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Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces with variable exponent

by

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Dedicated to Professor Paul Butzer on the occasion of his 75th birthday

Abstract

The Hardy type inequality

$$\left\| |x-x_0|^{\beta-\alpha} \int\limits_{\Omega} \frac{f(y)\, dy}{|y-x_0|^{\beta}|x-y|^{n-\alpha}} \right\|_{L^{p(\cdot)}(\Omega)} \leq C\, \|f\|_{L^{p(\cdot)}(\Omega)}, \qquad 0<\alpha<n, \qquad x_0 \in \overline{\Omega}$$

is proved for the spaces $L^{p(\cdot)}(\Omega)$ with variable exponent $p(x)$ in the case of bounded domains Ω in R^n, $-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}$.

Key words: *Hardy inequality, Lebesgue spaces with variable exponents, maximal function*

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1 Introduction

For the Riesz potential

$$I^{\alpha} f(x) \;=\; \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-\alpha}} \tag{1.1}$$

the following weighted $p \to q$-estimate is known

$$\left(\int\limits_{\mathbb{R}^n} |I^{\alpha} f(x)|^q \;\; |x|^{\mu}\, dx \right)^{\frac{1}{q}} \leq C \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \;\; |x|^{\gamma}\, dx \right)^{\frac{1}{p}},$$

where $0<\alpha<n,\ 1<p<\infty,\ 1<q<\infty,\ \frac{1}{p}-\frac{\alpha}{n}\le\frac{1}{q}\le\frac{1}{p},$ and

$$\frac{\mu+n}{q}=\frac{\gamma+n}{p}-\alpha, \qquad \alpha p-n<\gamma<n(p-1), \tag{1.2}$$

which is due to H.G.Hardy and J.E.Littlewood [9] in the one-dimensional case and to E.M.Stein and G.Weiss [22] in the case $n\ge 1$.

We establish such type of inequality in the limiting case $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ for the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent $p(x)$ over bounded domains in $\mathbb{R}^n$, see Theorem A in Section 3. We refer to the papers [8], [16], [20], [21] on the spaces $L^{p(\cdot)}$ and to the papers [2], [3], [4], [5], [6], [7], [10], [12], [13], [14], [17] on the recent progress in the study of the operator theory and harmonic analysis in $L^{p(\cdot)}$, the theory of these spaces and the corresponding Sobolev spaces $W^{m,p(\cdot)}$ with variable exponent being rapidly developing last decade, influenced by applications, see [18] and references therein.

N o t a t i o n:
S_{n-1} is the unit sphere in $\mathbb{R}^n$,
$|S_{n-1}|$ is its measure;
$e_1=(1,0,0,...,0)$;
inf and sup will everywhere stand for $esssup$ and $essinf$;
$p_0=\inf\limits_{x\in\Omega}p(x), \quad P=\sup\limits_{x\in\Omega}p(x),\ \overline{p}(x)=\frac{p(x)}{p(x)-1}$;
by c or C we denote various positive absolute constants not depending on the parameters involved.

2 Preliminaries.

When considering the operator I^α in the spaces $L^{p(\cdot)}(\Omega)$, we admit that its order α may be also variable, so we deal with the operator

$$I^{\alpha(\cdot)}f(x) \ = \int\limits_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}, \qquad x\in\Omega. \tag{2.1}$$

We refer to [8], [16], [20], [21] for details on the spaces $L^{p(\cdot)}(\Omega)$, but give the basic definitions. Let Ω be a bounded domain in $\mathbb{R}^n$ and $p:\Omega\to[1,\infty)$ a measurable function on Ω. By $L^{p(\cdot)}(\Omega)$ we denote the set of all measurable functions f on Ω such that

$$\int\limits_\Omega |f(x)|^{p(x)}\,dx<\infty.$$

Under the condition $1\le p(x)\le P<\infty$ on Ω, this is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0:\ \int\limits_\Omega\left|\frac{f(x)}{\lambda}\right|^{p(x)}\,dx\le 1\right\}. \tag{2.2}$$

The notation $L^{p(\cdot)}(\Omega,\rho)$ will stand for the corresponding weighted space

$$L^{p(\cdot)}(\Omega,\rho)=\left\{f:[\rho(x)]^{\frac{1}{p(x)}}\in L^{p(\cdot)}(\Omega)\right\},$$

$$\|f\|_{L^{p(\cdot)}(\Omega,\rho)}=\inf\left\{\lambda>0:\int\limits_{\Omega}\rho(x)\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\},\tag{2.3}$$

where $\rho(x)\geq 0$ a.e. and $|\{x\in\Omega:\ \rho(x)=0\}|=0$.

Definition 2.1. By $\mathbb{P}(\Omega)$ we denote the set of functions $p:\overline{\Omega}\to(1,\infty)$ satisfying the conditions

$$1<p_0\leq p(x)\leq P<\infty \qquad \text{on} \qquad \Omega,\tag{2.4}$$

$$|p(x)-p(y)|\leq\frac{A}{ln\frac{1}{|x-y|}} \qquad \text{for all} \quad x,y\in\overline{\Omega} \qquad \text{with} \qquad |x-y|\leq\frac{1}{2},\tag{2.5}$$

where $A>0$ does not depend on x and y.

Observe that condition (2.5) may be also written in the form

$$|p(x)-p(y)|\leq\frac{NA}{ln\frac{N}{|x-y|}} \qquad x,y\in\overline{\Omega},\tag{2.6}$$

where $N=2\ diam\ \Omega$.

By $\overline{p}(x)$ we denote the conjugate exponent,

$$\frac{1}{p(x)}+\frac{1}{\overline{p}(x)}\ =\ 1.$$

For the conjugate space $\left[L^{p(\cdot)}(\Omega,\rho)\right]^*$ we have

$$\left[L^{p(\cdot)}(\Omega,\rho)\right]^*=L^{\overline{p}(\cdot)}\left(\Omega,[\rho(x)]^{\frac{1}{1-p(x)}}\right)\tag{2.7}$$

which is an immediate consequence of the fact that $\left[L^{p(\cdot)}(\Omega)\right]^*=L^{\overline{p}(\cdot)}(\Omega)$ under (2.4), see [16], [21].

From the Hölder inequality for the $L^{p(\cdot)}$-spaces

$$\left|\int_{\Omega}u(x)v(x)\,dx\right|\leq k\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{\overline{p}(\cdot)}(\Omega)}, \qquad \frac{1}{p(x)}+\frac{1}{\overline{p}(x)}\equiv 1,$$

it follows that

$$\left|\int_{\Omega}u(x)v(x)\,dx\right|\leq k\|u\|_{L^{\overline{p}(\cdot)}\left(\Omega,[\rho(x)]^{\frac{1}{1-p(x)}}\right)}\|v\|_{L^{p(\cdot)}(\Omega,\rho)}.\tag{2.8}$$

Let

$$M^{\delta}f(x)=|x-x_0|^{\delta}\sup_{r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)\cap\Omega}\frac{|f(y)|}{|y-x_0|^{\delta}}\,dy\tag{2.9}$$

be the weighted maximal function, $x_0 \in \overline{\Omega}$. The following statement was proved in [10], [11], see also [15].

Theorem 2.2. *Let Ω be a bounded domain and $p \in \mathbb{P}$. The operator M^δ with $x_0 \in \Omega$ is bounded in $L^{p(x)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \delta < \frac{n}{\overline{p}(x_0)}\,. \tag{2.10}$$

If $x_0 \in \partial\Omega$, condition (2.10) is sufficient for the boundedness of M^δ. If $x_0 \in \partial\Omega$ and $|\{y \in \Omega : \quad r < |y - x_0| < 2r\}| \sim r^n$, then condition (2.10) is also necessary for the boundedness of M^δ.

3 The main statement.

We assume that the exponent $\alpha(x)$ in (2.1) satisfies the assumptions

$$\inf_{x\in\Omega} \alpha(x) > 0 \qquad \text{and} \qquad \sup_{x\in\Omega} \alpha(x)p(x) < n \tag{3.1}$$

and the logarithmic condition (2.5), that is,

$$|\alpha(x) - \alpha(y)| \le \frac{A_1}{ln\frac{1}{|x-y|}}, \qquad x, y \in \overline{\Omega} \qquad |x - y| \le \frac{1}{2}. \tag{3.2}$$

Theorem A. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$, let $p \in \mathbb{P}(\Omega)$ and α satisfy conditions (3.1). Then the following estimate is valid for operator (2.1)*

$$\left\| I^{\alpha(\cdot)} f \right\|_{L^{q(\cdot)}(\Omega,|x-x_0|^{\mu})} \le C \left\| f \right\|_{L^{p(\cdot)}(\Omega,|x-x_0|^{\gamma})} \tag{3.3}$$

where

$$\frac{1}{q(x)} \equiv \frac{1}{p(x)} - \frac{\alpha(x)}{n}, \tag{3.4}$$

$$\alpha(x_0)p(x_0) - n < \gamma < n[p(x_0) - 1] \tag{3.5}$$

and

$$\mu = \frac{q(x_0)}{p(x_0)}\,\gamma. \tag{3.6}$$

The proof of Theorem A will be based on the following crucial points:

1) the boundedness of the weighted maximal operator in the spaces $L^{p(\cdot)}(\Omega)$ proved in [10], [15];

2) Hedberg's approach to prove the non-weighted Sobolev theorem with constant p and α based on the domination of the Riesz potential by the maximal function;

3) a development of the technique of estimation of $L^{p(\cdot)}$-norms of power functions of distance truncated to exterior of a ball, given in [19] in the non-weighted case, for the weighted case.

The above mentioned development constitutes the essential body of the paper, see the next section. Theorem A itself is proved in Section 5.

Remark. *When $0 \le \gamma < n[p(x_0) - 1]$, condition (3.2) may be weakened: it suffices to assume its validity only at the point to which the weight is fixed, that is, $|\alpha(x) - \alpha(x_0)| \le \frac{A_1}{ln \frac{1}{|x-x_0|}}$, $|x - x_0| \le \frac{1}{2}$. Indeed, condition (3.2) is used in the proof of Theorem A only in (5.8) (where there was chosen $x_0 = 0$). As for negative values of γ, condition (3.2) was used in its generality when passing to the conjugate operator in Subsection 5.2.*

4 Estimation of weighted $L^{p(\cdot)}$-norms of power functions of distance truncated to exterior of a ball

4.1 Setting of the problem.

Let $\chi_r(x) = \begin{cases} 1, & \text{if } |x| > r \\ 0, & \text{if } |x| < r \end{cases}$ be the characteristic function of the ball centered at the origin of radius r and let

$$g_\beta(x, y, r) = |c - y|^{\beta(x)} \chi_r(x - y) \tag{4.1}$$

where $\beta(x)$ in future will be chosen as $\beta(x) = \alpha(x) - n$.

We are interested in estimation of the weighted norms

$$n_{\beta,\nu,p}(x, r) = \|g_\beta(x, y, r)\|_{L^{p(\cdot)}\left(\Omega, |y|^{\nu(y)}\right)} \tag{4.2}$$

(taken with respect to y) as $r \to 0$, where we suppose that $0 \in \overline{\Omega}$ and $\nu(y)$ is some variable exponent. In future, we will need this norm with $p(\cdot)$ replaced by $\overline{p}(\cdot)$ and $\nu(y)$ chosen as $\nu(y) = \frac{\gamma}{1-p(y)}$.

The next subsections are aimed to derive the required estimate of $n_{\beta,\nu,p}(x, r)$ as $r \to 0$.

4.2 An auxiliary estimate.

Lemma 4.1. *Let $e_1 = (1, 0, ..., 0)$. For the integral*

$$J_{a,b}(t) = \int\limits_{|y|<t} \frac{dy}{|y|^a |y - e_1|^b}, \qquad 0 < t < \infty, \tag{4.3}$$

where $a < n$, $b < n$, $a + b < n$, the following estimate is valid

$$J_{a,b}(t) \le C \frac{6^{|a|+|b|}}{(n-a)(n-b)(n-a-b)} \frac{t^{n-a}}{(1+t)^b}, \qquad 0 < t < \infty, \tag{4.4}$$

where $C > 0$ is an absolute constant not depending on t, a and b (depending only on n).

Proof. We separate the cases $0 < t < 1/2,\ 1/2 < t < 2$ and $2 < t < \infty$.

The case $0 < t < 1/2$. We have $|y - e_1| \le t + 1 < 2$ and $|y - e_1| \ge 1 - |y| \ge 1 - t \ge \frac{1}{2}$. Therefore, $\frac{1}{|y-e_1|^b} \le 2^{|b|}$ and then $J_{a,b}(t) \le 2^{|b|} \int\limits_{|y|<t} \frac{dy}{|y|^a}$ which gives the estimate

$$J_{a,b}(t) \le C \frac{2^{|b|}}{n-a}\, t^{n-a}, \qquad 0 < t < 1/2 \tag{4.5}$$

(with $C = |S_{n-1}|$).

The case $1/2 < t < 2$. We have

$$J_{a,b}(t) = \int\limits_{|y|<1/2} \frac{dy}{|y|^a |y-e_1|^b} + \int\limits_{1/2<|y|<t} \frac{dy}{|y|^a |y-e_1|^b} = I_1 + I_2.$$

Making use of (4.5) with $t = 1/2$, we see that

$$I_1 \le C \frac{2^{|b|}}{n-a}. \tag{4.6}$$

In the integral I_2 we have $\frac{1}{|y|^a} \le 2^{|a|}$ so that

$$I_2 \le 2^{|a|} \int\limits_{1/2<|y|<|t} \frac{dy}{|y-e_1|^b} = 2^{|a|} \int\limits_{1/2<|y-e_1|<|t} \frac{dy}{|y|^b}.$$

Observe that $|y - e_1| \le t$ implies that $|y| \le t+1 \le 3$ so that $\{y : |y-e_1| \le t\} \subset \{y : |y| \le 3\}$ and then

$$I_2 \le C \frac{2^{|a|} 3^{|b|}}{n-b}. \tag{4.7}$$

From (4.6)-(4.7) we have

$$J_{a,b}(t) \le C \left(\frac{2^{|b|}}{n-a} + \frac{2^{|a|} 3^{|b|}}{n-b} \right), \qquad \frac{1}{2} \le t \le 2. \tag{4.8}$$

The case $2 < t < \infty$. We have

$$J_{a,b}(t) = \int\limits_{|y|<2} \frac{dy}{|y|^a |y-e_1|^b} + \int\limits_{2<|y|<t} \frac{dy}{|y|^a |y-e_1|^b} = I_3 + I_4.$$

By (4.8) with $t = 2$ we have

$$I_3 \le C \left(\frac{2^{|b|}}{n-a} + \frac{2^{|a|} 3^{|b|}}{n-b} \right). \tag{4.9}$$

To estimate I_4, we observe that $\frac{|y|}{2} \le |y-e_1| \le 2|y|$, so that

$$I_4 \le 2^{|b|} \int\limits_{2<|y|<t} \frac{dy}{|y|^{|a|+|b|}} \le 2^{|b|} \int\limits_{|y|<t} \frac{dy}{|y|^{|a|+|b|}} = |S_{n-1}| \frac{2^{|b|}}{n-a-b}\, t^{n-a-b}. \tag{4.10}$$

Then from (4.9)-(4.10)

$$J_{a,b}(t) \le C\left(\frac{2^{|b|}}{n-a} + \frac{2^{|a|}3^{|b|}}{n-b} + \frac{2^{|b|}}{n-a-b}\, t^{n-a-b}\right), \qquad t \ge 2. \tag{4.11}$$

Unifying estimates (4.5), (4.8) and (4.11), we get (4.4). □

Corollary. *Let* $0 < r < \infty,\ 0 \in \overline{\Omega}, \gamma > -n$ *and a function* $h(x)$ *defined on* Ω *satisfy the conditions*

$$\sup_{x\in\Omega} |h(x)| := H < \infty, \tag{4.12}$$

$$\sup_{x\in\Omega} [h(x)+n] := -d_0 < 0, \tag{4.13}$$

and

$$\sup_{x\in\Omega} [h(x)+n+\gamma] := -d_1 < 0. \tag{4.14}$$

Then

$$\int\limits_{|y-x|>r} |y-x|^{h(x)} |y|^{\gamma}\, dy \ \le\ C r^{h(x)+n} (r+|x|)^{\gamma}, \qquad x \in \Omega, \tag{4.15}$$

where $C > 0$ *does not depend on* x *and* r.

Proof. We transform the integral $A(x,r) := \int\limits_{|y-x|>r} |y-x|^{h(x)} |y|^{\gamma}\, dy$ by the dilatation change of variables $y = |x|z$:

$$A(x,r) = |x|^{h(x)+n+\gamma} \int\limits_{|z-\frac{x}{|x|}|>\frac{t}{|x|}} \left|z - \frac{x}{|x|}\right|^{h(x)} |z|^{\gamma}\, dz.$$

The translation $z - \frac{x}{|x|} \to z$ and rotation arguments yield

$$A(x,r) = |x|^{h(x)+n+\gamma} \int\limits_{|z|>\frac{t}{|x|}} |z-e_1|^{\gamma}\, |z|^{h(x)}\, dz$$

where $e_1 = (1, 0, ..., 0)$. Finally, after the inversion change of variables

$$z = \frac{u}{|u|^2} \qquad \text{with} \qquad |z| = \frac{1}{|u|}, \qquad |z-e_1| = \frac{|u-e_1|}{|u|} \qquad \text{and} \quad dz = \frac{du}{|u|^{2n}}$$

we get

$$A(x,r) = |x|^{h(x)+n+\gamma} \int\limits_{|u|<\frac{|x|}{r}} \frac{|u-e_1|^{-\gamma}\, du}{|u|^{h(x)+2n+\gamma}}. \tag{4.16}$$

We arrived at the integral $J_{a,b}(t)$ estimated in (4.4). Making use of (4.4) with

$$t = \frac{|x|}{r}, \quad a = h(x) + 2n + \gamma, \quad b = -\gamma,$$

we arrive at (4.15) after easy evaluations with conditions (4.12)-(4.14) taken into account.
□

4.3 Estimation of $n_{\beta,\nu,p}(x,r)$.

Theorem 4.2. *Let Ω be a bounded domain, $0 \in \overline{\Omega}$, let $p \in \mathbb{P}(\Omega)$, $\nu \in L^\infty(\Omega)$ and $\beta \in L^\infty(\Omega)$ and let also $\nu(x)$ satisfy the logarithmic condition at the origin*

$$|\nu(y) - \nu(0)| \le \frac{A_2}{ln\frac{1}{|y|}}, \qquad y \in \Omega, \qquad |y| \le \frac{1}{2} \tag{4.17}$$

and let $\nu(0) > -n$. If

$$\sup_{x\in\Omega}[\beta(x)p(x) + n] := -d_0 < 0, \tag{4.18}$$

$$\sup_{x\in\Omega}[\beta(x)p(x) + \nu(x) + n] := -d_1 < 0, \tag{4.19}$$

and

$$\sup_{x\in\Omega}[\beta(x)p(x) + \nu(0) + n] := -d_2 < 0, \tag{4.20}$$

then

$$n_{\beta,\nu,p}(x,r) \le C r^{\beta(x)+\frac{n}{p(x)}} (r + |x|)^{\frac{\nu(x)}{p(x)}}. \tag{4.21}$$

for all $x \in \Omega$, $0 < r < diam\ \Omega$, where $C > 0$ does not depend on x and r.

Proof. For the norm $n_{\beta,\nu,p} = n_{\beta,\nu,p}(x,r)$ as defined in (4.2) we have

$$\int\limits_{\substack{y\in\Omega \\ |y-x|>r}} \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}}\right)^{p(y)} |y|^{\nu(y)}\, dy = 1 \tag{4.22}$$

by definition (2.3).

1st step: values $n_{\beta,\nu,p}(x,r) \ge 1$ are only of interest. First we observe that the right-hand side of (4.21) is bounded from below:

$$\inf_{\substack{x\in\Omega \\ 0<r<diam\ \Omega}} r^{\beta(x)+\frac{n}{p(x)}} (r + |x|)^{\frac{\nu(x)}{p(x)}} := c_1 > 0. \tag{4.23}$$

To verify (4.23), suppose first that $\nu(x) \geq 0$. Then by (4.19)

$$r^{\beta(x)+\frac{n}{p(x)}}(r+|x|)^{\frac{\nu(x)}{p(x)}} \geq r^{\beta(x)+\frac{n}{p(x)}+\frac{\nu(x)}{p(x)}} = r^{-\frac{|\beta(x)p(x)+\nu(x)+n|}{p(x)}} \geq D^{-\frac{|\beta(x)p(x)+\nu(x)+n|}{p(x)}}$$

where $D = diam\ \Omega$. The right hand side here is bounded from below since $\frac{|\beta(x)p(x)+\nu(x)+n|}{p(x)} \in L^\infty(\Omega)$. When $\nu(x) \leq 0$, we observe that

$$r^{\beta(x)+\frac{n}{p(x)}}(r+|x|)^{\frac{\nu(x)}{p(x)}} \geq r^{\beta(x)+\frac{n}{p(x)}} D^{\frac{\nu(x)}{p(x)}} = r^{-\left|\beta(x)+\frac{n}{p(x)}\right|} D^{\frac{\nu(x)}{p(x)}}$$

where (4.18) was taken into account. The right hand side here is also bounded from below.

From (4.23) we conclude that to prove (4.21), we may suppose that

$$n_{\beta,\nu,p}(x,r) \geq 1.$$

2nd step: small values of r are only of interest. We assume that r is small enough, $0 < r < \varepsilon_0$. To show that this assumption is possible, we have to check that the right-hand side of (4.21) is bounded from below and $n_{\beta,\nu,p}(x,r)$ is bounded from above when $r \geq \varepsilon_0$. The former is obvious, to verify the latter, we observe that from (4.22) it follows that

$$1 \geq \int\limits_{\substack{y\in\Omega \\ |y-x|>\varepsilon_0}} \frac{|y-x|^{\beta(x)p(y)}}{n_{\beta,\nu,p}} |y|^{\nu(y)}\, dy$$

whence

$$n_{\beta,\nu,p}(x,r) \leq \int\limits_{\substack{y\in\Omega \\ |y-x|>\varepsilon_0}} |y-x|^{\beta(x)p(x)}\, |y|^{\nu(0)} u(x,y)v(y)\, dy$$

where $u(x,y) = |y-x|^{\beta(x)[p(y)-p(x)]}$ and $v(y) = |y|^{\nu(y)-\nu(0))}$. By direct estimation of $ln\ u(x,y)$ and $ln\ v(y)$ we obtain that

$$e^{-NAB} \leq u(x,y) \leq e^{NAB}, \qquad x,y \in \Omega \tag{4.24}$$

where N and A are the constants from (3.2) and $B = \sup\limits_{x\in\Omega} |\beta(x)|$, and

$$e^{-c} \leq |y|^{\nu(y)-\nu(0))} \leq e^{c}, \qquad x,y \in \Omega \tag{4.25}$$

with some constant $c > 0$ (one may take $c = max\{2A_2, N \sup\limits_{x\in\Omega} |\nu(x)|\}$, where A_2 is the constant from (4.17)).

Therefore,

$$n_{\beta,\nu,p}(x,r) \leq e^{c+NAB} \int\limits_{\substack{y\in\Omega \\ |y-x|>\varepsilon_0}} |y-x|^{\beta(x)p(x)}\, |y|^{\nu(0)}\, dy \leq e^{c+NAB} \varepsilon_0^{-BP} \int\limits_{\Omega} \frac{dy}{|y|^{\nu(0)}} = const$$

which proves the boundedness of $n_{\beta,\nu,p}(x,r)$ from above.

The value of ε_0 will be chosen later.

3rd step: a rough estimate. First, we derive a weaker estimate

$$n_{\beta,\nu,p}(x,r) \le C r^{\beta(x)} \tag{4.26}$$

which will be used later do obtain the final estimate (4.21). To this end, we note that always $\lambda^{p(y)} \le \lambda^{\inf p(y)} + \lambda^{\sup p(y)}$, so that from (4.22) and (4.25) we have

$$1 \le \int\limits_{\substack{y\in\Omega \\ |y-x|>r}} \left[\left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{p_0} + \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{P} \right] |y|^{\nu(0)}\, dy.$$

Since $|y-x| > r$ and $\beta(x) < 0$, we obtain

$$1 \le \left[\left(\frac{r^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{p_0} + \left(\frac{r^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{P} \right] \int\limits_{y\in\Omega} |y|^{\nu(0)}\, dy.$$

Hence $\left(\frac{r^{\beta(x)}}{n_{\beta,\nu,p}}\right)^{p_0} + \left(\frac{r^{\beta(x)}}{n_{\beta,\nu,p}}\right)^{P} \ge c$ which yields $\frac{r^{\beta(x)}}{n_{\beta,\nu,p}} \ge C$ and we arrive at the estimate in (4.26).

4rd step. We split integration in (4.22) as follows

$$1 = \left(\int\limits_{\Omega_1(x,\varepsilon_0)} + \int\limits_{\Omega_2(x,\varepsilon_0)} + \int\limits_{\Omega_3(x,\varepsilon_0)} \right) \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{p(y)} |y|^{\nu(y)}\, dy := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \tag{4.27}$$

where

$$\Omega_1(x,\varepsilon_0) = \left\{ y \in \Omega : \ r < |y-x| < \varepsilon_0, \frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} > 1 \right\},$$
$$\Omega_2(x,\varepsilon_0) = \left\{ y \in \Omega : \ r < |y-x| < \varepsilon_0, \frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} < 1 \right\},$$
$$\Omega_3(x,\varepsilon_0) = \{ y \in \Omega : \ |y-x| > \varepsilon_0 \}.$$

5th step: Estimation of $\mathcal{I}_1$. We have

$$\mathcal{I}_1 = \int\limits_{\Omega_1(x,\varepsilon_0)} \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{p(x)} |y|^{\nu(y)}\, u_r(x,y)\, dy \tag{4.28}$$

where

$$u_r(x,y) = \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} \right)^{p(y)-p(x)}.$$

The estimate

$$e^{-NA} \leq u_r(x,y) \leq e^{NA}. \tag{4.29}$$

is valid. In its proof below we follow a similar estimation in [19], p. 266. We have

$$|ln\ u_r(x,y)| \leq A \left| \frac{ln\ \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}}\right)}{ln\ \frac{N}{|x-y|}} \right| .$$

Since $\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}} \geq 1$, we obtain

$$|ln\ u_r(x,y)| \leq A \frac{|\beta(x)| ln\ \frac{1}{|y-x|} - ln\ n_{\beta,\nu,p}}{ln\ \frac{N}{|x-y|}} \leq A \frac{|\beta(x)| ln\ \frac{1}{|y-x|}}{ln\ \frac{N}{|x-y|}} \leq AB$$

where $B = \sup\limits_{x\in\Omega} |\beta(x)|$ (without loss of generality we may assume that $N \geq 1$). Hence (4.29) follows.

By (4.29) and (4.25) we obtain from (4.30)

$$\mathcal{I}_1 \leq \frac{C}{n_{\beta,\nu,p}^{p(x)}} \int\limits_{\Omega_1(x,\varepsilon_0)} |y-x|^{\beta(x)p(x)}\ |y|^{\nu(0)}\ dy \leq \frac{C}{n_{\beta,\nu,p}^{p(x)}} \int\limits_{|y-x|>r} |y-x|^{\beta(x)p(x)}\ |y|^{\nu(0)}\ dy. \tag{4.30}$$

Now we make use of the estimate obtained in (4.15) which gives

$$\mathcal{I}_1 \leq \frac{C}{n_{\beta,\nu,p}^{p(x)}} r^{\beta(x)p(x)+n} (r+|x|)^{\nu(0)}. \tag{4.31}$$

The validity of conditions (4.12)-(4.14) under which the estimate (4.15) was obtained, follows from assumptions of our theorem.

6th step: Estimation of $\mathcal{I}_2$ and the choice of ε_0. In the integral $\mathcal{I}_2$ we have

$$\mathcal{I}_2 \leq C \int\limits_{\Omega_2(x,\varepsilon_0)} \left(\frac{|y-x|^{\beta(x)}}{n_{\beta,\nu,p}}\right)^{p_{\varepsilon_0}(x)} |y|^{\nu(0)}\ dy \tag{4.32}$$

where

$$p_{\varepsilon_0}(x) = \min_{|y-x_0|<\varepsilon_0} p(y)$$

and (4.25) was taken into account. Then

$$\mathcal{I}_2 \leq \frac{C}{n_{\beta,\nu,p}^{p_{\varepsilon_0}(x)}} \int\limits_{\Omega_2(x,\varepsilon_0)} |y-x|^{\beta(x)p_{\varepsilon_0}(x)}\ |y|^{\nu(0)}\ dy$$

and consequently

$$\mathcal{I}_2 \leq \frac{C}{n_{\beta,\nu,p}^{p_{\varepsilon_0}(x)}} \int\limits_{|y-x|>r} |y-x|^{\beta(x)p_{\varepsilon_0}(x)}\ |y|^{\nu(0)}\ dy. \tag{4.33}$$

We wish to apply estimate (4.15), but to this end we have to guarantee the validity of conditions (4.12)-(4.14). This may be achieved by a choice of ε_0 sufficiently small so that

$$\beta(x)p_{\varepsilon_0}(x)+n \le -\delta_1 < 0 \qquad \text{and} \qquad \beta(x)p_{\varepsilon_0}(x)+n+\nu(0) \le -\delta_2 < 0$$

which is easily derived from conditions (4.18)-(4.20) and continuity of $p(x)$ (compare with Lemma 1.7 from [19]). Conditions (4.12)-(4.14) being satisfied, we make use of (4.15) and get

$$\mathcal{I}_2 \le \frac{C}{n_{\beta,\nu,p}^{p_{\varepsilon_0}(x)}} r^{\beta(x)p_{\varepsilon_0}(x)+n}(r+|x|)^{\nu(0)} \tag{4.34}$$

where C does not depend on x and r.

7th step: Estimation of $\mathcal{I}_3$. We have

$$\mathcal{I}_3 \le \frac{C}{n_{\beta,\nu,p}^{p_0}}\mathcal{I}_4, \qquad \mathcal{I}_4 = \mathcal{I}_4(x) = \int\limits_{\substack{y\in\Omega \\ |y-x|>\varepsilon_0}} |y-x_0|^{\beta(x)p(y)}|y|^{\nu(y)}\,dy.$$

The integral $\mathcal{I}_4(x)$ is a bounded function of x. Indeed, by (4.24)-(4.25) we obtain

$$\mathcal{I}_4(x) \le C \int\limits_{\substack{y\in\Omega \\ |y-x|>\varepsilon_0}} |y-x_0|^{\beta(x)p(x)}|y|^{\nu(0)}\,dy \le C \int\limits_{|y-x|>\varepsilon_0} |y-x_0|^{\beta(x)p(x)}|y|^{\nu(0)}\,dy$$

which is bounded by (4.15). Therefore,

$$\mathcal{I}_3 \le \frac{C}{n_{\beta,\nu,p}^{p_0}}. \tag{4.35}$$

8th step. Gathering estimates (4.31), (4.34) and (4.35), we have from (4.27)

$$1 \le C_0\left(\frac{r^{\beta(x)p(x)+n}}{n_{\beta,\nu,p}^{p(x)}}(r+|x|)^{\nu(0)} + \frac{r^{\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,\nu,p}^{p_{\varepsilon_0}(x)}}(r+|x|)^{\nu(0)} + \frac{1}{n_{\beta,\nu,p}^{p_0}}\right) \tag{4.36}$$

with a certain constant C_0 not depending on x and r. We may assume that

$$n_{\beta,\nu,p}(x,r) \ge \left(\frac{1}{2C_0}\right)^{\frac{1}{p_0}} := C_1 \tag{4.37}$$

because for those x and r where $n_{\beta,\nu,p}(x,r) \le C_1$ there is nothing to prove, the right-hand side of (4.21) being bounded from below according to (4.23). In the situation (4.37) we derive from (4.36) the inequality

$$1 \le C_0\left(\frac{r^{\beta(x)p(x)+n}}{n_{\beta,\nu,p}^{p(x)}} + \frac{r^{\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,\nu,p}^{p_{\varepsilon_0}(x)}}\right)(r+|x|)^{\nu(0)}. \tag{4.38}$$

Since $n_{\beta,\nu,p}(x,r) \geq 1$ we observe that $\left(\frac{1}{n_{\beta,\nu,p}}\right)^{p_{\varepsilon_0}(x)} \leq \left(\frac{1}{n_{\beta,\nu,p}}\right)^{p(x)}$ and $\left(\frac{r^{\beta(x)}}{n_{\beta,\nu,p}}\right)^{p_{\varepsilon_0}(x)} \leq C\left(\frac{r^{\beta(x)}}{n_{\beta,\nu,p}}\right)^{p(x)}$ by (4.26). Hence,

$$\frac{r^{\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,\nu,p}^{p_{\varepsilon_0}(x)}} \leq \frac{r^{\beta(x)p(x)+n}}{n_{\beta,\nu,p}^{p(x)}}.$$

Therefore, from (4.38) we derive the estimate

$$\frac{r^{\beta(x)p(x)+n}}{n_{\beta,\nu,p}^{p(x)}}(r+|x|)^{\nu(0)} \geq C$$

which yields (4.21), because

$$e^{-c} \leq (r+|x|)^{\frac{\nu(x)-\nu(0)}{p(x)}} \leq e^{C}$$

with some $C > 0$, the latter inequality being easily obtained by estimating $\ln\,(r+|x|)^{\frac{\nu(x)-\nu(0)}{p(x)}}$ with (4.17) taken into account. $\square$

5 Proof of Theorem A

5.1 The case $\gamma \geq 0$.

We base ourselves on the well known Hedberg's approach to reduce the boundedness of the Riesz potential to that of the maximal operator which requires an information about the behaviour of the norms $\|\,|y-x|^{\beta(x)}\|_{L^{p(\cdot)}(\Omega\setminus B(x_0,r))}$ as $r \to 0$, which was obtained in the preceding section.

We have

$$I^{\alpha(\cdot)}f(x) = \int\limits_{\substack{x\in\Omega \\ |x-y|<r}} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}} + \int\limits_{\substack{x\in\Omega \\ |x-y|<r}} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}} := A_r(x) + B_r(x). \tag{5.1}$$

We make use of the inequality

$$|A_r(x)| \;\leq\; \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)}-1}\, Mf(x) \tag{5.2}$$

which is known in the case of $\alpha(x) = const$ (see for instance, [1], p. 54) and remains valid in case it is variable.

By (5.2) and the first condition in (3.1) we have

$$|A_r(x)| \leq cr^{\alpha(x)}Mf(x) \tag{5.3}$$

with some absolute constant $c > 0$ not depending on x and r.

We assume for simplicity that $x_0 = 0$. Let $f(x) \geq 0$ and $\|f\|_{L^{p(\cdot)}(\Omega,|x|^\gamma)} \leq 1$. Applying the Hölder inequality (2.8) in the integral $B_r(x)$, we obtain

$$|B_r(x)| \leq k\; n_{\beta,\nu,\overline{p}}(x,r)\|f\|_{L^{p(\cdot)}(\Omega,|x|^\gamma)} \leq n_{\beta,\nu,\overline{p}}(x,r) \tag{5.4}$$

where

$$\beta(x) = \alpha(x) - n \qquad \text{and} \qquad \nu(x) = \frac{\gamma}{1-p(x)}.$$

We make use of our estimate (4.21) and obtain

$$|B_r(x)| \leq C\; r^{-\frac{n}{q(x)}}(r+|x|)^{-\frac{\gamma}{p(x)}}, \tag{5.5}$$

the assumptions of Theorem 4.2 being satisfied by (3.1) and the fact that $\gamma \geq 0$. From (5.5) we obtain

$$|B_r(x)| \leq C\; |x|^{-\frac{\gamma}{p(x)}}\; r^{-\frac{n}{q(x)}}, \tag{5.6}$$

since $\gamma \geq 0$.

Therefore, taking into account (5.3) and (5.6) in (5.1), we arrive at

$$I^{\alpha(\cdot)}f(x) \leq C\left[r^{\alpha(x)}Mf(x) + |x|^{-\frac{\gamma}{p(x)}}\; r^{-\frac{n}{q(x)}}\right] . \tag{5.7}$$

It remains to choose the value of r which minimizes the right-hand side. A direct calculation provides

$$r = \left[\frac{n}{q(x)\alpha(x)}\right]^{\frac{p(x)}{n}} |x|^{-\frac{\gamma}{n}}\,[Mf(x)]^{-\frac{p(x)}{n}} .$$

Substituting this into (5.7), after easy evaluations we get

$$I^{\alpha(\cdot)}f(x) \leq C|x|^{-\frac{\gamma\alpha(x)}{n}}[Mf(x)]^{\frac{p(x)}{q(x)}} .$$

Hence,

$$\int\limits_\Omega |x|^\mu \left|I^{\alpha(\cdot)}f(x)\right|^{q(x)}\, dx \leq C\int\limits_\Omega |x|^{\mu-\frac{\gamma}{n}\alpha(x)q(x)}|Mf(x)|^{p(x)}\, dx.$$

Since $\alpha(x)$ and $p(x)$ satisfy the logarithmic Dini conditions (and, consequently, $q(x)$ as well), we see that

$$C_1|x|^\gamma \leq |x|^{\mu-\frac{\gamma}{n}\alpha(x)q(x)} \leq C_2|x|^\gamma \tag{5.8}$$

under the choice $\mu = \frac{q(0)}{p(0)}\gamma$ according to (3.6). Therefore,

$$\int\limits_\Omega |x|^\mu \left|I^{\alpha(\cdot)}f(x)\right|^{q(x)}\, dx \leq C\int\limits_\Omega |x|^\gamma|Mf(x)|^{p(x)}\, dx$$

or

$$\int\limits_\Omega |x|^\mu \left|I^{\alpha(\cdot)}f(x)\right|^{q(x)}\, dx \leq C\int\limits_\Omega \left[|x|^{\frac{\gamma}{p(x)}}\cdot M\left(\frac{f_0(x)}{|x|^{\frac{\gamma}{p(x)}}}\right)\right]^{p(x)} dx$$

where

$$f_0(x) = |x|^{\frac{\gamma}{p(x)}} f(x) \in L^{p(\cdot)}(\Omega), \qquad \|f_0\|_{L^{p(\cdot)}} \leq 1.$$

We again refer to the logarithmic condition for $p(x)$ which provides the equivalence

$$|x|^{\frac{\gamma}{p(x)}} \sim |x|^{\frac{\gamma}{p(0)}} \tag{5.9}$$

and gives

$$\int\limits_{\Omega} |x|^{\mu} \left|I^{\alpha(\cdot)} f(x)\right|^{q(x)} dx \leq C \int\limits_{\Omega} \left[|x|^{\frac{\gamma}{p(0)}} \cdot M\left(\frac{f_0(x)}{|x|^{\frac{\gamma}{p(0)}}}\right)\right]^{p(x)} dx = \int\limits_{\Omega} |M^{\delta} f_0(x)|^{p(x)} dx \tag{5.10}$$

where $\delta = \frac{\gamma}{p(0)}$ and M^{δ} is the weighted maximal operator (2.9).

It remains to make use of Theorem 2.2. Condition (2.10) of that theorem with $\delta = \frac{\gamma}{p(0)}$ means that $-n < \gamma < n[p(0) - 1]$ which is satisfied by (3.5). By Theorem 2.2 we have $\left\|M^{\delta} f_0\right\|_{L^{p(\cdot)}} \leq C\|f_0\|_{L^{p(\cdot)}} \leq C$. Then $\int\limits_{\Omega} |M^{\delta} f_0(x)|^{p(x)} dx \leq C$ and by (5.10) we obtain that

$$\int\limits_{\Omega} |x|^{\mu} \left|I^{\alpha(\cdot)} f(x)\right|^{q(x)} dx \leq C \quad \text{for all} \quad f \in L^{p(\cdot)}(\Omega, |x|^{\gamma}) \quad \text{with} \quad \|f\|_{L^{p(\cdot)}(\Omega,|x|^{\gamma})} \leq 1$$

which is equivalent to (3.3).

5.2 The case of negative γ.

This case is reduced to the previous case by the duality arguments. First we observe that the operator conjugate to $I^{\alpha(\cdot)}$ has the form

$$\left(I^{\alpha(\cdot)}\right)^{*} g(x) = I_{\alpha(\cdot)} g(x) := \int\limits_{\Omega} \frac{g(y)\, dy}{|x - y|^{n - \alpha(y)}} \sim \int\limits_{\Omega} \frac{g(y)\, dy}{|x - y|^{n - \alpha(x)}} = I^{\alpha(\cdot)} g(x) \tag{5.11}$$

where the equivalence

$$C_1 |x - y|^{n - \alpha(x)} \leq |x - y|^{n - \alpha(y)} \leq C_2 |x - y|^{n - \alpha(x)}$$

follows from the logarithmic condition for $\alpha(x)$.

We pass to the duality statement in Theorem A considering that $\gamma \geq 0$ there. By (5.11) we obtain from (3.3) that

$$\left\|I^{\alpha(\cdot)} g\right\|_{\left(L^{p(\cdot)}(\Omega,|x|^{\gamma}\right)^{*}} \leq C\|g\|_{\left(L^{q(\cdot)}(\Omega,|x|^{\mu}\right)^{*}}.$$

In view of (2.7) this takes the form

$$\left\|I^{\alpha(\cdot)} g\right\|_{L^{\overline{p}(\cdot)}\left(\Omega,|x|^{\frac{\gamma}{1-p(0)}}\right)} \leq C\|g\|_{L^{\overline{q}(\cdot)}\left(\Omega,|x|^{\frac{\mu}{1-q(0)}}\right)} \tag{5.12}$$

where the passage of the type (5.9) has been used. Now there is a sense to redenote

$$\frac{\mu}{1-q(0)} = \gamma_1, \qquad \frac{\gamma}{1-p(0)} = \mu_1, \qquad \overline{q}(x) = p_1(x)$$

where γ_1 is already negative.

For the exponent $p_1(x)$ we have

$$p_1(x) = \frac{np(x)}{n[p(x)-1]+\alpha(x)} \qquad \text{and} \qquad n-\alpha(x)p_1(x) = n\,\frac{[n-\alpha(x)][p(x)-1]}{n[p(x)-1]+\alpha(x)} \geq c > 0.$$

Its Sobolev exponent is

$$q_1(x) = \frac{np_1(x)}{n-p_1(x)\alpha(x)} = \overline{p}(x)$$

and it is easy to see that the new weight exponents γ_1 and μ_1 are related to each other by the necessary relation of type (3.6), that is

$$\mu_1 = \frac{q_1(0)}{p_1(0)}\gamma_1.$$

In the new notation, estimate (5.12) has the form

$$\left\|I^{\alpha(\cdot)}g\right\|_{L^{q_1(\cdot)}(\Omega,|x|^{\mu_1})} \leq C\|g\|_{L^{p_1(\cdot)}(\Omega,|x|^{\gamma_1})}. \tag{5.13}$$

Note also that

$$0 \leq \gamma < n[p(0)-1] \iff \alpha(0)p_1(0)-n < \gamma_1 \leq 0$$

so that the estimate in (5.13) is nothing else but our Theorem A for the negative subinterval of possible values of γ.

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