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On Sobolev theorem for Riesz type potentials in the Lebesgue spaces with Variable Exponent

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Abstract

The Riesz potential operator of variable order $\alpha(x)$ is shown to be bounded from the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(x)$ into the weighted space $L^{q(\cdot)}_\rho(\mathbb{R}^n)$, where $\rho = (1+|x|)^{-\gamma}$ with some $\gamma > 0$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$ when $p(x)$ is not necessarily constant at infinity. It is assumed that the exponent $p(x)$ satisfies the logarithmic continuity condition both locally and at infinity and $1 < p(\infty) \le p(x) \le P < \infty,\ x \in \mathbb{R}^n$.

Key words: *variable exponent, Lebesgue spaces, Riesz potential, maximal function, weighted estimates*

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1 Introduction

We consider the Riesz potential operator

$$I^{\alpha(\cdot)}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\, dy \tag{1.1}$$

in the Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with the variable exponent $p(x)$. We refer for instance to the papers [20], [15], [19], [18] for the spaces $L^{p(\cdot)}$.

(The order $\alpha(x)$ of the potential operator is also assumed to be variable.) Nowadays there is an evident increase of investigations related to both the theory of the spaces $L^{p(\cdot)}(\Omega)$ themselves and the operator theory in these spaces. This is caused by possible applications to models with non-standard local growth (in elasticity theory, fluid mechanics, differential equations, see for example [16], [6] and references therein) and is based on recent breakthrough result on boundedness of the Hardy-Littlewood maximal operator in these spaces. We refer, for example, to the papers [2], [3], [4], [5], [6],[7], [9], [10], [11], [12], [13], [14], see also references therein.

The boundedness of the operator $I^{\alpha(\cdot)}$ from the space $L^{p(\cdot)}(\mathbb{R}^n)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$ with the limiting Sobolev exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \tag{1.2}$$

was an open problem for a long time. It was solved in the case of bounded domains. First, in [17], in the case of bounded domains Ω there was proved a conditional result: the Sobolev theorem is valid for the potential operator $I^{\alpha(\cdot)}$ within the framework of the spaces $L^{p(\cdot)}(\Omega)$ with $p(x)$ satisfying the logarithmic Dini condition, if the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$. After L.Diening [3], [5] proved the boundedness of the maximal operator, the validity of the Sobolev theorem for bounded domains became an unconditional statement.

We refer also to the paper D.E.Edmunds and A.Meskhi [8] where some weighted statements on $L^{p(\cdot)} - L^{p(\cdot)}$- boundedness for the one-dimensional fractional integrals were obtained.

This problem still remains open for unbounded domains in the general case.

Recently, L. Diening [4] proved Sobolev's theorem for the potential I^{α} on the whole space $\mathbb{R}^n$ assuming that $p(x)$ is constant at infinity ($p(x) \equiv const$ outside some large ball) and satisfies the same logarithmic condition as in [17]. Another progress for unbounded domains is the recent result of D.Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer [2] on the boundedness of the maximal operator in unbounded domains for exponents $p(x)$ satisfying the logarithmic smoothness condition both locally and at infinity.

In this paper we prove Sobolev-type theorem for the potential $I^{\alpha(\cdot)}$ from the space $L^{p(\cdot)}(\mathbb{R}^n)$ into the weighted space $L^{q(\cdot)}_{\rho}(\mathbb{R}^n)$ with the power weight ρ fixed to infinity, under the logarithmic condition for $p(x)$ satisfied locally and at infinity, not supposing that $p(x)$ is constant at infinity but assuming that

$1 < p(\infty) \leq p(x) \leq P < \infty$ (Theorem **A**). The crucial points of the proof are the usage of the above mentioned result on maximal functions obtained in [2] and the estimates for $\| \, |x - x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))}$ as $r \to 0$ and $r \to \infty$ obtained in [17], see Propositions 1 and 2 in Section 3.

N o t a t i o n :
$\chi_\Omega(x)$ is the characteristic function of a set Ω in $\mathbb{R}^n$;
$|\Omega|$ is the Lebesgue measure of Ω;
$B(x_0, r)$ is the ball centered at x_0 and of radius r, $|B_n| = |B(0,1)|$;
$p(x) : \mathbb{R}^n \longrightarrow [1, \infty)$ is a measurable function, $p_0 = \inf\limits_{x \in \mathbb{R}^n} p(x)$, $P = \sup\limits_{x \in \mathbb{R}^n} p(x)$; everywhere inf and sup stand for *ess inf* and *ess sup*.

2 Statement of the main result.

By $L^{p(\cdot)}$ we denote the space of functions $f(x)$ on Ω such that

$$A_p(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty,$$

where $p(x)$ is a measurable function on $\mathbb{R}^n$ with values in $[1, \infty)$ and

$$1 \leq p_0 \leq p(x) \leq P < \infty, \quad x \in \mathbb{R}^n. \tag{2.1}$$

This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : A_p \left(\frac{f}{\lambda} \right) \leq 1 \right\} \tag{2.2}$$

(see e.g. [15]).

We assume that the exponent $p(x)$ satisfies the condition

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n; \tag{2.3}$$

we shall also use the assumption, introduced in [18], Definitions 3.2-3.3, that there exists $p(\infty) = \lim\limits_{|x| \to \infty} p(x)$ and

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{ln(e + |x|)}, \qquad x \in \mathbb{R}^n. \tag{2.4}$$

Note that (2.4) is equivalent to the condition

$$|p(x) - p(y)| \leq \frac{C}{ln[e + \min(|x|, |y|)]} \tag{2.5}$$

introduced by D. Cruz-Uribe, A. Fiorenza and C.J.Neugebauer [2] to treat the maximal functions in spaces with variable exponent on $\mathbb{R}^n$.

Condition (2.4) is obviously fulfilled for functions $p(x)$ satisfying the Hölder condition at infinity:

$$|p(x) - p(\infty)| \leq \frac{C}{(1 + |x|)^\lambda}, \qquad 0 < \lambda \leq 1, \qquad x \in \mathbb{R}^n. \tag{2.6}$$

The order $\alpha(x)$ of the Riesz potential operator is not supposed to be continuous. We assume that it is a measurable function on $\mathbb{R}^n$ satisfying the following assumptions

$$\alpha_0 := \inf_{x \in \mathbb{R}^n} \alpha(x) > 0, \tag{2.7}$$

and

$$\sup_{x \in \mathbb{R}^n} p(x)\alpha(x) < n, \qquad \sup_{x \in \mathbb{R}^n} p(\infty)\alpha(x) < n. \tag{2.8}$$

Theorem A. *Let assumptions (2.3), (2.4), (2.7) and (2.8) be satisfied and let*

$$1 < p(\infty) \leq p(x) \leq P < \infty. \tag{2.9}$$

Then the following weighted Sobolev-type estimate is valid for the operator $I^{\alpha(\cdot)}$:

$$\left\| (1 + |x|)^{-\gamma(x)} I^{\alpha(\cdot)} f \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c \, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{2.10}$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \tag{2.11}$$

is the Sobolev exponent and

$$\gamma(x) = A_\infty \alpha(x) \left[1 - \frac{\alpha(x)}{n} \right] \leq \frac{n}{4} A_\infty, \tag{2.12}$$

A_∞ *being the Dini-Lipschitz constant from (2.4).*

Corollary. Under the assumptions of Theorem A, estimate (2.11) is valid also for the fractional maximal operator

$$M^{\alpha(\cdot)}f(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{n-\alpha(x)}}\int\limits_{B(x,r)}|f(y)|\,dy.$$

Remark. 1. *If $\alpha(x)$ satisfies the condition of type (2.4): $|\alpha(x)-\alpha(\infty)|\le\frac{C}{ln(e+|x|)},\quad x\in\mathbb{R}^n$, then the weight $(1+|x|)^{-\gamma(x)}$ is equivalent to the weight $(1+|x|)^{-\gamma(\infty)}$.*

2. *One can also treat operator (1.1) with $\alpha(x)$ replaced by $\alpha(y)$. In the case of potentials over bounded domains Ω such potentials differ unessentially, if the function $\alpha(x)$ satisfies the smoothness logarithmic condition as in (2.3), since*

$$c_1|x-y|^{n-\alpha(y)}\le|x-y|^{n-\alpha(x)}\le c_2|x-y|^{n-\alpha(y)}$$

in this case, see [17], p.277.

3 Preliminaries

3.1 Estimates of $L^{p(\cdot)}$-norms of powers of distance truncated to exterior of a ball.

In this subsection we reproduce some results from [18]-[17] with slight modifications.

Let $\beta(x)$ be a function on $\mathbb{R}^n$ and $x_0\in\mathbb{R}^n$ and consider

$$\mu_\beta=\mu_\beta(x_0,r)=\|\;|x-x_0|^{\beta(x_0)}\|_{L^{p(\cdot)}(\mathbb{R}^n\setminus B(x_0,r))}\tag{3.1}$$

so that

$$\int_{|y|\ge r}\left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(x_0+y)}dy=1\tag{3.2}$$

by the definition of the norm in (2.2).

Lemma 3.1. *The function $\mu_\beta(x_0,r)$ is decreasing in r. If conditions (2.1), (2.3) are satisfied and $n+\beta(x_0)p(x_0)\le 0$, then $\lim_{r\to 0}\mu_\beta(x_0,r)=\infty$.*

Proof. The proof is straightforward. □

In [17]-[18] the estimation of $\mu_\beta(x_0, r)$ as $r \to 0$ and $r \to \infty$ was obtained under the following assumptions

$$B := \sup_{x \in \mathbb{R}^n} |\beta(x)| < \infty, \tag{3.3}$$

$$-d_1 := \sup_{x \in \mathbb{R}^n} [n + \beta(x)p(x)] < 0, \tag{3.4}$$

$$-d_2 := \sup_{x \in \mathbb{R}^n} [n + \beta(x)p(\infty)] < 0. \tag{3.5}$$

a). The "norming" value r_0. To reproduce the estimates for $\mu_\beta(x_0, r)$ and distinguish between "small" values $0 < r < r_0$ and "large" values of $r > r_0$, we need the number

$$r_0 = r_0(x_0) \qquad \text{for which} \qquad \mu_\beta(x_0, r_0) = 1.$$

This number is the root of the equation

$$\int\limits_{|x| > r_0} |x|^{\beta(x_0)p(x+x_0)}\, dx = 1;$$

a positive root of this equation certainly exists for $p(x)$ satisfying (2.3), if

$$n + \beta(x_0)p(x_0) \leq 0, \qquad n + \beta(x_0)p(\infty) < 0,$$

see [17], Lemma 1.3.

Lemma 3.2. *(**[17], Lemmas 1.4 and 1.5**). The number r_0 as function of x_0 is bounded from above and below:*

$$0 < c_1 \leq r_0(x_0) \leq c_2 < \infty \tag{3.6}$$

where c_1 and c_2 are constants not depending on x_0, if assumptions (2.1), (2.3), (3.3)-(3.5) are satisfied and there exists the limit $p(\infty) = \lim\limits_{|x| \to \infty} p(x)$.

b). Estimates for $\mu_\beta(x_0, r)$ as $r \to 0$ and $r \to \infty$. In [17] the following statements were proved.

PROPOSITION 1 (**an estimate as** $r \to 0$**, [17], Th.1.8**). *Let* $p(x)$ *and* $\beta(x)$ *satisfy assumptions (2.1), (2.3) and (3.3)-(3.5). Then*

$$\| \, |x-x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))} \le C r^{\beta(x_0)+\frac{n}{p(x_0)}}, \qquad 0 < r \le r_0, \tag{3.7}$$

where $C > 0$ *does not depend on* r *and* x_0*.*

PROPOSITION 2 (**an estimate as** $r \to \infty$**, [17], Th.1.10**). *Let* $p(x)$ *and* $\beta(x)$ *satisfy assumptions (2.1), (2.3) and (3.3)-(3.5). Then*

$$\frac{C_1}{K(x_0)} r^{\beta(x_0)+\frac{n}{p(\infty)}} \ \le \ \| \, |x-x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))} \ \le \ C_2 K(x_0) r^{\beta(x_0)+\frac{n}{p(\infty)}} \ , \tag{3.8}$$

for large r $\left(r \ge \max \left\{ 2^{\frac{1}{n}}, \frac{1}{|B_n|^{\frac{1}{n}}}, r_0 \right\} \right)$*, where* C_1 *and* C_2 *do not depend on* r *and* x_0*, while*

$$K(x_0) = (1+|x_0|)^{\frac{A_\infty |\beta(x_0)|}{p(\infty)}},$$

A_∞ *being the Dini-Lipschitz constant from (2.4); in the case where* $p(x) \ge p(\infty)$ *one may take* $K(x_0) \equiv 1$ *in (3.8).*

We shall prove Proposition 2 in the next section, since in [17] it was proved with the worse exponent for the factor $K(x_0)$.

Lemma 3.3. *Under the assumptions of Lemma 3.2, there exist absolute constants* $0 < c_1 < c_2 < \infty$ *not dependent on* x_0 *such that*

$$\mu_\beta(x_0,r) \le 1 \qquad \text{for} \qquad r \ge c_2 \tag{3.9}$$

and

$$\mu_\beta(x_0,r) \ge 1 \qquad \text{for} \qquad r \le c_1 \tag{3.10}$$

uniformly in x_0*, and* $\mu_\beta(x_0,r)$ *is uniformly bounded from above and below for* $c_1 \le r \le c_2$*:*

$$0 < m_1 \le \mu_\beta(x_0,r) \le m_2 < \infty \qquad \text{for} \qquad c_1 \le r \le c_2 \tag{3.11}$$

with m_1 *and* m_2 *not depending on* x_0*.*

Proof. Statements (3.9)- (3.10) follow immediately from (3.6) with the same constants c_1 and c_2. The bounds (3.11) are obtained from (3.2) by easy estimations. □

Corollary (to Propositions 1 and 2). *Let $p(x)$ and $\beta(x)$ satisfy assumptions (2.1), (2.3) and (3.3)-(3.5). Then*

$$\| \, |x-x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))} \le C r^{\beta(x_0)+\frac{n}{p(x_0)}}, \qquad 0 < r \le 1, \tag{3.12}$$

and

$$\| \, |x-x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))} \le C K(x_0) r^{\beta(x_0)+\frac{n}{p(\infty)}}, \qquad r \ge 1, \tag{3.13}$$

where $C > 0$ is an absolute constant nor depending on r and x_0. The estimate (3.12) is valid for all $0 < r < \infty$, if $p(x) \le p(\infty)$, $x \in \mathbb{R}^n$.

Proof. Corollary follows directly from Propositions 1 and 2 in view of Lemma 3.2. □

3.2 Boundedness of the maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$.

The boundedness of the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy, \tag{3.14}$$

was proved by L.Diening [3] and [5] for bounded domains, and also for $\mathbb{R}^n$ but in the case when $p(x)$ is constant at infinity (that is, outside some large ball). Recently, D. Cruz-Uribe, A. Fiorenza and C.J.Neugebauer [2] proved the boundedness of the maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$ under condition (2.4) on the behaviour of $p(x)$ at infinity. We shall use that result which runs as follows.

PROPOSITION 3 (**boundedness of the maximal operator, [2], Th.1.4**). *Let Ω be an arbitrary open set in $\mathbb{R}^n$ and let $p : \Omega \longrightarrow [1,\infty)$ satisfy the condition $1 < p_0 \le p(x) \le P < \infty$, $x \in \Omega$ and conditions (2.3) and (2.4) on Ω. Then the maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.*

4 Proof of the main result

a). A rough estimate of $\mu_\beta(x_0, r)$ from below.

We make use of the following rough estimate of $\mu_\beta = \mu_\beta(x_0, r)$ from below:

$$\mu_\beta(x_0, r) \geq 2^{-\frac{B}{n}} r^{\beta(x_0)} \qquad \text{for} \qquad r \geq |B_n|^{-\frac{1}{n}}, \tag{4.1}$$

(see [17], Lemma 1.9). Its proof is straightforwardly derived from (3.2):

$$1 \geq \int\limits_{r<|y|<\mu_\beta^{\frac{1}{\beta}}} \left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(x_0+y)} dy \geq \int\limits_{r<|y|<\mu_\beta^{\frac{1}{\beta}}} dy = |B_n| \left(\mu_\beta^{\frac{n}{\beta}} - r^n\right)$$

from which (4.1) easily follows (in the above estimates we assumed that $\mu_\beta \leq r^\beta$, since in the contrary case there is nothing to prove).

b). Proof of Proposition 2. We rewrite relation (3.2) as

$$\int\limits_{|y|>r} \left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(\infty)} \omega_r(y, x_0)\, dy = 1, \tag{4.2}$$

where

$$\omega_r(y, x_0) = \omega_r(y, x_0) = \left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(x_0+y)-p(\infty)}.$$

To derive estimates (3.8) from (4.2), we need the following lemma.

Lemma 4.1. *Let $p(x) : \mathbb{R}^n \longrightarrow [1, \infty)$ and $\beta(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^1$ be bounded functions satisfying conditions (2.1), (2.3), (2.4), and (3.3-(3.5). Then*

$$\frac{1}{c}(1 + |x_0|)^{-A_\infty|\beta(x_0)|} \leq \omega_r(y, x_0) \leq c(1 + |x_0|)^{A_\infty|\beta(x_0)|}, \qquad x_0 \in \mathbb{R}^n \tag{4.3}$$

for all $r \geq \max\left(c_2, \frac{1}{|B_n|^{\frac{1}{n}}}\right)$, where $c > 0$ does not depend on r and x_0.

Proof. We have

$$\omega_r(y, x_0) \leq 2^{\frac{B(P-p_0)}{n}} \left(\frac{|y|^{\beta(x_0)}}{2^{\frac{B}{n}} \mu_\beta}\right)^{p(y+x_0)-p(\infty)}$$

where $\frac{|y|^{\beta(x_0)}}{2^{\frac{B}{n}} \mu_\beta} \leq 1$ by (4.1). Therefore,

$$ln\ \omega_r(y, x_0) \leq \ln\ C \ + \ [p(y + x_0) - p(\infty)] \cdot ln\ \left(\frac{|y|^{\beta(x_0)}}{2^{\frac{B}{n}} \mu_\beta}\right)$$

$$= \ln\ C\ +\ |p(y+x_0)-p(\infty)|\cdot ln\ \frac{2^{\frac{B}{n}}\mu_\beta}{|y|^{\beta(x_0)}}$$

with $C = 2^{\frac{B(P-p_0)}{n}}$. Since $\beta(x_0) < 0$ by (3.5), we have

$$ln\ \omega_r(y,x_0) \le \ln\ C\ +\ |p(y+x_0)-p(\infty)|\left[\frac{B}{n} ln\ 2 + |\beta(x_0)| ln\ |y| + ln\ \mu_\beta\right].$$

We observe that $\mu_\beta \le 1$ for $r \ge c_2$ by Lemma 3.3. Consequently,

$$ln\ \omega_r(y,x_0) \le \ln\ C_1\ +\ |p(y+x_0)-p(\infty)|\cdot|\beta(x_0)|\cdot ln\ |y|.$$

Making use of (2.4), we obtain

$$ln\ \omega_r(y,x_0) \le \ln\ C_1\ +\ A_\infty|\beta(x_0)|\frac{ln\ |y|}{ln(e+|y+x_0|)}. \tag{4.4}$$

The inequality

$$\frac{ln\ |y|}{ln(e+|y+x_0|)} \le ln\ (e+|x_0|), \qquad x_0, y \in \mathbb{R}^n \tag{4.5}$$

is valid. Indeed $\frac{ln\ |y|}{ln(e+|y+x_0|)} \le \frac{ln\ (|x_0|+|x_0+y|)}{ln(e+|y+x_0|)}$ and, to obtain (4.5), it remains to note that the maximum of the function $g(t) = \frac{ln\ (t+|x_0|)}{ln\ (t+e)}, t \ge 0$, is reached at the point $t = 0$ when $|x_0| \ge e$ and at the point $t = \infty$ when $|x_0| \le e$. Then from (4.4) the right-hand side inequality in (4.3) follows. The left-hands side inequality is proved in a similar way. □

To prove now estimates (3.8), we observe that from (4.2) and (4.3) it follows that

$$\frac{1}{c}(1+|x_0|)^{-A_\infty|\beta(x_0)|}\int\limits_{|y|>r}\left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(\infty)} dy \le 1 \tag{4.6}$$

and

$$1 \le c(1+|x_0|)^{A_\infty|\beta(x_0)|}\int\limits_{|y|>r}\left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(\infty)} dy. \tag{4.7}$$

Evidently,

$$\int\limits_{|y|>r}\left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(\infty)} dy = \frac{c_1}{\mu_\beta^{p(\infty)}}\ \frac{r^{\beta(x_0)p(\infty)+n}}{|\beta(x_0)p(\infty)+n|}$$

where c_1 is an absolute constant. Then from (4.6) and (4.7), we obtain estimates (3.8) for $\mu_\beta = \| \, |x - x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))}$.

c) Proof of Theorem A. We use the well known approach to reduce the boundedness of the Riesz potential to that of the maximal operator which requires an information about the behaviour of the norms $\| \, |x - x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0,r))}$ as $r \to 0$ and $r \to \infty$. This information is provided by Propositions 1 and 2.

We have

$$I^{\alpha(\cdot)} f(x) = \int\limits_{|x-y| \le r} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}} + \int\limits_{|x-y| \ge r} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}} := A_r(x) + B_r(x). \tag{4.8}$$

We make use of the inequality

$$|A_r(x)| \;\le\; \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)} - 1}\, Mf(x) \tag{4.9}$$

which is known in case of $\alpha(x) = const$ (see for instance, [1], p. 54) and remains valid in case it is variable.

By (4.9) and (2.7) we have

$$|A_r(x)| \le c r^{\alpha(x)} Mf(x) \tag{4.10}$$

with some absolute constant $c > 0$ not depending on x and r.

We assume that $\|f\|_{p(\cdot)} \le 1$. Applying the Hölder inequality for the $L^{p(\cdot)}$-spaces

$$\left| \int_\Omega u(x) v(x)\,dx \right| \le k \|u\|_p \|v\|_{p'}, \qquad p' = \frac{p}{p-1} \tag{4.11}$$

in the integral $B_r(x)$, we obtain

$$|B_r(x)| \le k \mu_\beta(x,r) \|f\|_{p(y)} \le \mu_\beta(x,r) \tag{4.12}$$

where

$$\mu_\beta(x,r) = \left\| |x-y|^{\beta(x)} \chi \right\|_{s(y)}, \qquad \frac{1}{s(x)} + \frac{1}{p(x)} = 1. \tag{4.13}$$

and χ is the characteristic function of $\{y \in \mathbb{R}^n : |x - y| > r\}$ and $\beta(x) = \alpha(x) - n$. We make use of Corollary to Propositions 1 and 2, which is possible since the assumptions of that Corollary with $\beta(x) = \alpha(x)$ are satisfied by conditions of Theorem A. Applying that Corollary with $p(x)$ replaced by $s(x)$, we obtain

$$|B_r(x)| \leq c_3 K(x) r^{-\frac{n}{q(x)}}, \quad x \in \mathbb{R}^n, \tag{4.14}$$

with

$$K(x) = (1+|x|)^{\frac{[n-\alpha(x)]A_\infty}{p(\infty)}} \tag{4.15}$$

and c_3 not depending on r and x. Then from (4.8), (4.10) and (4.14), we have

$$\left|I^{\alpha(\cdot)} f(x)\right| \leq c_4 \left[r^{\alpha(x)} Mf(x) + K(x) r^{-\frac{n}{q(x)}}\right], \quad 0 < r < \infty, \qquad x \in \mathbb{R}^n. \tag{4.16}$$

Minimizing the right-hand side with respect to r we see that its minimum is reached at

$$r_{\min} = \left[\frac{\alpha(x) q(x)}{nK(x)} Mf(x)\right]^{-\frac{p(x)}{n}}$$

and easy evaluations yield

$$\left|I^{\alpha(\cdot)} f(x)\right| \leq c_5 \left[K(x)\right]^{\frac{\alpha(x)p(x)}{n}} \left[Mf(x)\right]^{\frac{p(x)}{q(x)}} .$$

Since $p(x)$ satisfies the logarithmic condition (2.4) at infinity, we may replace $p(x)$ in $[K(x)]^{\frac{\alpha(x)p(x)}{n}}$ by $p(\infty)$. Then

$$\left|I^{\alpha(x)} f(x)\right| \leq c_6 (1+|x|)^{\alpha(x)\left(1-\frac{\alpha(x)}{n}\right)A_\infty} \left[Mf(x)\right]^{\frac{p(x)}{q(x)}} = c_6 (1+|x|)^{\gamma(x)} \left[Mf(x)\right]^{\frac{p(x)}{q(x)}} .$$

Then

$$A_q\left((1+|x|)^{-\gamma(x)} I^{\alpha(x)} f(x)\right) \leq c_6 \int\limits_{\mathbb{R}^n} |Mf(x)|^{p(x)}\, dx \leq c_7$$

by Proposition 3. The theorem is proved.

Proof of Corollary to Theorem A. The statement of the corollary follows from the pointwise estimate

$$M^{\alpha(\cdot)} f(x) \leq c I^{\alpha(x)} |f|(x) \tag{4.17}$$

where c does not depend on f and x. To prove (4.17), we observe that for any $x \in \mathbb{R}^n$ there exists an $r = r_x$ such that

$$M^{\alpha(\cdot)}f(x) \leq \frac{2}{|B(x,r_x)|^{n-\alpha(x)}} \int\limits_{B(x,r_x)} |f(y)|\,dy$$

and on the other hand

$$I^{\alpha(x)}f(x) \geq \int\limits_{B(x,r_x)} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}} \geq \frac{c}{|B(x,r_x)|^{n-\alpha(x)}} \int\limits_{B(x,r_x)} |f(y)|\,dy.$$

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