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ON MULTIDIMENSIONAL ANALOGUE OF MARCHAUD FORMULA FOR FRACTIONAL RIESZ-TYPE DERIVATIVES IN DOMAINS IN $\mathbb{R}^n$

Humberto Rafeiro [1] **and Stefan Samko** [2]

Abstract

There is given a generalization of the Marchaud formula for one-dimensional fractional derivatives on an interval (a,b), $-\infty < a < b \leq \infty$, to the multidimensional case of functions defined on a region in $\mathbb{R}^n$:

$$\mathbb{D}^\alpha_\Omega f(x) = c(\alpha)\left[a_\Omega(x)f(x) + \int\limits_\Omega \frac{f(x)-f(y)}{|x-y|^{n+\alpha}}dy\right], \qquad x\in\Omega, \qquad 0<\alpha<1,$$

which is the Riesz fractional derivative of the zero continuation of $f(x)$ from Ω to the whole space $\mathbb{R}^n$, $c(\alpha)$ being a certain constant. A special attention is paid to the role of the coefficient $a_\Omega(x)$, which in the multidimensional case is estimated in terms of the power of the distance of the point x to the boundary $\partial\Omega$. In the case when Ω is a ball, this function is calculated explicitly in terms of the Gauss hypergeometric function.

It is also shown that the operator $\mathbb{D}^\alpha_\Omega$ acts boundedly from the range of the Riesz potential operator $I^\alpha_\Omega(L_p(\Omega))$ to $L_p(\Omega), 1<p<\frac{1}{\alpha}$.

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1. Introduction

The Marchaud formula

$$\mathbb{D}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{f(x)}{(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^{x-a}\frac{f(x)-f(x-t)}{t^{1+\alpha}}\,dt, \qquad x>a, \tag{1.1}$$

for fractional derivatives of order $0<\alpha<1$, is well known, see [5], Subsection 13.1, which is a "difference form" of the Riemann-Liouville fractional derivative.

We introduce a multidimensional analogue of this formula in domains in $\mathbb{R}^n$ adjusted for the Riesz fractional derivatives, see about Riesz fractional derivatives $\equiv$ hypersingular integrals in [5], Section 26.

This generalized Marchaud formula for a domain Ω has the form

$$\mathbb{D}^\alpha_\Omega f(x) = c(\alpha)\left[a_\Omega(x)f(x) + \int\limits_\Omega \frac{f(x)-f(y)}{|x-y|^{n+\alpha}}dy\right], \qquad x\in\Omega, \tag{1.2}$$

where $0<\alpha<1$ and

$$a_\Omega(x) = \int\limits_{R^n\backslash\Omega}\frac{dy}{|x-y|^{n+\alpha}} \qquad \text{and} \qquad c(\alpha) = \frac{2^\alpha\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)\sin\frac{\alpha\pi}{2}}{\pi^{1+\frac{n}{2}}}.$$

We prove a property of the function $a(x)$ important for further applications, namely, that the function $a(x)$ behaves, generally speaking as $[\delta(x)]^{-\alpha}$, as x approaches the boundary $\partial\Omega$, where $\delta(x) = dist(x,\partial\Omega)$ is the distance of a point $x\in\Omega$ to the boundary. We also show that in the case of the ball $\Omega = \{x\in\mathbb{R}^n : |x|<1\}$ the function $a(x)$ may be explicitly calculated. Finally, we show that the operator $\mathbb{D}^\alpha_\Omega$ has some features of the operator inverse to the Riesz potential operator over Ω.

2. Definition

As is known([5], Section 26), for functions defined on the whole Euclidean space, the Riesz derivative $\mathbb{D}^\alpha f = F^{-1}|\xi|^\alpha Ff$, where F stands for the Fourier transforms, in the case $0<\alpha<1$ has the form

$$\mathbb{D}^\alpha f = c(\alpha)\int\limits_{\mathbb{R}^n}\frac{f(x)-f(x-y)}{|y|^{n+\alpha}}\,dy.$$

Let now Ω be a domain in $\mathbb{R}^n$ and $f(x)$ a function defined in Ω. We introduce the fractional Riesz-type derivative $\mathbb{D}^\alpha_\Omega f(x)$ of f as a restriction onto Ω of the Riesz derivative of the zero extension of f to the whole space $\mathbb{R}^n$. Namely, let

$$\mathcal{E}_\Omega f(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \backslash \Omega \end{cases} =: \tilde{f}(x).$$

Then, by definition

$$\mathbb{D}^\alpha_\Omega f(x) := r_\Omega \mathbb{D}^\alpha \mathcal{E}_\Omega f(x) = c(\alpha) \int\limits_{\mathbb{R}^n} \frac{f(x) - \tilde{f}(x-y)}{|y|^{n+\alpha}}\, dy, \qquad x \in \Omega, \quad (2.1)$$

where r_Ω stands for the operator of restriction onto Ω. By splitting the integration in (2.1) to $\int_\Omega + \int_{\mathbb{R}^n \backslash \Omega}$, we easily arrive at (1.2).

In what follows, the convergence of the integral in (1.2) is interpreted as

$$\int\limits_\Omega \frac{f(x) - f(y)}{|x-y|^{n+\alpha}}\, dy = \lim_{\varepsilon \to 0} \int\limits_{\substack{y \in \Omega \\ |y-x| > \varepsilon}} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}}\, dy, \qquad x \in \Omega. \quad (2.2)$$

Obviously, this integral absolutely converges on functions f satisfying the Hölder condition of order $\lambda > \alpha$ in $\overline{\Omega}$. The integral, defining $a(x)$ is always convergent, see estimation in (3.3).

3. Boundary behavior

We recall that a domain Ω is said to have the cone property, if for every $x \in \overline{\Omega}$ there exists a finite cone C_x centered at the point x, contained in Ω and congruent to a finite cone of fixed aperture centered at the origin (a finite cone C_0 is the intersection of open ball centered at the origin with the set $\{\lambda x; \lambda > 0, |x - z_0| < r\}$, where $z_0 \neq 0$ and $r > 0$ are fixed), see for instance, [1], p. 300.

Proposition 3.1. *Let Ω be an arbitrary domain in $\mathbb{R}^n$. Then for all $x \in \Omega$*

$$a_\Omega(x) \leq \frac{c_1}{[\delta(x)]^\alpha}, \qquad c_1 = \frac{|S^{n-1}|}{\alpha} = \frac{2\pi^{\frac{n}{2}}}{\alpha \Gamma(\frac{n}{2})}. \quad (3.1)$$

If the domain $\mathbb{R}^n \backslash \overline{\Omega}$ has the cone property, then there exists a constant $c_2 > 0$ such that

$$\frac{c_2}{[\delta(x)]^\alpha} \leq a_\Omega(x) \leq \frac{c_1}{[\delta(x)]^\alpha}. \quad (3.2)$$

P r o o f. For $x \in \Omega$, we have

$$\int\limits_{\mathbb{R}^n\setminus\Omega} \frac{dy}{|x-y|^{n+\alpha}} \leq \int\limits_{\substack{y\in\mathbb{R}^n\\ |x-y|\geq\delta(x)}} \frac{dy}{|x-y|^{n+\alpha}} = \int\limits_{\substack{y\in\mathbb{R}^n\\ |y|>\delta(x)}} \frac{dy}{|y|^{n+\alpha}}\,. \tag{3.3}$$

Passing to polar coordinates, we arrive at the estimate in (3.1).

To prove the left-hand side estimate in (3.2), we choose the boundary point $x_0 \in \partial\Omega$ (depending on x and not necessarily unique) at which $|x - x_0| = \delta(x)$. Then,

$$|x-y| \leq |x-x_0| + |x_0-y|$$

and

$$a_\Omega(x) \geq \int\limits_{R^n\setminus\Omega} \frac{dy}{[|x_0-y|+\delta(x)]^{n+\alpha}}\,.$$

Since $\mathbb{R}^n\setminus\overline{\Omega}$ has the cone property, there exists a finite cone $\Gamma_\Omega(x_0,\theta)$ with vertex at x_0 and fixed aperture $\theta(= arctg\frac{r}{|z_0|}) > 0$, such that $\Gamma_\Omega(x_0,\theta) \subset \mathbb{R}^n\setminus\overline{\Omega}$. Then

$$\int\limits_{R^n\setminus\Omega} \frac{dy}{[|x_0-y|+\delta(x)]^{n+\alpha}} \geq \int\limits_{B(x_0,\delta(x))\cap\Gamma_\Omega(x_0,\theta)} \frac{dy}{[|x_0-y|+\delta(x)]^{n+\alpha}}\,.$$

After translation to the origin and passing to polar coordinates, we obtain

$$a_\Omega(x) \geq \int\limits_0^{\delta(x)} \frac{\rho^{n-1}d\rho}{[\rho+\delta(x)]^{n+\alpha}} \int\limits_{S^{n-1}\cap\Gamma_\Omega(0,\theta)} d\sigma = \frac{c_2}{[\delta(x)]^\alpha}\,,$$

where

$$c_2 = C(\Omega)\int\limits_0^1 \frac{t^{n-1}dt}{(t+1)^{n+\alpha}} = \frac{C(\Omega)}{n}F(n,n+\alpha;n+1;-1), \quad C(\Omega) = |S^{n-1}\cap\Gamma_\theta(0)|$$

where $F = {}_2F_1$ is the Gauss hypergeometric function. ■

4. The function $a_\Omega(x)$ in the case of the ball

THEOREM 4.1. *In case Ω is the ball $B(0,R)$, the function $a_\Omega(x)$ has the form*

$$a_{B(0,R)}(x) = \frac{|S_{n-1}|}{\alpha R^\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+n}{2}; \frac{n}{2}; \frac{|x|^2}{R^2}\right), \qquad |x| < R. \tag{4.1}$$

It has also the following representation

$$a_{B(0,R)}(x) = \frac{\mathcal{A}_R(|x|)}{(R^2-|x|^2)^\alpha} \tag{4.2}$$

with $\mathcal{A}_R(0) = \frac{1}{R^\alpha}|S^{n-1}|$ *and* $\mathcal{A}_R(R) = (2R)^\alpha \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\alpha\Gamma\left(\frac{n+\alpha}{2}\right)}$, *where*

$$\mathcal{A}_R(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}\int\limits_0^1 s^{\frac{\alpha}{2}-1}(1-s)^{\frac{n-\alpha}{2}-1}(R^2-r^2+sr^2)^{\frac{\alpha}{2}}\,ds \tag{4.3}$$

$$= \frac{|S^{n-1}|}{\alpha}(R^2-|x|^2)^{\frac{\alpha}{2}} F\left(-\frac{\alpha}{2},\frac{\alpha}{2};\frac{n}{2};-\frac{|x|^2}{R^2-|x|^2}\right).$$

P r o o f. The function $a_{B(0,R)}(x)$ depends on $|x|$ only. Let us denote $a(r) := a_{B(0,R)}(x), r = |x|$, for brevity. It suffices to consider the case $R = 1$. After passing to polar coordinates and using the Catalan formula

$$\int\limits_{S^{n-1}} f(x\cdot\sigma)\,d\sigma = |S^{n-2}|\int\limits_{-1}^{1} f(|x|t)(1-t^2)^{\frac{n-3}{2}}, \quad x\in\mathbb{R}^n,$$

we arrive at

$$\begin{aligned} a(r) &= \frac{|S_{n-2}|}{r^\alpha}\int\limits_{-1}^{1}(1-t^2)^{\frac{n-3}{2}}dt\int\limits_{\frac{1}{r}}^{\infty}\frac{s^{n-1}}{(s^2-2st+1)^{\frac{n+\alpha}{2}}}ds \\ &= \frac{|S_{n-2}|}{r^\alpha}\int\limits_{-1}^{1}(1-t^2)^{\frac{n-3}{2}}dt\int\limits_{0}^{r}\frac{s^{\alpha-1}}{(s^2-2st+1)^{\frac{n+\alpha}{2}}}ds. \end{aligned}$$

Since $(1-2st+s^2)^{-\lambda} = \sum\limits_{k=0}^{\infty} C_k^\lambda(t)s^k$, where $C_k^\lambda(t)$ are the Gegenbauer polynomials, we obtain

$$a(r) = 2|S_{n-2}|\sum_{k=0}^{\infty}\frac{r^{2k}}{2k+\alpha}\int\limits_0^1(1-t^2)^{\frac{n-3}{2}}C_{2k}^{\frac{n+\alpha}{2}}(t)dt, \tag{4.4}$$

where we took into account that the Gegenbauer polynomials of odd order are odd. The formula

$$\int\limits_0^1(1-t^2)^{\frac{n-3}{2}}C_{2k}^{\frac{n+\alpha}{2}}(t)\,dt = \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{2k!\Gamma\left(\frac{n}{2}\right)}\frac{\left(\frac{n+\alpha}{2}\right)_k\left(\frac{\alpha}{2}+1\right)_k}{\left(\frac{n}{2}\right)_k}$$

holds, see 2.21.2.3 in [3]. Making use of this formula, after easy calculations with the duplication formula for the Gamma-function taken into account, we arrive at

$$a(r) = \frac{1}{\alpha}|S_{n-1}| \sum_{k=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_k \left(\frac{\alpha+n}{2}\right)_k}{k! \left(\frac{n}{2}\right)_k} r^{2k}$$

which is nothing else but (4.1) with $R = 1$.

To get (4.3) from (4.1), it suffices to make use of the transformation formula for the Gauss hypergeometric function:

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right).$$

■

Corollary 4.2. *In the case $n = 2m+1$ is odd, $a_{B(0,R)}(x)$ is an elementary function for any $\alpha \in (0,1)$:*

$$a_{B(0,1)}(x)\Big|_{|x|=r} = d_{n,\alpha}(1-r^2)^{\frac{1-\alpha}{2}} \frac{d^m}{dr^{2m}} \left\{(1-r^2)^{m-\frac{\alpha+1}{2}} \left[(1+r)^\alpha + (1-r)^\alpha\right]\right\} \tag{4.5}$$

where $d_{n,\alpha} = \frac{\alpha^{-1}\pi^{\frac{n-1}{2}}\Gamma\left(1-\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}$.

In particular, when $n = 3$ one has

$$a_{B(0,1)}(x) = \frac{2\pi}{\alpha^2-1}\left\{\left(2+\frac{1}{\alpha}\right)\left[(1+|x|)^{-\alpha} + (1-|x|)^{-\alpha}\right] + \frac{(1-|x|)^{-\alpha} - (1+|x|)^{-\alpha}}{|x|}\right\}.$$

P r o o f. When n is odd, we could write $n = 2m+1$, then

$$a(x) = \frac{1}{\alpha}|S_{n-1}| F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}+m; \frac{1}{2}+m; r^2\right).$$

Using the formula

$$\frac{d^n}{dz^n}\left[(1-z)^{a+n-1}F(a,b;c;z)\right] = (-1)^n \frac{(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}F(a+n, b; c+n; z),$$

see for example, 7.2.1.13 from [4], and the fact that the hypergeometric function is symmetric with respect to a and b, we get

$$F\left(\frac{\alpha+1}{2}+m,\frac{\alpha}{2};\frac{1}{2}+m;r^2\right)=(-1)^m\frac{\left(\frac{1}{2}\right)_m}{\left(\frac{\alpha+1}{2}\right)_m\left(\frac{1-\alpha}{2}\right)_m}$$
$$\times(1-r^2)^{\frac{1-\alpha}{2}}\frac{d^m}{dr^{2m}}\left[(1-r^2)^{\frac{\alpha-1}{2}+m}F\left(\frac{\alpha+1}{2},\frac{\alpha}{2};\frac{1}{2};r^2\right)\right]. \tag{4.6}$$

Then by the formula

$$F\left(a,a+\frac{1}{2};\frac{1}{2};z\right)=\frac{1}{2}\left[(1+\sqrt{z})^{-2a}+(1-\sqrt{z})^{-2a}\right], \tag{4.7}$$

see [4], formula 7.3.1.106, after simplifications we arrive at (4.5). ■

Remark 4.3. When $n=2m$ is even, the function $a(r)$ may not be expressed in terms of elementary functions, being given by

$$a(r)=\frac{|S_{n-1}|}{\alpha}\cdot(-1)^{m-1}\frac{(m-1)!}{(\frac{\alpha}{2}+1)_{m-1}(1-\frac{\alpha}{2})_{m-1}}(1-r^2)^{-\frac{\alpha}{2}} \tag{4.8}$$
$$\times\left(\frac{d}{dr^2}\right)^{m-1}\left\{(1-r^2)^{m-2}\left[P^0_{-\frac{\alpha}{2}-1}\left(\frac{1+r^2}{1-r^2}\right)-\frac{2r}{\alpha}P^1_{\frac{\alpha}{2}-1}\left(\frac{1+r^2}{1-r^2}\right)\right]\right\},$$

where $P^0_{-\frac{\alpha}{2}-1}\equiv P_{-\frac{\alpha}{2}-1}$ and $P^1_{-\frac{\alpha}{2}-1}$ are the Legendre polynomials and the associated Legendre function of the 1st kind. In particular, for $n=2$ one has

$$a(r)=\frac{2\pi}{\alpha}(1-r^2)^{-\frac{\alpha}{2}-1}\left[P^0_{-\frac{\alpha}{2}-1}\left(\frac{1+r^2}{1-r^2}\right)-\frac{2r}{\alpha}P^1_{-\frac{\alpha}{2}-1}\left(\frac{1+r^2}{1-r^2}\right)\right]. \tag{4.9}$$

To obtain this, we use

$$F(a,b;a-b+2;z)$$
$$=\frac{\Gamma(a-b+2)}{b-1}z^{(b-a-1)/2}(1-z)^{-b}\left[aP^{(b-a-1)/2}_{-b}\left(\frac{1+z}{1-z}\right)-\sqrt{z}P^{b-a}_{-b}\left(\frac{1+z}{1-z}\right)\right],$$

see [4], formula 7.3.1.62, and (4.7) formula.

5. The operator $\mathbb{D}^\alpha_\Omega$ as "quasi"- inverse to the operator I^α_Ω

Definition 5.1. The function $\mu(x)$ is called a multiplier in the space X, if $\mu f\in X$ and $\|\mu f\|_X\le c\|f\|_X$, for all $f\in X$.

Definition 5.2. Let Ω be an open set in $\mathbb{R}^n$. We say that Ω satisfies the *Strichartz condition* if there exist a coordinate system in $\mathbb{R}^n$ and an integer $N > 0$ such that almost every line parallel to the axes intersects Ω in at most N components.

The following statement shows that although the operator $\mathbb{D}^\alpha_\Omega$ is not inverse to the operator I^α_Ω in the cases where $\Omega \neq \mathbb{R}^n$, it possesses some property of the inverse operator.

Theorem 5.3. *Let $f = I^\alpha_\Omega \varphi$ where $\varphi \in L_p(\Omega), 1 \leq p < \frac{1}{\alpha}$ and Ω is a bounded domain satisfying the Strichartz condition. Then*

$$\mathbb{D}^\alpha_\Omega f \in L_p(\Omega) \qquad \text{and} \qquad \|\mathbb{D}^\alpha_\Omega f\|_{L_p(\Omega)} \leq C\|\varphi\|_{L_p(\Omega)}, \tag{5.1}$$

where $C > 0$ does not depend on f.

P r o o f. By the definition in (2.1) we have

$$\mathbb{D}^\alpha_\Omega I^\alpha_\Omega \varphi(x) = r_\Omega \mathbb{D}^\alpha \chi_\Omega I^\alpha \mathcal{E}_\Omega \varphi(x), \qquad x \in \Omega. \tag{5.2}$$

Then the statement of the theorem is derived from the following three facts:

1) hypersingular integral operator is the left inverse operator to the Riesz potential operator in the case of the whole space $\mathbb{R}^n$:

$$\mathbb{D}^\alpha I^\alpha \varphi \equiv \varphi, \qquad \varphi \in L_p(R^n), \qquad 1 \leq p < n/\alpha,$$

see [5], p. 517.

2) the characteristic function χ_Ω of the domain Ω satisfying the Strichartz condition is a multiplier in the space $I^\alpha(L_p), L_p = L_p(\mathbb{R}^n)$ (see [6] for the case Bessel potentials and [2] for the case of Riesz potentials):

$$\|\mathbb{D}^\alpha \chi_\Omega I^\alpha \varphi\|_{L_p(\mathbb{R}^n)} \leq C\|\varphi\|_{L_p(\mathbb{R}^n)}, \qquad 1 < p < \frac{1}{\alpha}. \tag{5.3}$$

3) the condition $\|\mathbb{D}^\alpha f\|_{L_p(\mathbb{R}^n)} < \infty$ is sufficient for a function $f \in L_p(\mathbb{R}^n)$ to belong to $I^\alpha(L_p)$ and $f = I^\alpha \mathbb{D}^\alpha f$, see [5], Section 26. Observe that here we have used the fact that the domain Ω is bounded: in case Ω is unbounded, the function $f = \mathcal{E}_\Omega I^\alpha_\Omega \varphi$ is not necessarily in $L_p(\mathbb{R}^n)$.

Indeed, by (5.2) and (5.3) we have

$$\|\mathbb{D}^\alpha \mathcal{E}_\Omega I^\alpha_\Omega \varphi\|_{L_p(\mathbb{R}^n)} = \|\mathbb{D}^\alpha \chi_\Omega I^\alpha \mathcal{E}_\Omega \varphi\|_{L_p(\mathbb{R}^n)} \leq C\|\mathcal{E}_\Omega \varphi\|_{L_p(\mathbb{R}^n)} = C\|\varphi\|_{L_p(\Omega)}.$$

Therefore, by 3) there exists a function $\psi \in L_p(\mathbb{R}^n)$ such that $\mathcal{E}_\Omega I_\Omega^\alpha \varphi(x) = I^\alpha \psi(x),\ x \in \mathbb{R}^n$ with $\psi = \mathbb{D}^\alpha \mathcal{E}_\Omega I_\Omega^\alpha \varphi$. Observe that $\|\psi\|_{L_p(\mathbb{R}^n)} \leq C\|\varphi\|_{L_p(\Omega)}$ by (5.3). Then

$$\|\mathbb{D}_\Omega^\alpha I_\Omega^\alpha f\|_{L_p(\Omega)} = \|r_\Omega \mathbb{D}^\alpha \mathcal{E}_\Omega I_\Omega^\alpha \varphi\|_{L_p(\Omega)} = \|r_\Omega \mathbb{D}^\alpha I^\alpha \psi\|_{L_p(\Omega)}.$$

Consequently, by 1),

$$\|\mathbb{D}_\Omega^\alpha I_\Omega^\alpha \varphi\|_{L_p(\Omega)} = \|\psi\|_{L_p(\Omega)} \leq \|\varphi\|_{L_p(\Omega)}$$

and (5.1) thus having been proved. ■

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Universidade do Algarve
Faculdade de Ciências e Tecnologia
Campus de Gambelas, Faro 8000 – PORTUGAL

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e-mails: [1] (*H. Rafeiro*) *a19423@ualg.pt*
[2] *ssamko@ualg.pt* , *http://w3.ualg.pt/˜ssamko/*