqrt{t}$ and $\sin\varphi = \sqrt{t}$ in these integrals yield

$$g_\alpha(\mu, \nu) = \frac{1}{2\alpha}\left(\mu^\alpha \int_{\mu^2}^1 \frac{dt}{t^{\frac{1+\alpha}{2}}\sqrt{1-t}} + \nu^\alpha \int_{\nu^2}^1 \frac{dt}{t^{\frac{1+\alpha}{2}}\sqrt{1-t}}\right)$$

which is transformed to (4.63) via the substitutions $t = \mu^2 + \nu^2 s$ in the first integral and $t = \nu^2 + \mu^2 s$ in the second one.

Formula (4.64) for $\alpha = 1$ follows from the known relation

$$F(1, 1; \frac{3}{2}; z) = \frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}}.$$

□

Remark 4.15. Making use of the formula

$$F(a, b; c; -z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{-a} F\left(a, 1-c+a; 1-b+a; -\frac{1}{z}\right) \quad (4.65)$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}z^{-b}F\left(b,1-c+b;1-a+b;-\frac{1}{z}\right)$$

for hypergeometric functions (see [7], formula 9.132.2), we may represent $g_\alpha(\mu,\nu)$ in the case $\alpha \neq 1$ also in the form

$$g_\alpha(\mu,\nu)=\frac{\sqrt{\pi}\Gamma\left(\frac{1-\alpha}{2}\right)}{2\alpha\Gamma\left(1-\frac{\alpha}{2}\right)}\mu^\alpha \tag{4.66}$$

$$+\frac{\mu}{\alpha\nu}\left[F\left(\frac{1+\alpha}{2},1;\frac{3}{2};-\frac{\mu^2}{\nu^2}\right)+\frac{1}{\alpha-1}F\left(1,\frac{1}{2};\frac{3-\alpha}{2};-\frac{\mu^2}{\nu^2}\right)\right].$$

Now we pass to the study of the asymptotics of the term $a_m(y)$ defined in (4.57). First we note that it has the form

$$a_m(y)=h_\alpha(m;y)+o(1) \tag{4.67}$$

as $m\to\infty$, where

$$h_\alpha(m;y)=\frac{1}{\pi}\int_0^{2\pi m}\frac{ds}{(s^2+y^2)^{1-\frac{\alpha}{2}}} \tag{4.68}$$

and $o(1)$ is uniform in y.

Lemma 4.16. *Let $0<\alpha<2$. The integral $h_\alpha(m;y)$ admits the following representation in terms of the Gauss hypergeometric function*

$$h_\alpha(m;y)=2m|y|^{\alpha-2}F\left(\frac{2-\alpha}{2},\frac{1}{2};\frac{3}{2};\ -\left(\frac{2\pi m}{y}\right)^2\right); \tag{4.69}$$

in particular,

$$h_1(m;y)=\frac{1}{\pi}\,ln\,\left(\frac{2\pi m+\sqrt{(2\pi m)^2+y^2}}{|y|}\right) \tag{4.70}$$

when $\alpha=1$. The function $h_\alpha(m;y)$ has the following asymptotics

$$h_\alpha(m;y)=\frac{1}{2\pi}\left[\frac{A}{|y|^{1-\alpha}}+\frac{B}{(2\pi m)^{1-\alpha}}+O\left(\frac{y^2}{m^{3-\alpha}}\right)\right] \tag{4.71}$$

as $m\to\infty$, with $A=\frac{\sqrt{\pi}\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}$ and $B=\frac{\Gamma\left(\frac{\alpha-1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}$ in the case $\alpha\neq 1$ and

$$h_1(m;y)=\frac{1}{\pi}\left[ln\,\frac{1}{|y|}+ln\,(4\pi m)+O\left(\frac{|y|}{m}\right)\right]. \tag{4.72}$$

in the case $\alpha=1$.

Proof. Representation (4.69) follows directly from the integral representation of the hypergeometric function, see (4.62). The particular case (4.70) is obtained from (4.69) by the known formula $F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\ -z^2\right)=\frac{ln(z+\sqrt{z^2+1})}{z}$.

To obtain the asymptotics (4.71), we make use of formula (4.65) and in the case where $\alpha \neq 1$, obtain

$$F\left(\frac{2-\alpha}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{A}{2z} + \frac{B}{2} z^{\alpha-2} F\left(\frac{2-\alpha}{2}, \frac{1-\alpha}{2}; \frac{3-\alpha}{2}; -\frac{1}{z^2}\right).$$

Since $F(a, b; c; \frac{1}{z}) = 1 + \frac{M(z)}{z}$ where $M(z)$ is bounded for $|z| \geq 2$, we obtain

$$F\left(\frac{2-\alpha}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{A}{2z} + \frac{B}{2z^{2-\alpha}} + \frac{C(z)}{z^{4-\alpha}} \tag{4.73}$$

where $\sup\limits_{|z|\geq 2} |C(z)| < \infty$. Then (4.71) follows from (4.69) and (4.73).

Finally, (4.72) is a direct consequence of (4.70) since $ln \left(1 + \sqrt{1 + \left(\frac{y}{2\pi m}\right)^2}\right) = ln\, 2 + O(\frac{|y|}{m})$ as $m \to \infty$.

□

d) Periodization of the Riesz kernel $k_\alpha(x, y)$; the final statements.
Making use of the asymptotics obtained above for the terms c_{mn} and $a_m(\cdot)$ involved in (4.55) and (4.56), we provide an exactification of relations (4.55)-(4.56) in Theorems 4.17 and 4.19 below.

Theorem 4.17. *Let* $0 < \alpha < 1$. *Then*

$$\mathcal{K}_\alpha(x, y) = \frac{1}{\gamma_2(\alpha)} \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\leq m} \sum_{|\ell|\leq n} \frac{1}{[(x + 2\pi j)^2 + (y + 2\pi\ell)^2]^{1-\frac{\alpha}{2}}} - (m^2 + n^2)^{\frac{\alpha}{2}} \Omega\left(\frac{m}{n}\right)\right] \tag{4.74}$$

where $(x, y) \in S_{00}$ *and the "angular coefficient"*

$$\Omega\left(\frac{m}{n}\right) = \frac{2^\alpha}{\alpha\pi^{2-\alpha}} \left[\frac{m}{n} F\left(\frac{1+\alpha}{2}, 1, ; \frac{3}{2}; -\frac{m^2}{n^2}\right) + \frac{n}{m} F\left(\frac{1+\alpha}{2}, 1, ; \frac{3}{2}; -\frac{n^2}{m^2}\right)\right] \tag{4.75}$$

satisfies the estimates

$$\frac{2^{\alpha-1}}{\alpha\pi^{1-\alpha}} \left[\min\left(\frac{m}{\sqrt{m^2+n^2}}, \frac{n}{\sqrt{m^2+n^2}}\right)\right]^\alpha \leq \Omega\left(\frac{m}{n}\right) \leq \frac{2^{\alpha-1}}{\alpha\pi^{1-\alpha}}; \tag{4.76}$$

the limit in (4.74) exists for any $(x, y) \in S_{00}$. *Under the symmetrical* ($m = n$) *passage to the limit one has*

$$\mathcal{K}_\alpha(x, y) = \frac{1}{\gamma_2(\alpha)} \lim_{m\to\infty} \left[\sum_{|j|\leq m} \sum_{|\ell|\leq m} \frac{1}{[(x + 2\pi j)^2 + (y + 2\pi\ell)^2]^{1-\frac{\alpha}{2}}} - \lambda(\alpha) m^\alpha\right] \tag{4.77}$$

with

$$\lambda(\alpha) = 2^{\frac{\alpha}{2}} \Omega(1) = \frac{2^{1+\frac{3\alpha}{2}}}{\alpha\pi^{2-\alpha}} \beta(\alpha), \qquad \beta(\alpha) = F\left(\frac{1+\alpha}{2}, 1, ; \frac{3}{2}; -1\right). \tag{4.78}$$

P r o o f. Relation (4.74)-(4.75) follows from (4.55) in view of (4.58), (4.60) and (4.63). The estimates (4.76) are obtained from (4.61), since $\Omega\left(\frac{m}{n}\right) = \frac{2^\alpha}{\pi^{2-\alpha}} g_\alpha(\mu,\nu)$. □

Remark 4.18. The number $\beta(\alpha)$ from (4.78) satisfies the estimates $\frac{\pi}{4}\cdot\frac{1}{2^{\frac{\alpha}{2}}} \leq \beta(\alpha) \leq \frac{\pi}{4}$ which follows from (4.76), so that

$$\frac{1}{\alpha}(2\pi)^{\alpha-1} \leq \lambda(\alpha) \leq \frac{1}{\alpha}(2\pi)^{\alpha-1}\, 2^{\frac{\alpha}{2}}.$$

Corollary to Theorem 4.17. For $\alpha = 1$, from (4.74) and (4.82) we have

$$\mathcal{K}_1(x,y) = \frac{1}{2\pi} \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\leq m}\sum_{|\ell|\leq n} \frac{1}{\sqrt{(x+2\pi j)^2+(y+2\pi\ell)^2}} \;-\; \sqrt{m^2+n^2}\,\Omega\left(\frac{m}{n}\right)\right] \tag{4.79}$$

$$= \frac{1}{2\pi} \lim_{m\to\infty} \left[\sum_{|j|\leq m}\sum_{|\ell|\leq m} \frac{1}{\sqrt{(x+2\pi j)^2+(y+2\pi\ell)^2}} \;-\; \lambda m\right]$$

with

$$\Omega(t) = \frac{2}{\pi}\frac{(1+t)\, ln\,(1+\sqrt{1+t^2}) - t ln\, t}{\sqrt{1+t^2}} \qquad \text{and} \qquad \lambda = \frac{4}{\pi}\, ln\,(1+\sqrt{2}). \tag{4.80}$$

Similarly, in the case of the repeated periodization, we make use also the asymptotics (4.56), (4.71)-(4.72) for $a_m(y)$ and from (4.56) derive the following statement.

Theorem 4.19. *Let* $0 < \alpha < 2$ *and* $(x,y) \in S_{00}$. *Then for* $\alpha \neq 1$

$$\mathbb{K}_\alpha(x,y) = \frac{1}{\gamma_2(\alpha)} \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\leq m}\sum_{|\ell|\leq n} \frac{1}{[(x+2\pi j)^2+(y+2\pi\ell)^2]^{1-\frac{\alpha}{2}}}\right. \tag{4.81}$$

$$\left. -\frac{A}{2\pi}\left(|x|^{\alpha-1}+|y|^{\alpha-1}\right) - \frac{B}{(2\pi)^{1-\alpha}}\left(m^{\alpha-1}+n^{\alpha-1}\right) + (m^2+n^2)^{\frac{\alpha}{2}}\Omega\left(\frac{m}{n}\right)\right]$$

where $\Omega\left(\frac{m}{n}\right)$ *is the same as in (4.74),* A *and* B *are the constants from (4.71) and the terms with* B *may be omitted in the case* $\alpha < 1$. *In the case* $\alpha = 1$ *one has*

$$\mathbb{K}_1(x,y) = \frac{1}{2\pi} \lim_{\min(m,n)\to\infty} \left[\sum_{|j|\leq m}\sum_{|\ell|\leq n} \frac{1}{\sqrt{(x+2\pi j)^2+(y+2\pi\ell)^2}}\right. \tag{4.82}$$

$$\left. -\frac{1}{\pi}\, ln\,\frac{1}{xy} \;+\; \frac{1}{\pi} ln\,[(4\pi)^2 mn] + \sqrt{m^2+n^2}\;\Omega\left(\frac{m}{n}\right)\right]$$

$$= \frac{1}{2\pi} \lim_{m\to\infty} \left[\sum_{|j|\leq m}\sum_{|\ell|\leq m} \frac{1}{\sqrt{(x+2\pi j)^2+(y+2\pi\ell)^2}} - \frac{1}{\pi}\, ln\,\frac{1}{xy} \;+\; \frac{2}{\pi} ln\,(4\pi m) + \lambda m\right]$$

with $\Omega(t)$ and λ given in (4.80).

Theorem 4.20. *The Weyl-Riesz kernel $\Psi^\alpha(x,y)$ is related to the periodizations $\mathcal{K}^\alpha(x,y)$ and $\mathbb{K}^\alpha(x,y)$ of the Riesz kernel by the formulas*

$$\Psi_\alpha(x,y) = (2\pi)^2\mathcal{K}^\alpha(x,y), \qquad 0<\alpha<1, \tag{4.83}$$

$$\Psi_\alpha(x,y) = \Psi_\alpha(x)+\Psi_\alpha(y)+(2\pi)^2\mathbb{K}^\alpha(x,y), \qquad 0<\alpha<2, \tag{4.84}$$

where $\Psi_\alpha(x)$ is the one-dimensional Weyl-Riesz kernel (2.13).

P r o o f. Relations (4.83)-(4.84) follow from the general Theorem 4.8, see also its Corollary, under the choice $k(x,y)=k_\alpha(x,y)$. In the case $1\le\alpha<2$, when considering (4.84), one should take into account that periodic convolutions of $f(x,y)$ with the one-dimensional terms $\Psi^\alpha(x)$ and $\Psi^\alpha(y)$ are identical zeroes by Lemma 4.1. □

Theorem 4.21. *Let $0<\alpha<1$. Then the Weyl-Riesz fractional integral (4.5) coincides with the Riesz potential:*

$$\frac{1}{(2\pi)^2}\iint_{S_{00}}\Psi_\alpha(\xi,\eta)f(x-\xi,y-\eta)\,d\xi d\eta = \frac{1}{\gamma_2(\alpha)}\iint_{\mathbb{R}^2}\frac{f(x-\xi,y-\eta)}{(\xi^2+\eta^2)^{1-\frac{\alpha}{2}}}\,d\xi d\eta, \tag{4.85}$$

for all doubly periodic functions $f(x,y)\in L_1(S_{00})$ with $f_{00}=0$, the convergence of the integral on the right-hand side being interpreted as in (4.43). Relation (4.85) is also valid for $1\le\alpha<2$, if $f(x,y)$ satisfies additionally conditions (4.10).

P r o o f. Relation (4.85) is a consequence of general Theorem 4.9 and Theorem 4.20. □

5. Final remarks.

Remark 5.1. All the results for the Riesz kernel remain valid for complex values of α ($0<\Re\alpha<2$ for the repeated periodization and $0<\Re\alpha<1$ for the double one) with α replaced by $\Re\alpha$ when we write estimating inequalities in Subsection 4.6. For example, the estimation in (4.61) should run as $\frac{\pi}{2\Re\alpha}[\min(\mu,\nu)]^{\Re\alpha}\le|g_\alpha(\mu,\nu)|\le\frac{\pi}{2\Re\alpha}$ and so on.

Remark 5.2. Instead of 2π-periodicity in each variable, we could deal with the different periodicity in each variable, say with the period T_1 in x and T_2 in y, working with the net of the corresponding rectangles instead of squares. Such a slight generalization is just a matter of easy recalculations.

Remark 5.3. Similar results may be developed for the periodization of two-dimensional Riesz differentiation, formally corresponding to the case of negative α in kernel (4.7). In this case the kernel $k_{-\alpha}(x,y), 0<\alpha<2$, has a non-integrable singularity at the origin, so the general approach presented in Subsections 4.2-4.5 for locally integrable kernels, is not applicable. However, we can arrange a similar process within the framework of distributions and it is always possible to consider the pointwise convergence of the series defining the periodization beyond the points $(2\pi m, 2\pi n),\ m, n \in \mathbb{Z}$. When (x, y) lies far away from those points, this series converges faster for $\alpha < 0$ than for $\alpha > 0$ because of nicer behavior of the kernel at infinity. But, of course we have troubles with the behavior of the series when (x, y) may reach those points.

Remark 5.4. Of a special interest is to consider periodization of other forms of fractional integration of many variables. We dealt with the fractional power of the Laplace operators. One could proceed in the same way for fractional powers of other differential operators in partial derivatives, for example, wave or heat operators, Schrödinger operators and others, see [17], Ch. 9, for the space versions of these fractional powers.

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