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Sonine Integral Equations of the First Kind in $L_p(0,b)$.

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Abstract

A Volterra integral equation of the first kind

$$\mathbf{K}\varphi(x) :\equiv \int_0^x k(x-t)\varphi(t)\,dt = f(x), \qquad 0 < x < b \le \infty$$

with a kernel $k(x) \in L_1(0,b)$ is called **Sonine equation** if there exists another locally integrable kernel $\ell(x)$ such that $\int_0^x k(x-t)\ell(t)\,dt \equiv 1$. We construct the real inverse operator, within the framework of the spaces $L_p(0,b)$, in Marchaud form:

$$\mathbf{K}^{-1}f(x) = \ell(x)f(x) + \int_0^x \ell\,'(t)[f(x-t)-f(x)]\,dt$$

with the interpretation of the convergence of this "hypersingular" integral in L_p-norm. The description of the range $K(L_p)$ is given.

Key Words and Phrases: Sonine kernels, integral equations of the first kind, special functions, Marchaud constructions, Hardy inequality

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1. Introduction

We study integral equations of the first kind

$$\mathbf{K}\varphi := \int_0^x k\,(x-t)\,\varphi\,(t)\,dt = f\,(x)\,, \quad x \in (0,b), \tag{1.1}$$

where $0 < b \le \infty$, and $k(x)$ is the so called Sonine kernel, see Definition 2.1. A formal solution of such equations is known long ago [5]-[6], see also [4], p. 85 about Sonine equations.

In [3] there was given the solution of a similar equation on the whole real line $\mathbb{R}^1$ in the space $L_p(\mathbb{R}^1)$ by constructing the inverse operator bounded from the range $\mathbf{K}(L_p)$ into $L_p(\mathbb{R}^1), 1 < p < \infty$.

Meanwhile, Sonine equations occur in applications in the case of a finite interval $[0, b]$. The results obtained in [3] may be applied to the case of the half-axis $\mathbb{R}^1_+$ via the natural truncation. As for the case of a finite interval, the results of [3] may be also obviously applied due to the Volterra kind of the equation, but under assumptions made in [3]. In [3] it was assumed that the kernel $k(x)$ and the so called associate kernel $\ell(x)$ (see Definition 2.1) are differentiable for large x and the derivatives $k'(x), \;\; \ell'(x)$ are absolutely integrable at infinity:

$$\int_N^\infty |k'(t)|\, dt < \infty, \qquad \int_N^\infty |\ell'(t)|\, dt < \infty \tag{1.2}$$

for some $N > 0$, and the inverse operator involved values of the kernel $\ell(x)$ up to infinity. However, we have to obtain the inversion of the equation in natural terms, related to the values of the kernel on $[0, b]$ only, and to avoid usage of assumptions of type (1.2). This is moreover important that assumption (1.2) on $\ell(x)$ proves to be restrictive and is not satisfied for various examples, for instance, for the following important particular case

$$k(x) = \frac{A + ln\frac{1}{x}}{x^{1-\alpha}}$$

or even the case $k(x) = 1 - \frac{a}{x^{1-\alpha}}, \;\; a > 0$, and many others.

Therefore, there arose the necessity to deal with equations (1.1) anew, avoiding assumptions of type (1.2). This is the goal of this paper. Under some natural assumptions on the kernel $k(x)$, we construct the inversion of equation (1.1) in $L_p(0, b)$ and characterize the range $K(L_p(0, b))$ of the operator K.

Examples involving important cases of elementary and special functions defining the kernel $k(x)$ are considered.

2. Preliminaries.

2.1 Sonine kernels

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Definition 2.1. *A kernel $k(x) \in L_1(0,b)$ is called a Sonine kernel, if there exists a kernel $\ell(x) \in L_1(0,b)$ such that the relation*

$$\int_0^x \ell\,(x-t)\,k\,(t)\,dt = 1, \tag{2.1}$$

is valid for almost all $x \in (0,b)$. Correspondingly, the operator $\mathbf{K}$ with a Sonine kernel $k(x)$ is called Sonine integral operator.

The kernel $\ell(x)$ will be referred to as the kernel *associated* to the kernel $k(x)$. Obviously, $\ell(x)$ is also a Sonine kernel.

From relation (2.1) it follows that the formal solution of equation (1.1) is given by

$$\varphi(x) = \frac{d}{dx}\int_0^x \ell(x-t)f(t)\,dt. \tag{2.2}$$

For an equation of type (1.1) on the whole line $\mathbb{R}^1$, the inversion operator adjusted for L_p-solutions, was given in [3] in the form

$$\varphi(x) = \mathbb{K}^{-1}f := \ell(\infty)f(x) + \int\limits_0^\infty \ell'(t)[f(x-t)-f(x)]\,dt. \tag{2.3}$$

For functions $f(x)$ supported on the half-axis $\mathbb{R}^1_+$ (as in (1.1)), operator (2.3) may be also represented in the form

$$\mathbb{K}^{-1}f = \ell(x)f(x) + \int\limits_0^x \ell'(t)[f(x-t)-f(x)]\,dt \tag{2.4}$$

which suits well for the goals of this paper. Compare (2.4) with the Marchaud-type form of the fractional differentiation "with the starting point $x=0$"

$$D^\alpha_{0+}f(x) = \frac{f(x)}{\Gamma(1-\alpha)x^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^x \frac{f(x)-f(x-t)}{t^{1+\alpha}}\,dt,$$

see [4], p.225, formula (13.2), which corresponds to the case $k(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$. We show that the construction (2.4), with the integral term in (2.4) treated as convergent in L_p-norm, provides the inversion operator for equation (1.1) with L_p-solutions.

Sonine kernels have the following property.

Lemma 2.2. *([3]). Let a Sonine kernel $k(x)$ and its associate $\ell(x)$ have distributional derivatives in $L_1(\delta, b)$ for any $\delta > 0$ and satisfy the conditions*

$$\lim_{t\to 0} tk(t) = 0, \qquad \lim_{t\to 0} t\ell(t) = 0. \tag{2.5}$$

Then the formula is valid

$$\int_0^x \ell'(t)\,[k(x) - k(x-t)]\,dt = k(x)\,\ell(x)\,. \tag{2.6}$$

for almost all $x > 0$.

As in [3], we introduce some assumptions on behaviour of the kernels $k(x)$ and $\ell(x)$ near the origin:

monotonicity near the origin: *there exists a neighborhood $0 < x < \varepsilon_0$ where*

$$k(x) \geq 0, \quad \ell(x) \geq 0 \qquad \text{and} \qquad k(x)\downarrow, \quad \ell(x)\downarrow, \qquad 0 < x \leq \varepsilon_0. \tag{2.7}$$

absolute integrability of $k'(x)$ and $\ell'(x)$ beyond the origin: it is assumed that derivatives exist in the generalized sense and

$$\int_\delta^b |k'(x)|\,dx < \infty, \qquad \int_\delta^b |\ell'(x)|\,dx < \infty. \tag{2.8}$$

for any $0 < \delta < b$.

In [3] the following statement was proved.

Lemma 2.3. *Any Sonine kernel satisfying assumption (2.7), has the following properties:*

$$xk(x)\,\ell(x) \leq 1, \qquad 0 < x \leq \varepsilon_0 \tag{2.9}$$

$$k(x)\int_0^x l(t)\,dt \leq 1 \qquad \textit{and} \qquad \ell(x)\int_0^x k(t)\,dt \leq 1 \tag{2.10}$$

for all $x \in (0, \varepsilon_0)$, and

$$\lim_{x\to 0+} k(x) = \lim_{x\to 0+} \ell(x) = +\infty, \tag{2.11}$$

$$\lim_{x\to 0+} xk(x) = \lim_{x\to 0+} x\ell(x) = 0, \tag{2.12}$$

$$\sup_{0<\varepsilon<\varepsilon_0}\left[\ell(\varepsilon)\int_\varepsilon^b |k(x) - k(x-\varepsilon)|\,dx\right] < \infty. \tag{2.13}$$

$$\sup_{0<\varepsilon<\varepsilon_0}\left[\int_0^\varepsilon |k'(t)|\,dt \int_{\varepsilon-t}^\varepsilon l(s)\,ds\right] \leq 1 \tag{2.14}$$

and

$$\sup_{0<\varepsilon<\varepsilon_0}\left[\int_0^{\varepsilon}\ell(t)\,dt\int_{\varepsilon}^{b}|k'(t)|\,dt\right]<\infty. \tag{2.15}$$

Remark 2.4. For the first inequality in (2.10) to be valid, it suffices to suppose that only the function $k(x)$ satisfies condition (2.7); similarly, only $\ell(x)$ has to satisfy (2.7) for the second inequality. Property (2.13) was proved in [3] in the case $b=\infty$. The proof for $b<\infty$ is slightly different, so we give it in Appendix.

2.2 Inversion of Sonine equations on the whole line.

For the operator

$$\mathbf{K}\varphi := \int_{\mathbb{R}^1} k(x-t)\varphi(t)\,dt = f(x), x\in\mathbb{R}^1$$

in [3] the theorem below was proved, in which the following condition is used:

$$\int_N^{\infty}\frac{|k^{**}(x)|}{x^{\frac{1}{p}}}dx<\infty, \tag{2.16}$$

where

$$k^{**}(t)=\frac{1}{t}\int_0^t k^*(s)\,ds$$

$k^*(s)=inf\{t>0: m(k,t)\le s\}$ being the non-increasing rearrangement of $|k(x)|$, $m(k,t)=mes\{x\in\mathbb{R}^1: |k(x)|>t\}$.

Theorem 2.5. *Let $k(x)$ be a Sonine kernel satisfying condition (2.16) for some $N>0$, assumption (2.7) and conditions (2.8) with $b=\infty$. Then*

$$\mathbf{K}^{-1}\mathbf{K}\varphi=\varphi, \qquad \varphi\in L_p(\mathbb{R}^1), \qquad 1<p<\infty, \tag{2.17}$$

where

$$(\mathbf{K}^{-1}f)(x)=\ell(\infty)f(x)+\lim_{\substack{\varepsilon\to 0\\(L_p)}}\int_{\varepsilon}^{\infty}[f(x-t)-f(x)]\ell'(t)dt. \tag{2.18}$$

In the next theorem we use one of the following assumptions on $k(x)$

i)
$$\overline{\lim_{x\to 0}}\frac{\int_x^{\infty}k^*(t)\left(\frac{x}{t}\right)^{\frac{1}{p}}dt}{\int_0^x k(t)\,dt}<\infty \tag{2.19}$$

and

$$ii) \quad \textit{there exists an} \quad \alpha \in \left(0, \frac{1}{p}\right) \quad \textit{such that} \quad \sup_{\mathbb{R}^1_+} |x^{\alpha-1}k(x)| < \infty \tag{2.20}$$

and we denote for brevity

$$Y(\mathbb{R}^1) = \begin{cases} L_\Phi(\mathbb{R}^1), & \text{if} \quad k(x) \quad \text{satisfies (2.16) and (2.19)} \\ L_q(\mathbb{R}^1),\ q = \frac{p}{1-\alpha p}, & \text{if} \quad k(x) \quad \text{satisfies (2.20)} \end{cases} \tag{2.21}$$

where the Young function $\Phi(u)$ defining the Orlicz space $L_\Phi(\mathbb{R}^1)$ is given by

$$\Phi^{-1}(x) = \int_{\frac{1}{x}}^{\infty} \frac{k^{**}(t)}{t^{\frac{1}{p}}} dt = p\left[x^{\frac{1}{p}} \int_0^{\frac{1}{x}} |k(s)|\, ds + \int_{\frac{1}{x}}^{\infty} \frac{k^*(s)}{s^{\frac{1}{p}}} ds\right]. \tag{2.22}$$

Theorem 2.6. *Let $k(x)$ satisfy conditions (2.16),(2.7) and (2.8) for $b = \infty$. Then $f(x) \in \mathbf{K}(L_p)$, if and only if $f(x) \in Y(\mathbb{R}^1)$ and one of the following conditions is satisfied*

$$\lim_{\substack{\varepsilon\to 0\\(L_p)}} \mathbf{K}_\varepsilon^{-1} f \in L_p(\mathbb{R}^1) \qquad or \qquad \sup_{\varepsilon>0} \left\|\mathbf{K}_\varepsilon^{-1} f\right\|_{L_p(\mathbb{R}^1)} < \infty. \tag{2.23}$$

2.3 On identity approximations.

An operator A_ε is said to be an identity approximation in a Banach space X if $\|A_\varepsilon f - f\|_X \to 0$ as $\varepsilon \to 0$. Let A_ε be a convolution operator:

$$A_\varepsilon \varphi = \int_{\mathbb{R}^n} a_\varepsilon(t)\, \varphi(x-t)\, dt, \qquad \varepsilon > 0. \tag{2.24}$$

Statements, providing sufficient conditions on the family $a_\varepsilon(t)$ for operator (2.24) to be an identity approximation are well known, at the least for L_p-spaces. The following version of such a statement in the general Banach function spaces setting was proved in [3].

Let $X = X(\mathbb{R}^n)$ be an arbitrary space of functions defined on $\mathbb{R}^n$, satisfying the following axioms:

1) X is translation invariant: $\|f(x-h)\|_X \leq C\|f\|_X$ with C not depending on $h \in \mathbb{R}^n$ and the mean continuity with respect to the norm $\|\cdot\|_X$ holds:

$$\omega_X(f,\delta) := \sup_{|h|<\delta} \|f(x-h) - f(x)\|_X \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.25}$$

2) Minkowsky integral inequality is valid

$$\left\| \int_{\mathbb{R}^n} f(\cdot,t)\,dt \right\|_X \le \int_{\mathbb{R}^n} \|f(\cdot,t)\|_X \,dt. \tag{2.26}$$

Lemma 2.7. *Let a Banach function space X satisfy assumptions 1) and 2) and functions $a_\varepsilon(t)$ satisfy the conditions*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} a_\varepsilon(t)\,dt = 1, \qquad \text{and} \qquad \int_{\mathbb{R}^n} |a_\varepsilon(t)|\,dt \le M < \infty \tag{2.27}$$

with M not depending on ε and

$$\lim_{\varepsilon\to 0}\int_{|t|\ge\delta} |a_\varepsilon(t)|\,dt = 0 \tag{2.28}$$

for any $\delta > 0$. Then A_ε is an identity approximation in the space X.

2.4 The generalized Hardy inequality.

Theorem 2.8. *Let a kernel $k(x) \in L_1([0,b])$, $0 < b < \infty$, satisfy the monotonicity condition (2.7) in a neighbourhood of the origin. Then the following Hardy-type inequality is valid:*

$$\left\| \ell(x)\int_0^x k(x-t)\varphi(t)\,dt \right\|_{L^p(0,b)} \le A\,\|\varphi\|_{L^p(0,b)}, \qquad 1<p<\infty, \qquad A>0, \tag{2.29}$$

for any function $\ell(x)$ such that

$$\sup_{0<x<\varepsilon_0}\left|\ell(x)\int_0^x k(t)dt\right| < \infty \tag{2.30}$$

and $\ell(x)$ is bounded beyond $[0,\varepsilon_0]$:

$$|\ell(x)| \le C, \qquad \varepsilon_0 \le x \le b < \infty. \tag{2.31}$$

Proof. The proof follows the well known idea that the operator

$$A\varphi(x) = \ell(x)\int_0^x k(x-t)\varphi(t)\,dt \tag{2.32}$$

may be dominated, in a neighbourhood of the origin, by the maximal Hardy operator

$$(M\varphi)(x) = \sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h} |\varphi(t)|\,dt, \qquad 0 \le x \le b, \tag{2.33}$$

due to the monotonicity of the kernel near the origin. In (2.33) $\varphi(t)$ is assumed to be continued as zero beyond the interval $[0,b]$: $f\varphi(t)\equiv 0$, if $t<0$ or $t>b$. As is well known, the maximal operator M is bounded in the space $L^p(0,b)$:

$$\|M\varphi\|_{L^p(0,b)} \le C\,\|\varphi\|_{L^p(0,b)}\,, \qquad 1<p<\infty, \tag{2.34}$$

see for example, [7].

Under assumptions (2.7) and (2.30)-(2.31), the following pointwise estimate

$$\left|\ell(x)\int_0^x k(x-t)\varphi(t)\,dt\right| \le CM\varphi(x) \tag{2.35}$$

is valid for almost all $x\in(0,\varepsilon_0]$. To prove (2.35), we represent the integral as follows:

$$\int_0^x k(x-t)\varphi(t)\,dt = \sum_{m=1}^{\infty}\int_{2^{-m}x}^{2^{-m+1}x} k(t)\varphi(x-t)\,dt. \tag{2.36}$$

Making use of the monotonicity property (2.7), we obtain:

$$\left|\int_0^x k(x-t)\varphi(t)\,dt\right| \le \sum_{m=1}^{\infty} k\left(2^{-m}x\right)\int_{2^{-m}x}^{2^{-m+1}x}|\varphi(x-t)|\,dt$$

$$\le \sum_{m=1}^{\infty} k\left(2^{-m}x\right)\int_{x-2^{-m+1}x}^{x+2^{-m+1}x}|\varphi(t)|\,dt$$

$$= \sum_{m=1}^{\infty} k\left(2^{-m}x\right)x2^{2-m}\cdot\left(\frac{1}{x2^{2-m}}\int_{x-2^{-m+1}x}^{x+2^{-m+1}x}|\varphi(t)|\,dt\right). \tag{2.37}$$

Consequently,

$$\left|\int_0^x k(x-t)\varphi(t)\,dt\right| \le A(x)(M\varphi)(x) \tag{2.38}$$

with $A(x)=\sum_{m=1}^{\infty} k\left(2^{-m}x\right)x2^{2-m}$. The function $A(x)$ may be estimated via the kernel $k(x)$. To this end, we return to (2.36), make use of the monotonicity of $k(x)$ again and obtain that

$$\int_0^x k(t)\,dt = \sum_{m=1}^{\infty}\int_{2^{-m}x}^{2^{-m+1}x} k(t)\,dt \ge \sum_{m=1}^{\infty} k\left(2^{-m+1}x\right)2^{-m}x$$

$$= \sum_{m=0}^{\infty} k\left(2^{-m}x\right)2^{-m-1}x \ge \frac{1}{2}\sum_{m=1}^{\infty} k\left(2^{-m}x\right)2^{-m}x = \frac{1}{8}A(x).$$

Therefore,

$$A(x) \le 8 \int_0^x k(t)\, dt, \qquad 0 < x \le \varepsilon_0. \tag{2.39}$$

Then from (2.38) and (2.39) we obtain

$$\left| \ell(x) \int_0^x k(x-t)\varphi(t)\, dt \right| \le 8\ell(x) \int_0^x k(t)\, dt \cdot (M\varphi)(x) \le C(M\varphi)(x)$$

by condition (2.30).

Then the statement (2.29) follows from (2.35) in view of the boundedness (2.34) applied to the interval $[0, \varepsilon_0]$; on the interval $[\varepsilon_0, b]$ the boundedness is trivial since $\ell(x)$ is bounded on $[\varepsilon_0, b]$ and $k(x)$ is an integrable kernel. □

Corollary 2.9. *Let a Sonine kernel $k(x) \in L_1([0, b])$ and its associate $\ell(x)$ satisfy the monotonicity condition (2.7) and let $\ell(x)$ be bounded beyond $[0, \varepsilon_0]$. Then inequality (2.29) holds.*

Example 2.10. Let $k(x) = \frac{|ln^m x|}{x^{1-\alpha}}, 0 < \alpha < 1$. In this case we have

$$\left\| \frac{1}{x^\alpha (1 + |ln^m x|)} \int_0^x \frac{|ln^m (x-t)|}{(x-t)^{1-\alpha}} \varphi(t)\, dt \right\|_{L^p(0,b)} \le A \left\| \varphi \right\|_{L^p(0,b)} \tag{2.40}$$

with $1 < p < \infty$ and $0 < b < \infty$.

Indeed, we have $\int x^{\alpha-1} ln^m x\, dx = x^\alpha \sum\limits_{j=0}^m c_j ln^j x$, so that $\int_0^x |k(t)|\, dt = \int\limits_0^x t^{\alpha-1} |ln^m t|\, dt \le x^\alpha \sum\limits_{j=0}^m a_j |ln^j x|$ for all $x \in \mathbb{R}^1_+$, a_j being constants. Therefore, the condition

$$\frac{1}{x^\alpha (1 + |ln^m x|)} \int_0^x |k(t)|\, dt \le C < \infty$$

is satisfied.

3. Statements of the main results.

Following the approach in [4], p. 226, see also [3], we consider the truncation of operator (2.4) in the form

$$\mathbf{K}_\varepsilon^{-1} f(x) = \begin{cases} \ell(x) f(x) + \int\limits_\varepsilon^x \ell'(t)[f(x-t) - f(x)]\, dt, & x > \varepsilon \\ \\ \ell(x) f(x), \quad 0 < x < \varepsilon. \end{cases} \tag{3.1}$$

For further goals we find it convenient to denote

$$\Psi_\varepsilon f(x) = \begin{cases} \int\limits_\varepsilon^x \ell'(t)[f(x-t)-f(x)]\,dt, & x > \varepsilon \\ \\ 0, \quad 0 < x < \varepsilon \end{cases} \tag{3.2}$$

so that

$$\mathbf{K}_\varepsilon^{-1} f(x) = \ell(x)f(x) + \Psi_\varepsilon f(x), \quad 0 < x < b. \tag{3.3}$$

We prove the following theorems A and B.

Theorem A. *Let $k(x)$ be a Sonine kernel satisfying assumptions (2.7) and (2.8) on $[0,b], 0 < b < \infty$. Then for any $f = \mathbf{K}\varphi$ with $\varphi \in L_p(0,b), 1 < p < \infty$*

$$\varphi(x) = \mathbf{K}^{-1} f := \ell(x)f(x) + \int\limits_0^x \ell'(t)[f(x-t)-f(x)]\,dt \tag{3.4}$$

where the convergence of the integral in $\mathbf{K}^{-1}f = \lim\limits_{\varepsilon\to 0} \mathbf{K}_\varepsilon^{-1} f$ is treated in the L_p-sense:

$$\lim_{\varepsilon\to 0} \left\| \mathbf{K}_\varepsilon^{-1} f - \varphi \right\|_{L_p(0,b)}. \tag{3.5}$$

Theorem B. *Let a Sonine kernel $k(x)$ satisfy assumptions (2.7) and (2.8) on $[0,b], 0 < b < \infty$. A function $f \in L_1(0,b)$ is in the range $\mathbf{K}(L_p),\ 1 < p < \infty$, if and only if*

$$\ell(x)f(x) \in L_p(0,b) \tag{3.6}$$

and one of the following conditions is fulfilled:

i)
$$\lim_{\substack{\varepsilon\to 0 \\ (L_p)}} \Psi_\varepsilon f \in L_p(0,b), \tag{3.7}$$

or

ii)
$$\sup_{0<\varepsilon<b} \|\Psi_\varepsilon f\|_{L_p(0,b)} < \infty. \tag{3.8}$$

4. Proof of Theorem A.

First, we prove Lemma 4.1 providing an integral representation of the truncated operator (3.1) on the range $\mathbf{K}(L_p)$ of the operator $\mathbf{K}$. We make use of the kernel $N_\varepsilon(s)$

$$N_\varepsilon(s) = \ell(\varepsilon)k(s) + \begin{cases} \int\limits_\varepsilon^s \ell'(t)k(s-t)\,dt, & \varepsilon < s < b \\ 0, & 0 < s < \varepsilon \end{cases} \tag{4.1}$$

introduced in [3], see Subsection 5.2. It was also proved that this kernel has other representations

$$N_\varepsilon(s) = \begin{cases} \int_0^\varepsilon \ell'(t)\left[k(s) - k(s-t)\right]dt, & s > \varepsilon \\ k(s)\,\ell(\varepsilon), & 0 < s < \varepsilon \end{cases}, \tag{4.2}$$

$$N_\varepsilon(s) = \ell(\varepsilon)\left[k(s) - k_+(s-\varepsilon)\right] + \frac{d}{ds}\int_\varepsilon^s \ell(t)\,k(s-t)\,dt, \tag{4.3}$$

and

$$N_\varepsilon(s) = \ell(\varepsilon)\left[k(s) - k(s-\varepsilon)\right] - \int_{s-\varepsilon}^s \ell(s-t)\,k'(t)\;dt, \qquad s > \varepsilon, \tag{4.4}$$

see Lemma 5.3 in [3].

Lemma 4.1. *Let a Sonine kernel $k(x)$ satisfy assumptions (2.7) and (2.8) on $[0,b]$, $0 < b < \infty$. Then for $f = \mathbf{K}\varphi$ with $\varphi \in L_p(0,b), 1 < p < \infty$, the representation is valid*

$$\mathbf{K}_\varepsilon^{-1} f(x) = \begin{cases} \int\limits_0^x N_\varepsilon(s)\varphi(x-s)\,ds, & \varepsilon < x < b \\ \ell(x)f(x), & 0 < x < \varepsilon \end{cases}. \tag{4.5}$$

Proof. According to (3.1) we represent the operator $\mathbf{K}^{-1}$ as

$$\mathbf{K}_\varepsilon^{-1} f(x) = \ell(x)f(x) + \Psi_\varepsilon f(x) \tag{4.6}$$

where $\Psi_\varepsilon f$ is the operator defined in (3.2). To calculate $\Psi_\varepsilon f(x)$ for $f = \mathbf{K}\varphi$ we observe that

$$f(x-t) - f(x) = \int_0^x \left[k_+(s-t) - k(s)\right]\varphi(x-s)\,ds \tag{4.7}$$

where $k_+(s) = \begin{cases} k(s), & s > 0 \\ 0, & s < 0 \end{cases}$. By definition (3.1), it suffices to consider only the case $x > \varepsilon$. Substituting (4.7) into (3.2), we obtain

$$\Psi_\varepsilon f(x) = \int\limits_0^x A_\varepsilon(x,s)\varphi(x-s)\, ds \tag{4.8}$$

where

$$A_\varepsilon(x,s) = \int\limits_\varepsilon^x \ell'(t)[k_+(s-t) - k(t)]\, dt, \qquad x > \varepsilon,$$

justification of the interchange of order of integration being easy by Fubini theorem. Obviously,

$$A_\varepsilon(x,s) = \int\limits_\varepsilon^x \ell'(t)k_+(s-t)\, dt + k(s)[\ell(\varepsilon) - \ell(x)]$$

Therefore, according to (4.1) we have

$$A_\varepsilon(x,s) = N_\varepsilon(s) - k(s)\ell(x)$$

and then from (4.8)

$$\Psi_\varepsilon f(x) = \int\limits_0^x N_\varepsilon(s)\varphi(x-s)\, ds - \ell(x)f(x), \qquad x > \varepsilon$$

which transforms (4.6) into (4.5). □

To prove (3.5), we make use of (4.5) and for $f = \mathbf{K}\varphi$ obtain

$$\left\|\mathbf{K}_\varepsilon^{-1} f - \varphi\right\|_{L_p(0,b)} \leq \left\|\varphi(x) - \int\limits_0^x N_\varepsilon(s)\varphi(x-s)\, ds\right\|_{L_p(\varepsilon,b)} + \|\ell(x)f(x)\|_{L_p(0,\varepsilon)}$$

$$+ \|\varphi(x)\|_{L_p(0,\varepsilon)} \leq \left\|\varphi(x) - \int\limits_0^x N_\varepsilon(s)\varphi(x-s)\, ds\right\|_{L_p(\varepsilon,b)} + (A+1)\|\varphi(x)\|_{L_p(0,\varepsilon)}$$

by Theorem 2.8. The second term in the last line tends to zero as $\varepsilon \to 0$. We have to show the same for the first term

$$I_\varepsilon := \left\|\varphi(x) - \int\limits_0^x N_\varepsilon(s)\varphi(x-s)\, ds\right\|_{L_p(\varepsilon,b)} \leq \left\|\widetilde{\varphi}(x) - \int\limits_{\mathbb{R}^1} \widetilde{N}_\varepsilon(s)\widetilde{\varphi}(x-s)\, ds\right\|_{L_p(\mathbb{R}^1)} \tag{4.9}$$

where we denoted

$$\widetilde{\varphi}(s) = \begin{cases} \varphi(s), & \text{if} \quad 0 \le s \le b, \\ 0, & \text{if} \quad s \notin [0,b] \end{cases} \qquad \text{and} \qquad \widetilde{N}_\varepsilon(s) = \begin{cases} N_\varepsilon(s), & \text{if} \quad 0 \le s \le b, \\ 0, & \text{if} \quad s \notin [a,b] \end{cases}$$

Here the kernel $\widetilde{N}_\varepsilon(s)$ satisfies conditions of Lemma 2.7 under assumptions (2.7) and (2.8) which we assume to be satisfied. This fact was shown in [3] in the case $b = \infty$. For completeness of the presentation, we expose this proof also for the case $b \le \infty$ in Appendix, see Lemma 7.1. Therefore, according to Lemma 7.1 and Lemma 2.7, from (4.9) we see that $I_\varepsilon \to 0$ as $\varepsilon \to 0$. Then $\|\mathbf{K}_\varepsilon^{-1} f - \varphi\|_{L_p(0,b)} \to 0$ as $\varepsilon \to 0$, which proves Theorem A.

5. Proof of Theorem B.

N e c e s s i t y. Necessity of the condition $\ell(x)f(x) \in L_p$ follows from the Hardy-type inequality (2.29). Necessity of condition (3.7) has been proved in Theorem A. Necessity of (3.7) follows from representation (4.5) and condition (7.2).

S u f f i c i e n c y . Let $\ell(x)f(x) \in L_p$ and $f(x) \in L_1$. Then $\Psi_\varepsilon f(x)$ is well defined. Keeping (3.3) in mind, we denote

$$\varphi_\varepsilon(x) = \ell(x)f(x) + \Psi_\varepsilon f(x) \in L_p. \tag{5.1}$$

We shall show that $\mathbf{K}\varphi_\varepsilon$ has the following representation it the rems of the function f:

$$\mathbf{K}\varphi_\varepsilon(x) = A_\varepsilon f, \tag{5.2}$$

where

$$A_\varepsilon f = \begin{cases} \mu_\varepsilon(x) + \int\limits_0^x N_\varepsilon(t) f(x-t)\, dt, & \varepsilon < x < b \\ \nu(x), & 0 < x < \varepsilon \end{cases} \tag{5.3}$$

and $N_\varepsilon(x)$ is the identity approximation kernel (4.1) and

$$\mu_\varepsilon(x) = \int\limits_0^\varepsilon k(x-t)[\ell(t) - \ell(\varepsilon)] f(t)\, dt, \qquad \varepsilon < x < b \tag{5.4}$$

and

$$\nu(x) = \int\limits_0^x k(x-t)\ell(t) f(t)\, dt, \qquad 0 < x < \varepsilon \tag{5.5}$$

are "small" terms.

Let $x > \varepsilon$. To obtain the first line in (5.3), we make use of the representation (3.2) for the term $\Psi_\varepsilon f(x)$ in (5.1) and obtain

$$\mathbf{K}\varphi_\varepsilon(x) = \int\limits_0^x k(x-t)\ell(t)f(t)\,dt + \int\limits_0^x k(x-t)\,dt \int\limits_\varepsilon^t \ell'(s)[f(t-s)-f(t)]\,ds$$

$$= \int_0^\varepsilon k(x-t)\ell(t)f(t)\,dt + \ell(\varepsilon)\int\limits_\varepsilon^x k(x-t)f(t)\,dt + \int\limits_\varepsilon^x k(x-t)\,dt \int_0^{t-\varepsilon} \ell'(t-\xi)f(\xi)\,d\xi.$$

After the interchange of the order of integration in the last term, we have

$$\mathbf{K}\varphi_\varepsilon(x) = \int_0^\varepsilon k(x-t)\ell(t)f(t)\,dt + \ell(\varepsilon)\int\limits_\varepsilon^x k(x-t)f(t)\,dt \tag{5.6}$$

$$+ \int\limits_0^{x-\varepsilon} f(\xi)\,d\xi \int\limits_\varepsilon^{x-\xi} k(x-\xi-y)\ell'(y)\,dy.$$

By (4.1), we have

$$\int\limits_\varepsilon^x k(x-y)\ell'(y)\,dy = N_\varepsilon(x) - \ell(\varepsilon)k(x) \qquad \text{for} \qquad x > \varepsilon,$$

which transforms (5.6) into

$$\mathbf{K}\varphi_\varepsilon(x) = \int_0^\varepsilon k(x-t)[\ell(t)-\ell(\varepsilon)]f(t)\,dt$$

$$+\ell(\varepsilon)\int\limits_{x-\varepsilon}^x k(x-t)f(t)\,dt + \int_\varepsilon^x N_\varepsilon(t)f(x-t)\,dt$$

$$= \int_0^\varepsilon k(x-t)[\ell(t)-\ell(\varepsilon)]f(t)\,dt + \int_0^x N_\varepsilon(t)f(x-t)\,dt,$$

the last passage being made by using the fact that $N_\varepsilon(t) = \ell(\varepsilon)k(t)$ for $0 < t < \varepsilon$, see (4.1). This gives the first line in (5.3). The second line is obvious.

To complete the proof of the sufficiency part of Theorem B, it remains to pass to the limit in representation (5.2)-(5.3). We consider first the case *i)*.

i). Suppose that condition (3.7) is satisfied. Then the functions φ_ε converge in L_p and we put

$$\varphi(x) = \lim_{\substack{\varepsilon\to 0\\(L_p)}} \varphi_\varepsilon(x).$$

We shall show that $f = \mathbf{K}\varphi$. By continuity of the operator $\mathbf{K}$ in L_p, it is sufficient to check that

$$f = \lim_{\substack{\varepsilon\to 0\\(L_p)}} \mathbf{K}\varphi_\varepsilon. \tag{5.7}$$

From (5.2) we obtain

$$\|\mathbf{K}\varphi_\varepsilon(x) - f\|_{L_p(0,b)} \le \left\| \int_0^x N_\varepsilon(t) f(x-t)\, dt - f(x) \right\|_{L_p(0,b)} \tag{5.8}$$

$$+\|f\|_{L_p(0,\varepsilon)} + \|\mu_\varepsilon\|_{L_p(0,b)} + \|\nu\|_{L_p(0,\varepsilon)}.$$

Here the first term on the right-hand side tends to zero as $\varepsilon \to 0$, since $N_\varepsilon(t)$ is an identity approximating kernel according to Lemma 7.1, see also (4.9), the terms $\|f\|_{L_p(0,\varepsilon)}$ and $\|\nu\|_{L_p(0,\varepsilon)}$ obviously tend to zero since f and ν are in $L_p(0,b)$. To show that the term $\|\mu_\varepsilon\|_{L_p(0,b)}$ tends to zero, we observe that $\ell(\varepsilon) \le \ell(t)$ in the integral defining the function μ_ε, if $0 < \varepsilon < \varepsilon_0$. Therefore,

$$\mu_\varepsilon(x) \le \int_0^\varepsilon |k(x-t)\ell(t)f(t)|\, dt = \int_{x-\varepsilon}^x |k(t)\ell(x-t)f(x-t)|\, dt.$$

Hence the estimate

$$\|\mu_\varepsilon\|_{L_p(0,b)} \le \|k\|_{L_1(0,b)} \|\ell f\|_{L_p(0,\varepsilon)} \quad \longrightarrow 0$$

as $\varepsilon \to 0$. Therefore, from (5.8) we obtain (5.7) which proves Theorem B under condition (3.7).

ii). Suppose that condition (3.8) is fulfilled. Since the space L_p is weakly compact, from the boundedness of the set $\{\Psi_\varepsilon f\}_{\varepsilon>0}$ we conclude that there exists a sequence $\varepsilon_k \to 0$ such that φ_{ε_k} weakly converges to a certain function $\varphi \in L_p$. Then in the representation

$$\mathbf{K}\varphi_{\varepsilon_k} = A_{\varepsilon_k} f$$

which we have proved in (5.2), one may pass to the weak limit. Since $A_{\varepsilon_k} f$ strongly converges to f, moreover it converges in the weak sense and we obtain $\mathbf{K}\varphi = f$, which proves the theorem.

6. Examples.

Examples given below are treated in $L_p(0,b)$ with $0 < b < \infty$.

Example 1. The equation

$$\mathbf{K}\varphi :\equiv \frac{1}{\Gamma(\alpha)} \int\limits_0^x \frac{ln\,\frac{1}{x-t} + A}{(x-t)^{1-\alpha}} \varphi(t)\,dt = f(x), \quad 0 < \alpha < 1, \quad A \in \mathbb{R}^1 \quad (6.1)$$

was studied by Volterra [8], see also [9], who obtained its solution in the form

$$\varphi(x) = \frac{d}{dx} \int\limits_0^x \mu_\alpha(x-t) f(t)\,dt$$

where

$$\mu_\alpha(x) = \int\limits_0^\infty \frac{x^{t-\alpha} e^{ht}}{\Gamma(1-\alpha+t)}\,dt, \qquad h = \frac{\Gamma'(1)}{\Gamma(1)} - A,$$

the special function $\mu_\alpha(x)$ being known thereafter as Volterra function, see [1]. So in this example

$$k(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \left[ln \frac{1}{x} + A \right], \qquad \ell(x) = \mu_\alpha(x).$$

It is easily seen that

$$\ell'(x) = \mu'_\alpha(x) = \mu_{1+\alpha}(x).$$

Both conditions (2.7) and (2.8) are satisfied, the monotonicity of $\ell(x) = \mu_\alpha(x)$ near the origin follows from the fact that $\ell'(x) = \mu_{\alpha+1}(x)$ is negative near the origin, the latter being seen from the known asymptotics of the function $\mu_{\alpha+1}(x)$:

$$\mu_{\alpha+1}(x) = \frac{\Gamma(-\alpha)}{x^{\alpha+1} ln\,\frac{1}{x}} \left[1 + O\left(\frac{1}{ln\,\frac{1}{x}} \right) \right] \qquad \text{as} \qquad x \to 0,$$

see [1], Subsection 18.3.

Therefore, we obtain as a corollary to Theorems A and B, the following statement proved, in a slightly different form in [2] (in the case $A = 0$).

The unique solution of equation (6.1) in the space $L_p(0,b), 1 < p < \infty$, *is given by the formula*

$$\varphi(x) = \mu_\alpha(x) f(x) + \int\limits_0^x \mu_{\alpha+1}(t)[f(x-t) - f(x)]\,dt$$

with the convergence of the integral in the L_p-norm in accordance with (3.1) and (3.5). The range $\mathbf{K}(L_p)$ consists of those and only those function $f(x)$ for which

$$\sup_{\varepsilon>0}\left\|\int_{\varepsilon}^{x}\mu_{\alpha+1}(t)[f(x-t)-f(x)]\,dt\right\|_{L_p(\varepsilon,b)} < \infty.$$

Example 2. The following example is due to Sonine himself:

$$k(x)=\frac{J_{-\nu}(2\sqrt{x})}{(\sqrt{x})^{\nu}}, \qquad \ell(x)=\frac{I_{\nu-1}(2\sqrt{x})}{(\sqrt{x})^{1-\nu}}, \qquad 0<\nu<1.$$

In particular, when $\nu=\frac{1}{2}$, we have $k(x)=\frac{\cos(2\sqrt{x})}{\sqrt{\pi x}}$, $\ell(x)=\frac{ch(2\sqrt{x})}{\sqrt{\pi x}}$.

In this example $\ell'(x)=\frac{I_{\nu-2}(2\sqrt{x})}{(\sqrt{x})^{2-\nu}}$ and we see that both conditions (2.7) and (2.8) are satisfied. Therefore, for the equation

$$\int_0^x \frac{J_{-\nu}(2\sqrt{x-t})}{(\sqrt{x-t})^{\nu}}\varphi(t)\,dt=f(x), \tag{6.2}$$

from Theorems A and B we obtain the following new inversion statement.

The unique solution of equation (6.2) in the space $L_p(0,b), 1<p<\infty$, is given by the formula

$$\varphi(x)=\frac{I_{\nu-1}(2\sqrt{x})}{(\sqrt{x})^{1-\nu}}f(x)+\int\limits_0^x\frac{I_{\nu-2}(2\sqrt{t})}{(\sqrt{t})^{2-\nu}}[f(x-t)-f(x)]\,dt$$

with the convergence of the integral in the L_p-norm and the description of the range $\mathbf{K}(L_p)$ given by

$$\sup_{\varepsilon>0}\left\|\int_{\varepsilon}^{x}\frac{I_{\nu-2}(2\sqrt{t})}{(\sqrt{t})^{2-\nu}}[f(x-t)-f(x)]\,dt\right\|_{L_p(\varepsilon,b)} < \infty.$$

Example 3. Let

$$\Phi(\beta;\alpha;x)=\sum_{k=0}^{\infty}\frac{(\beta)_k}{(\alpha)_k}\frac{x^k}{k!}, \qquad x\in\mathbb{R}^1$$

be the confluent hypergeometric function (Kummer function). The functions

$$k(x)=\frac{x^{\alpha-1}}{\Gamma(\alpha)}\Phi(\beta;\alpha;\lambda x) \quad \text{and} \quad \ell(x)=\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\Phi(-\beta;1-\alpha;\lambda x) \tag{6.3}$$

where $0 < \alpha < 1$ and $\beta, \lambda \in \mathbb{R}^1$, form a pair of Sonine kernels. This may be easily verified via Laplace transforms. Indeed, relation (2.1) in Laplace transforms means that

$$\mathcal{K}(p)\mathcal{L}(p) \equiv \frac{1}{p}$$

and by direct calculation of the Laplace transforms of the series defining kernels (6.3), one can easily obtain that

$$\mathcal{K}(p) = \frac{p^{\beta-\alpha}}{(p-\lambda)^\beta} \qquad \text{and} \qquad \mathcal{L}(p) = \frac{(p-\lambda)^\beta}{p^{1+\beta-\alpha}}, \qquad p > \lambda.$$

By the known formula $\frac{d}{dx}\left[x^{b-1}\Phi(a;b;x) = (b-1)x^{b-2}\Phi(a;b-1;x)\right]$ we obtain that

$$\ell'(x) = \frac{1}{\Gamma(-\alpha)} \frac{\Phi(-\beta;-\alpha;\lambda x)}{x^{1+\alpha}}$$

and conditions (2.7) and (2.8) are fulfilled. Therefore, one can easily derive the correspondence corollary from Theorems A and B for the equation

$$\frac{1}{\Gamma(\alpha)} \int\limits_0^x \frac{\Phi(\beta;\alpha;\lambda(x-t))}{(x-t)^{1-\alpha}} \varphi(t)\, dt = f(x), \tag{6.4}$$

with the inversion formula

$$\varphi(x) = \frac{\Phi(-\beta;1-\alpha;\lambda x)}{\Gamma(1-\alpha)x^\alpha} f(x) + \frac{1}{\Gamma(-\alpha)} \int\limits_0^x \Phi(-\beta;-\alpha;\lambda t) \frac{f(x-t)-f(x)}{t^{1+\alpha}}\, dt.$$

Example 4. Let $k(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} - \lambda$, where $0 < \alpha < 1$ and $\lambda > 0$. This is the Sonine kernel with

$$\ell(x) = x^{-\alpha} E_{1-\alpha,1-\alpha}(\lambda x^{1-\alpha}),$$

where $E_{\alpha,\beta}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k+\beta)}$ is the generalized Mittag-Leffler function, which may be verified via Laplace transforms. In this case

$$\ell'(x) = \frac{E_{1-\alpha,-\alpha}(\lambda x^\alpha)}{x^{1+\alpha}}$$

by the known differentiation formula for the Mittag-Leffler function, see for instance, [10], p.48.

Then it is easily seen that assumptions (2.7) and (2.8) are fulfilled and one may apply Theorems A and B for inverting the equation

$$\int_0^x \left[\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} - \lambda\right] \varphi(t)\, dt = f(x), \tag{6.5}$$

within the framework of L_p-spaces:

$$\varphi(x) = E_{1-\alpha,1-\alpha}(\lambda x^{1-\alpha}) \frac{f(x)}{x^\alpha} + \int_0^x E_{1-\alpha,-\alpha}(\lambda t^\alpha) \frac{f(x-t)-f(x)}{t^{1+\alpha}}\, dt.$$

7. Appendix.

7.1 Proof of inequality (2.13).

Inequality (2.13) is derived from the inequality

$$\int_\varepsilon^b |k(x) - k(x-\varepsilon)|\, dx \le c\varepsilon + \int_0^\varepsilon k(t)\, dt, \qquad 0 < \varepsilon \le \varepsilon_0, \tag{7.1}$$

where $c = \int_{\varepsilon_0}^b |k'(t)|\, dt$. To prove (7.1), we proceed as follows

$$\int_\varepsilon^b |k(x) - k(x-\varepsilon)|\, dx = \int_\varepsilon^b \left| \int_0^\varepsilon k'(x-\varepsilon+t)\, dt \right| dx$$

$$\le \int_0^\varepsilon dt \int_\varepsilon^b |k'(x-\varepsilon+t)|\, dx \le \int_0^\varepsilon dt \int_t^b |k'(x)|\, dx = \int_0^\varepsilon dt \left(\int_t^{\varepsilon_0} + \int_{\varepsilon_0}^b \right) |k'(x)|\, dx$$

whence (7.1) follows:

$$\int_\varepsilon^b |k(x) - k(x-\varepsilon)|\, dx \le -\int_0^\varepsilon dt \int_t^{\varepsilon_0} k'(x)\, dx + \varepsilon \int_{\varepsilon_0}^b |k'(x)|\, dx$$

$$= c\varepsilon + \int_0^\varepsilon k(t)\, dt - \varepsilon k(\varepsilon_0) \le c\varepsilon + \int_0^\varepsilon k(t)\, dt.$$

It suffices to observe that (2.13) follows from (7.1) in view of properties (2.10) and (2.12).

7.2 Approximating property of the family $N_\varepsilon(x)$.

Lemma 7.1. *If the Sonine kernel $k(x)$ satisfies assumptions (2.7) and (2.8), then $\widetilde{N}_\varepsilon(x)$ has the properties of the identity approximation kernel:*

$$\sup_{0<\varepsilon<\varepsilon_0}\int_0^\infty |\widetilde{N}_\varepsilon(x)|\,dx = M < \infty, \tag{7.2}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \widetilde{N}_\varepsilon(t)\,dt = 1 \tag{7.3}$$

and

$$\lim_{\varepsilon\to 0}\int_\delta^\infty \left|\widetilde{N}_\varepsilon(s)\right| ds = 0, \quad \delta > 0. \tag{7.4}$$

Proof. By (4.2), we have

$$\int_0^\infty \left|\widetilde{N}_\varepsilon(s)\right| ds = \ell(\varepsilon)\int_0^\varepsilon k(s)\,ds + \int_\varepsilon^b |N_\varepsilon(s)|\,ds$$

for $0 < \varepsilon < \varepsilon_0$. Here the first term is bounded for $0 < \varepsilon \le \varepsilon_0$ by (2.10). The second term is estimated by means of (4.4) as follows

$$\int_\varepsilon^b |N_\varepsilon(s)|\,ds \le \ell(\varepsilon)\int_\varepsilon^b |k(s)-k(s-\varepsilon)|\,ds + \int_\varepsilon^b ds\int_{s-\varepsilon}^s |\ell(s-t)k'(t)|\,dt.$$

Here the first term is bounded by (2.13), while the second one is reduced after the change of order of integration to

$$\int_0^\varepsilon |k'(t)|\,dt\int_{\varepsilon-t}^\varepsilon |\ell(s)|\,ds + \int_\varepsilon^b |k'(t)|\,dt\int_0^\varepsilon \ell(s)\,ds$$

which is also uniformly bounded in view of (2.15) and (2.14).

To verify (7.3), we make use of representation (4.3) so that

$$\int_0^\infty \widetilde{N}_\varepsilon(s)\,ds = \ell(\varepsilon)\int_0^b [k(s) - k_+(s-\varepsilon)]\,ds + \int_\varepsilon^b \frac{d}{ds}A_\varepsilon(s)\,ds \tag{7.5}$$

where we denoted

$$A_\varepsilon(s) = \int_\varepsilon^s \ell(t)\,k(s-t)\,dt, \qquad s > \varepsilon.$$

For the first term on the right-hand side of (7.5) we have

$$\left|\ell(\varepsilon)\int_0^b [k(s) - k_+(s-\varepsilon)]\,ds\right| = \left|\ell(\varepsilon)\int_{b-\varepsilon}^b k(s)\,ds\right| \le c\varepsilon\ell(\varepsilon) \tag{7.6}$$

since $k(t)$ is continuous beyond the origin, see (2.8). For the second term in (7.5) we have

$$\int_{\varepsilon}^{b} \frac{d}{ds} A_{\varepsilon}(s)\, ds = A_{\varepsilon}(b) = 1 - \int_{0}^{\varepsilon} \ell(t) k(b-t)\, dt \tag{7.7}$$

by condition (2.1), which obviously tends to 1 as $\varepsilon \to 0$, which proves (7.3).

Finally, to prove (7.4), we make use of representation (4.4) taking into account that $\varepsilon < \delta < s$ and obtain

$$\lim_{\varepsilon\to 0} \int_{\delta}^{\infty} \left| \widetilde{N}_{\varepsilon}(s) \right| ds \leq \lim_{\varepsilon\to 0} \int_{\delta}^{b} \ell(\varepsilon) \left| k(s) - k(s-\varepsilon) \right| ds$$

$$+ \lim_{\varepsilon\to 0} \int_{\delta}^{b} ds \int_{0}^{\varepsilon} \ell(t) \left| k'(s-t) \right| dt =: \lim_{\varepsilon\to 0} I_{\varepsilon}^{1} + \lim_{\varepsilon\to 0} I_{\varepsilon}^{2}.$$

For the term I_{ε}^{1} we have

$$I_{\varepsilon}^{1} \leq \ell(\varepsilon) \int_{\varepsilon}^{b} |k(s) - k(s-\varepsilon)|\ ds \leq c\varepsilon\ell(\varepsilon) \tag{7.8}$$

as in (7.6). Then $\lim_{\varepsilon\to 0} I_{\varepsilon}^{1} = 0$ in view of (2.12).

For I_{ε}^{2} we have

$$I_{\varepsilon}^{2} \leq \int_{0}^{\varepsilon} \ell(t)\, dt \int_{\delta-t}^{b} |k'(s)|\, ds \leq c(\delta) \int_{0}^{\varepsilon} \ell(t)\, dt \longrightarrow 0$$

where $c(\delta) = \int_{\frac{\delta}{2}}^{b} |k'(s)|\, ds$ under the assumption that $\varepsilon \leq \frac{\delta}{2}$. □

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