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Hardy inequality in the generalized Lebesgue spaces

by

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Abstract

The Hardy type inequality

$$\left\| |x-x_0|^{\beta-\alpha} \int\limits_{\Omega} \frac{f(y)\,dy}{|y-x_0|^{\beta}|x-y|^{n-\alpha}} \right\|_{L^{p(\cdot)}(\Omega)} \leq C\,\|f\|_{L^{p(\cdot)}(\Omega)}\,, \quad 0<\alpha<n, \qquad x_0 \in \overline{\Omega}$$

is proved for the spaces $L^{p(\cdot)}(\Omega)$ with variable exponent $p(x)$ in the case of bounded domains Ω in R^n, $-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}$.

Key words: *Hardy inequality, Lebesgue spaces with variable exponents, maximal function*

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1. Introduction

The classical Hardy inequality ([9]) for fractional integrals states that

$$\left\| x^{\beta-\alpha} \int\limits_0^x \frac{f(y)\,dy}{y^{\beta}(x-y)^{1-\alpha}} \right\|_{L^p(0,b)} \leq C\,\|f\|_{L^p((0,b))}\,, \qquad 0<\alpha<1,$$

where $\alpha - \frac{1}{p} < \beta < \frac{1}{q}$ $\frac{1}{p}+\frac{1}{q}=1$ and $0<b\leq\infty$. Its generalization

$$\int_{\mathbb{R}^n} |x|^{\mu}\,|I^{\alpha}\varphi(x)|^p\;dx \leq C\int_{\mathbb{R}^n} |x|^{\gamma}|\varphi(x)|^p\;dx \tag{1.1}$$

for the n-dimensional fractional integration (Riesz potential operator)

$$I^{\alpha}\varphi(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{\varphi(y)\, dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n, \tag{1.2}$$

where $\gamma_n(\alpha) = \frac{2^{\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$ (see for instance [20], p.37), was proved by Stein-Weiss [23] under the natural assumptions on the parameters:

$$1 \le p < \infty, \qquad \alpha > 0, \qquad \alpha p - n < \gamma < n(p-1), \qquad \mu = \gamma - \alpha p. \tag{1.3}$$

We prove a Stein-Weiss-type generalization of the Hardy type inequality for the Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent $p(x)$, in the case of bounded domains Ω, see Theorem A below.

Nowadays there is an evident increase of interest to the theory of these spaces and Sobolev type spaces $W^{m,p(\cdot)}(\Omega)$ generated by them, as well as to the operator theory in these spaces. This is caused by possible applications to models with non-standard local growth (in elasticity theory, fluid mechanics, differential equations, see for example [17], [5] and references therein) and is based on recent breakthrough result on boundedness of the maximal Hardy-Littlewood operator in these spaces. We refer, for example, to the papers [1], [2], [3], [4], [14], [15], [11], [12], [10], [13], [16], [18], [19], [21] and references therein in connection with the generalized Lebesgue spaces.

The boundedness statement of Theorem A adjoins to weighted variable exponent estimates obtained in [14], see also [10] and [11].

It is worthwhile emphasizing that in the result of Theorem A the bounds obtained for the weight power function $|x - x_0|^{\beta}$ depend only on the value of the exponent $p(x)$ at the point x_0 to which the weight function is fixed.

2. Preliminaries.

The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent is introduced as a set of functions for which the following modular is finite

$$I_p(f) := \int_{\Omega} |f(x)|^{p(x)}\, dx < \infty, \qquad \Omega \subseteq \mathbb{R}^n. \tag{2.1}$$

When $1 \le p(x) \le P < \infty$ for $x \in \Omega$, this is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0: \ \ I_p\Big(\frac{f}{\lambda}\Big) \le 1\right\}.$$

The basics on the spaces $L^{p(\cdot)}$ may be found in [8], [16], [17], [18], [19], [21].

We make use of the following result on the boundedness of the weighted maximal function

$$M^\beta f(x) = |x - x_0|^\beta \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} \frac{|f(y)|}{|y - x_0|^\beta}\, dy, \tag{2.2}$$

in the spaces with variable exponent proved in [10], [11] (see also [14]) (the non-weighted case is due to L.Diening [2], [4]).

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $p(x)$ satisfy conditions*

$$1 < p_0 \le p(x) \le P < \infty, \ \ x \in \overline{\Omega} \tag{2.3}$$

and

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{|x-y|}}\,, \ \ |x - y| \le \frac{1}{2}, \ \ x,\, y \in \overline{\Omega}. \tag{2.4}$$

Then the operator M^β with $x_0 \in \overline{\Omega}$ is bounded in $L^{p(x)}(\Omega)$ if

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}\,. \tag{2.5}$$

3. The main statement.

In Theorem below we assume that $0 < \alpha < n$ and $x_0 \in \overline{\Omega}$. Note that a weaker estimate of the type (3.1) with $|x - x_0|^\beta$ instead of $|x - x_0|^{\beta-\alpha}$ was obtained in [14], see also [10] and [11], and the case $n = 1$ and $\beta = 0$ was proved in [7].

Theorem A. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $p(x)$ satisfy conditions (2.3) and (2.4). Then the Hardy-type inequality is valid*

$$\left\| |x - x_0|^{\beta-\alpha} \int\limits_\Omega \frac{f(y)\, dy}{|y - x_0|^\beta |x - y|^{n-\alpha}} \right\|_{p(\cdot)} \le c\, \|f\|_{p(\cdot)} \tag{3.1}$$

for all β in the interval

$$\alpha - \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}\,. \tag{3.2}$$

Proof.

For simplicity we take $x_0 = 0 \in \overline{\Omega}$. We may consider non-negative functions $f(y)$ and assume that $f(y)$ is continued as zero outside the domain Ω.

We denote

$$I^{\alpha}_{\beta} f(x) = |x|^{\beta-\alpha} \int\limits_{\Omega} \frac{f(y)\,dy}{|y|^{\beta}|x-y|^{n-\alpha}}. \tag{3.3}$$

The following pointwise estimate is valid

$$I^{\alpha}_{\beta} f(x) \le cM^{\beta} f(x) + Bf(x), \tag{3.4}$$

where $M^{\beta} f(x)$ is the weighted maximal function (2.2), c is an absolute constant not depending on x, f (and x_0, when writing this estimate for any point $x_0 \in \overline{\Omega}$) and

$$Bf(x) = |x|^{\beta-\alpha} \int\limits_{|y-x|\ge 2|x|} \frac{f(y)\,dy}{|y|^{\beta}|x-y|^{n-\alpha}}. \tag{3.5}$$

Indeed,

$$I^{\alpha}_{\beta} f(x) = |x|^{\beta-\alpha} \int\limits_{|y-x|\le 2|x|} \frac{f(y)\,dy}{|y|^{\beta}|x-y|^{n-\alpha}} + |x|^{\beta-\alpha} \int\limits_{|y-x|\ge 2|x|} \frac{f(y)\,dy}{|y|^{\beta}|x-y|^{n-\alpha}}$$

$$:= A_{\beta} f(x) + Bf(x). \tag{3.6}$$

To show that the operator $A_{\beta} f(x)$ admits the pointwise estimate

$$A_{\beta} f(x) \le M^{\beta} f(x), \tag{3.7}$$

we observe that it has the form

$$A_{\beta} f(x) = \frac{|x|^{\beta}}{|x|^{n}} \int\limits_{\frac{|x-y|}{|x|}\le 2} \left(\frac{|x-y|}{|x|}\right)^{\alpha-n} \varphi(y)\,dy$$

$$= |x|^{\beta} \frac{1}{\varepsilon^{n}} \int\limits_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) \varphi(y)\,dy \tag{3.8}$$

where we temporarily denoted $\varepsilon = |x|, \;\; \varphi(y) = \frac{f(y)}{|y|^\beta}$ and

$$k(x) = \begin{cases} |x|^{\alpha-n}, & |x| \leq 2 \\ 0, & |x| \geq 2 \end{cases}$$

Since the kernel $k(x)$ is radial, decreasing and integrable, the dilation $\frac{1}{\varepsilon^n} \int\limits_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) \varphi(y)\, dy$ may be dominated by the maximal operator, see [22] or [6], p. 31-32,

$$\frac{1}{\varepsilon^n} \int\limits_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) \varphi(y)\, dy \leq M\varphi(x)$$

where $M\varphi$ stands for $M^0\varphi = M^\beta\varphi\Big|_{\beta=0}$ (observe that this pointwise estimation admits a possibility for ε to depend on the point x, see the proof in [22], [6]). Therefore, from (3.8)

$$A_\beta f(x) \leq c|x|^\beta M\varphi(x) = cM^\beta f(x). \tag{3.9}$$

Thus, estimate (3.7) has been obtained.

Then, by Theorem 2.1, the operator A_β is bounded in the space $L^{p(\cdot)}(\Omega)$ if $-\frac{n}{p(0)} < \beta < \frac{n}{q(0)}$ which is satisfied by (3.11).

It remains to prove the boundedness of the operator B. Obviously, $|x - y| \geq 2|x|$ implies that

$$|y| \geq |x-y| - |x| \geq |x-y| - \frac{|x-y|}{2} = \frac{|x-y|}{2}.$$

Therefore,

$$Bf(x) = |x|^{\beta-\alpha} \int\limits_{|y-x|\leq 2|y|} \frac{f(y)\, dy}{|y|^\beta |x-y|^{n-\alpha}} := B_1 f(x).$$

The operator conjugate to B_1 has the form

$$B_1^* g(x) = |x|^{-\beta} \int\limits_{|y-x|\leq 2|x|} \frac{g(y)\, dy}{|y|^{\alpha-\beta} |x-y|^{n-\alpha}}$$

which is nothing else but the operator of the familiar type A_β which we had in (3.6), namely

$$B_1^* = A_{\alpha-\beta}.$$

According to (3.7) and Theorem 2.1, the operator B_1^* is bounded in the conjugate space $L^{q(\cdot)}(\Omega)$ if and only if $-\frac{n}{q(0)} < \alpha - \beta < \frac{n}{p(0)}$, that is, $\alpha - \frac{n}{p(0)} < \beta < \alpha + \frac{n}{q(0)}$, which is satisfied by (3.11). Therefore, the operator B_1 is bounded in $L^{p(\cdot)}(\Omega)$ and then B is bounded in this space. $\square$

Remark 3.1. Analysis of the proof of Theorem A shows that it is also valid in the case when the order α is variable as well, in the form

$$\left\| |x - x_0|^{\beta - \alpha(x_0)} \int\limits_{\Omega} \frac{f(y)\, dy}{|y - x_0|^{\beta} |x - y|^{n - \alpha(x)}} \right\|_{p(\cdot)} \leq c \, \|f\|_{p(\cdot)} \tag{3.10}$$

for all β in the interval

$$\alpha(x_0) - \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)} \tag{3.11}$$

if $\inf\limits_{x \in \Omega} \alpha(x) > 0$ and $\alpha(x)$ satisfies the same logarithmic condition as $p(x)$ in (2.4).

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