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On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators

S. SAMKO*

Universidade do Algarve, Faro 8000, Portugal

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This paper represents a broadened version of the plenary lecture presented by the author at the conference *Analytic Methods of Analysis and Differential Equations* (AMADE-2003), September 4–9, 2003, Minsk, Belarus. We give a survey of investigations on 'the variable exponent business', concentrating mainly on recent advances in the operator theory and harmonic analysis in the generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$.

Keywords: Variable exponent; Maximal operators; Singular operators; Potential operators; Hardy operators; Generalized Lebesgue and Sobolev spaces

1. Introduction

1.1 *The non-standard growth background*

Last decade, we witness a strong rise of interest to the study of various mathematical problems in the so-called spaces with non-standard growth. This expression mainly relates to the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$, $\Omega \subseteq R^n$, with variable order $p(x)$ (the generalized Lebesgue spaces with variable exponent), and to the corresponding generalized Sobolev spaces $W^{m,p(\cdot)}(\Omega)$. Such Sobolev spaces naturally arise when one deals with functionals of the form

$$\int_\Omega |\nabla f(x)|^{p(x)}\,\mathrm{d}x.$$

Such a functional appears, for instance, in the study of differential equations of the type

$$\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u\right) = |u|^{\sigma(x)-1}u(x) + f(x).$$

In this case, one deals with the Dirichlet integral of the form

$$\int_\Omega \left(|\nabla f(x)|^{p(x)} + |u(x)|^{\sigma(x)}\right)\,\mathrm{d}x$$

Such mathematical problems and spaces with variable exponent arise in applications to mechanics of the continuum medium. In some problems of mechanics, there arise variational

*Email: ssamko@ualg.pt

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problems with Lagrangians more complicated than is usually assumed in variational calculus, for example, of the form $|\xi|^{\alpha(x)}$ when the character of non-linearity varies from point to point (Lagrangians of the plasticity theory, Lagrangians of the mechanics of the so-called rheological fluids and others).

Investigation of variational problems with variable exponent started from the paper by Zhikov [1], related to the so-called Lavrentiev phenomenon; see for instance ref. [2] on this phenomenon. Nowadays, this topic of variational problems and differential equations with variable exponent is intensively developed worldwide by many researchers. We refer to the papers by Acerbi and Mingione [3, 4], Alkhutov [5], Cabada and Pouso [6], Chiadò and Coscia [7], Coscia and Mingione [8], Fan [9], Fan and Fan [10], Fan *et al.* [11], Fan and Zhang [12], Fan and Zhao [13–15], Kovácǐk [16], Marcellini [17], Zhikov [18] and references therein. Recently, there also appeared a group of researchers of variational problems with variable exponent in Finland (the 'Helsinki group'); *Petteri Harjulehto, Peter Hasto, Mika Koskenoja* and others; we refer to their web page http://www.math.helsinki.fi/analysis/varsobgroup and the paper by Harjulehto *et al.* [19].

We also refer to Antontsev [20, 21], where the problems of localization of solutions of elliptic and parabolic equations, in the spirit of the book by Antontsev *et al.* [22], were treated in the spaces with variable exponent.

We do not touch the non-standard growth in this interesting and large area of variational problems and differential equations; the aim of this survey is to represent the main results in the operator theory and harmonic analysis in the spaces $L^{p(\cdot)}$, as can be seen from the main sections 2–7.

1.2 *Feedback from applications*

Investigations in this topic during recent years were strongly stimulated by applications in various problems related to objects with non-standard local growth in which growth conditions of variable order arise (in elasticity theory, fluid mechanics, differential equations). In 1995–1999, there appeared a series of papers by Ružička on problems in the so-called rheological and electrorheological fluids which lead to spaces with variable exponent. The results developed in those papers were summarized in his book [23], see also refs. [24, 25] and also references in [23, 25]. Many mathematical models in fluid mechanics, elasticity theory, in differential equations and so on were shown to be naturally related to the problems with non-standard local growth. It is difficult to overestimate the impact of the earlier publications on investigation of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(x)}(R^n)$ which proved to be an appropriate tool applicable in this area.

1.3 *The generalized Lebesgue spaces: theoretical background*

However, the first interest to the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ was purely theoretical, being evoked just by mathematical curiosity. The first papers on investigation of such spaces were performed without any idea of possible vast applications, which happened very soon after the first theoretical papers.

The spaces $L^{p(\cdot)}([0, 1])$ probably first appeared in the book by Nakano [26], as an example illustrating the theory of modular spaces. The pioneer paper where the space $L^{p(\cdot)}$ was studied as a special object and as a Banach space was that by Sharapudinov [27], although the spaces $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$ are particular cases of the generalized Orlicz and Orlicz–Sobolev spaces introduced and investigated earlier by Hudzik [28]; see also ref. [29] However, that was namely the specifics of the spaces $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$ which attracted many researchers and allowed us to develop rather rich basic theory of these spaces, this interest also being roused by applications revealed in various areas.

The basics of the spaces $L^{p(\cdot)}$ were developed partially in the paper by Sharapudinov [27] (in the one-dimensional case, although most of the results from ref. [27] are automatically rewritten for the multi-dimensional case), and to a big extent in the paper by Kovácǐk and Rákosnǐk [30]; see also refs. [31, 32].

This paper represents a brief survey of results obtained recently for maximal and singular operators and potential-type operators in the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent. Such spaces and operators in these spaces are intensively studied nowadays. One may see an evident rise of interest to these spaces and to the corresponding Sobolev-type spaces $W^{m,p(\cdot)}$ during the last decade, especially the last years. The increase in studying both the spaces $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$ themselves and the operator theory in these spaces is observed.

The development of the operator theory in the spaces $L^{p(\cdot)}$ encountered essential difficulties from the very beginning. For example, the convolution operators in general are not bounded in these spaces, the Young's theorem not being valid in the general case.

A convolution operator may be bounded in this space if, roughly speaking, its kernel has singularity only at the origin; see ref. [33]. Singular operators are of this type. The maximal operator close to singular ones was also a candidate for being a bounded operator in the spaces $L^{p(\cdot)}$. However, the boundedness of the maximal and singular operators was an open problem for a long time. Recently, the breakthrough result by Diening appeared on boundedness of maximal operator [34]. After this paper, a certain progress followed for maximal and singular operators both in non-weighted and weighted cases. We present a brief survey of results in this direction.

After the paper was prepared for publication, many interesting and important papers appeared on this rapidly developing topic. The paper reflects the status of research in the area up to the beginning of 2004.

Notation: Ω is an open set in $\mathbb{R}^n$; $\mathbb{R}^1_+ = (0, \infty)$; $\chi_\Omega(x)$ is the characteristic function of a set Ω in $\mathbb{R}^n$; $|\Omega|$ is the Lebesgue measure of Ω; $B(x_0, r)$ is the ball centered at x_0 and of radius r, $|B_n| = |B(0, 1)|$; S_{n-1} is the unit sphere in $\mathbb{R}^n$, $p(x)$: $\mathbb{R}^n \to [1, \infty)$ is a measurable function, $p_0 = \inf_{x\in\mathbb{R}^n} p(x)$, $P = \sup_{x\in\mathbb{R}^n} p(x)$; everywhere inf and sup stand for 'ess inf' and 'ess sup'.

2. Definitions

We refer to the papers by Kovácǐk and Rákosnǐk [30], Samko [31, 35] and Fan and Zhao [32] for the proofs of the main results on the generalized Lebesgue spaces, but give the main definitions for the reader's convenience.

We also mention that in the papers of Edmunds and Nekvinda [36] and Nekvinda [37], where the spaces $\ell^{\{p_n\}}$ were studied which were discrete analogues of the spaces $L^{p(\cdot)}$.

By $L^{p(\cdot)}$, we denote the space of functions $f(x)$ on Ω such that

$$I_{p,\Omega}(f) = \int_\Omega |f(x)|^{p(x)} \mathrm{d}x < \infty,$$

where $p(x)$ is a measurable function on Ω with values in $[1, \infty)$. This is a linear space if and only if $\sup_{x\in\Omega} p(x) < \infty$ [27]. The case $\sup_{x\in\Omega} p(x) = \infty$ may also be admitted, but to keep the space linear, instead of the condition $I_{p,\Omega}(f) < \infty$, one should use the condition that $I_{p,\Omega}(f/\lambda) < \infty$, for some $\lambda = \lambda(f)$.

Let $\Omega_\infty = \{x \in \Omega\colon p(x) = \infty\}$. When $|\Omega_\infty| = 0$, then this is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0\colon I_{p,\Omega}\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{1}$$

When $|\Omega_\infty| > 0$, then the space $L^{p(\cdot)}(\Omega)$ is introduced as the space of functions with the finite norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0\colon I_{p,\Omega\backslash\Omega_\infty}\left(\frac{f}{\lambda}\right) \le 1\right\} + \sup_{x\in\Omega_\infty} |f(x)|. \tag{2}$$

In ref. [38, Proposition 1.3], it was observed that $L^{p(\cdot)}(\Omega)$ is a Banach function space in the well-known sense; see, for instance, ref. [39].

In this survey, we consider only bounded exponents $p(x)$. Thus, $p(x)$ is not allowed to tend to infinity. Similarly, when dealing with the conjugate space and considering singular and maximal operators, we have to exclude the tendency of $p(x)$ to 1. Therefore, in the sequel, we assume that

$$1 < p_0 \le p(x) \le P < \infty, \quad x \in \Omega. \tag{3}$$

In the case of a bounded set Ω, the function $p(x)$ will be supposed to satisfy, besides condition (3), the only assumption

$$|p(x) - p(y)| \le \frac{A}{\ln(1/|x-y|)}, \quad |x-y| \le \frac{1}{2}, \ \ x, y \in \Omega. \tag{4}$$

This condition always arises when one deals witth variable exponent; see in particular, ref. [40], where it appeared in connection with Hölder spaces $H^{\lambda(x)}$ of variable order.

In case Ω is unbounded, we shall also refer to the assumption [ref. 35, Definitions 3.2, 3.3] that there exists $p(\infty) = \lim_{|x|\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \le \frac{A_\infty}{\ln(e+|x|)}, \quad x \in \Omega \tag{5}$$

or to the condition [41, 42]

$$|p(x) - p(y)| \le \frac{C}{\ln[e+\min(|x|,|y|)]}, \quad x, y \in \Omega, \tag{6}$$

which are equivalent.

The Sobolev space $W^{m,p(\cdot)}(\Omega)$ with variable p is introduced as the space of functions $f(x) \in L^{p(\cdot)}(\Omega)$ which have all the distributional derivatives $D^j f(x) \in L^{p(\cdot)}(\Omega)$, $0 \le |j| \le m$, with the norm

$$\|f\|_{W^{m,p(x)}} = \sum_{|j|\le m} \|D^j f\|_{L^{p(\cdot)}}.$$

(Both the versions (1) and (2) are possible).

We shall also use the spaces with variable exponent on curves in the complex plane. Let Γ be a closed Jordan curve, and $L^{p(\cdot)}(\Gamma)$ the space of functions $f(t)$ on Γ such that

$$I_p(f) = \int_\Gamma |f(t)|^{p(t)}|dt| < \infty, \quad \|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0\colon I_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Similar to equations (3) and (4), it is assumed that

$$1 < p_0 \le p(t) \le P < \infty, \quad t \in \Gamma, \tag{7}$$

and

$$|p(t_1) - p(t_2)| \le \frac{A}{\ln(1/|t_1 - t_2|)}, \quad |t_1 - t_2| \le \frac{1}{2}, \ t_1, t_2 \in \Gamma. \tag{8}$$

Under condition (7), the space $L^{p(\cdot)}$ coincides with the space

$$\left\{ f(t): \left| \int_\Gamma f(t)\varphi(t)\,\mathrm{d}t \right| < \infty \quad \text{for all } \varphi \in L^{q(\cdot)}(\Omega) \right\} \tag{9}$$

where $(1/p(t)) + (1/q(t)) \equiv 1$.

Condition (8) may be imposed either on the function $p(t)$ or on the function $p_*(s) = p[t(s)]$:

$$\left| p_*(s_1) - p_*(s_2) \right| \le \frac{A}{\ln(1/|s_1 - s_2|)}, \quad |s_1 - s_2| \le \frac{1}{2}, \ s_1, s_2 \in [0, \ell]. \tag{10}$$

Condition (8) always implies equation (10). Conditions (8) and (10) are equivalent, for example on curves with the chord-arc condition $|(t(s) - t(\sigma))/(s - \sigma)| \ge m > 0$ (curves of bounded rotation without cusps satisfy the chord-arc condition).

3. Denseness of C_0^∞-functions

The class $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω is dense in the spaces $L^{p(\cdot)}(\Omega)$, which was established among the first basic properties of these spaces in ref. [30], see Theorem 2.11; the case $\Omega = \mathbb{R}^1$ being considered in ref. [43].

We also observe an interesting result by Sharapudinov [44] on Haar basis in the spaces $L^{p(\cdot)}([0, 1])$. Let $\{\chi_m\}$ be the Haar system on [0, 1] and $p(x)$ be a bounded function on [0, 1], $p(x) \ge 1$ which is piece-wise Dini-Lipschitz on [0, 1], that is, there exists an n such that

$$|f(x) - f(y)| \le \frac{C}{\ln^\alpha (1/|x - y|)} \tag{11}$$

for all $x, y \in [(k-1)/2^n, k/2^n]$, $k = 1, 2, \ldots, 2^n$.

THEOREM 3.1 (Sharapudinov) *The Haar system* $\{\chi_m\}$ *is a basis in the space* $L^{p(\cdot)}([0, 1])$ *with* $p(x)$ *satisfying the piece-wise condition* (11) *if and only if* $\alpha \ge 1$.

Denseness of C^∞-functions in $L^{p(\cdot)}(\Gamma)$-spaces, Γ a curve in the complex plane, with an arbitrary weight was recently proved in refs. [45, 46].

THEOREM 3.2 (Kokilashvili and Samko) *Let* Γ *be a Jordan curve. The set* $C^\infty(\Gamma)$ (*and even the set of bounded rational functions on* Γ) *is dense in* $L^{p(\cdot)}(\Gamma, \rho)$, *for any measurable bounded exponent* $p(x) \ge 1$ *and any weight* $\rho \ge 0$ *such that*

$$|\{t \in \Gamma: \rho(t) = 0\}| = 0 \quad \text{and} \quad [\rho(t)]^{p(t)} \in L^1(\Gamma).$$

Denseness of $C^\infty(\Omega)$-functions in the Sobolev spaces $W^{m,p(\cdot)}(\Omega)$ proved to be a more difficult problem. Such a denseness does not necessarily hold in the case of discontinuous exponents $p(x)$, according to the example of Zhikov: let $\Omega = \{x = (x_1, x_2): |x| < 1\}$, $p(x) = \alpha$ if $x_1 x_2 > 0$ and $p(x) = \beta$ if $x_1 x_2 < 0$, then $C^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ is not dense in $W^{1,p(\cdot)}(\Omega)$, if $1 < \alpha < 2 < \beta$, which is known as Zhikov's example; see ref. [32, p. 440, or 18, p. 107].

Sufficient conditions for the denseness of C^∞-functions in $W^{m,p(\cdot)}(\Omega)$ were first given by Edmunds and Rákosník [47] in terms of some monotonicity condition on the exponent $p(x)$. In their following result, $V = V_{\xi,h}$ stands for the following cone with vertex at the origin

$$V = V_{\xi,h} = \bigcup_{0<t\leq 1} B(t\xi, th), \quad 0 < h < |\xi|.$$

THEOREM 3.3 (Edmunds and Rákosník) *Let Ω be an open, non-empty set and $p\colon \Omega \to [1,\infty)$ a bounded measurable function. Let also $p(x)$ satisfy the condition that for every $x \in \Omega$ there exist numbers $r = r(x) \in (0,1]$, $h = h(x) \in (0,\infty)$ and a vector $\xi = \xi(x) \in \mathbb{R}^n\backslash\{0\}$ such that*

$$h < |\xi| \leq 1, \quad B(x,r) + V(x) \subset \Omega$$

and

$$p(x) \leq p(x+y) \quad \textit{for almost all} \quad x \in \Omega, \;\; y \in V(x) = V_{\xi(x),h(x)}.$$

Then the set $C^\infty(\Omega) \bigcap W^{m,p(\cdot)}(\Omega)$ is dense in $W^{m,p(\cdot)}(\Omega)$.

In particular, in the one-dimensional case, Theorem 3.3 provides denseness of C^∞-functions on an open set Ω in $\mathbb{R}^1$ for any bounded monotone function $p(x)$ with values in $[1,\infty)$.

The denseness of $C_0^\infty(\mathbb{R}^n)$ in the Sobolev spaces $W^{m,p(\cdot)}(\mathbb{R}^n)$ without monotonicity assumption was proved in refs. [48, 49] under the usual logarithmic smoothness condition.

THEOREM 3.4 (Samko) *Let $p(x)$ satisfy the assumption $1 \leq p(x) \leq P < \infty$ and condition* (4) *in $\mathbb{R}^n$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p(\cdot)}(\mathbb{R}^n)$.*

Similar statements for the spaces $W^{m,p(\cdot)}(\Omega)$ on domains in $\mathbb{R}^n$ were given in refs. [32, 50]. Their results run as follows.

THEOREM 3.5 (Fan and Zhao) *Let Ω be an open bounded set in $\mathbb{R}^n$ and the exponent $p(x)$ satisfy the condition*

$$1 \leq p(x) \leq P < \infty, \quad x \in \Omega \tag{12}$$

and condition (4). *Then $C_0^\infty(\Omega) \bigcap W^{m,p(x)}(\Omega)$ is dense in $W^{m,p(\cdot)}(\Omega)$ and the closure of $C_0^\infty(\Omega)$ in the norm of $W^{m,p(\cdot)}(\Omega)$ coincides with the space $W^{m,p(\cdot)}(\Omega) \cap W_0^{m,1}$.*

THEOREM 3.6 (Burenkov and Samko) *Let Ω be an open set in $\mathbb{R}^n$ and the exponent $p(x)$ satisfy condition* (2) *and let for each compact $G \subset \Omega$, there exist a constant $M_G > 0$ such that*

$$|p(x) - p(y)| \leq \frac{M_G}{\log(1/|x-y|)}, \quad x, y \in G, \;\; |x-y| \leq \frac{1}{2}. \tag{13}$$

Then $C^\infty(\Omega) \bigcap W^{m,p(\cdot)}(\Omega)$ is dense in $W^{m,p(\cdot)}(\Omega)$.

Also, Diening [51, Theorem 3.7] proved the following statement.

THEOREM 3.7 (Diening) *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. If the exponent $p(x)$ satisfies condition* (2) *and is such that the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$* [*particularly, if* (3) *and* (4) *hold*], *then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.*

Proofs in refs. [48–50] are based on a result on the uniform boundedness of the dilation convolution operators obtained in refs. [48, 49], which itself is of interest for the spaces $L^{p(\cdot)}$.

Let Ω be a bounded domain, $\mathcal{K}(x)$ a measurable function with support in the ball $B_R = B(0, R)$ of a radius $R < \infty$, and

$$K_\epsilon f = \frac{1}{\epsilon^n} \int_\Omega \mathcal{K}\left(\frac{x-y}{\epsilon}\right) f(y)\,\mathrm{d}y.$$

We define the larger domain $\Omega_R = \{x\colon \mathrm{dist}(x, \Omega) \leq R\} \supseteq \Omega$ and suppose that the exponent $p(x)$ is defined in Ω_R. Let

$$Q = \begin{cases} \sup\limits_{x \in \Omega_R} \dfrac{p(x)}{p(x)-1}, & \text{if } |E_1(p)| = 0 \\ \infty, & \text{if } |E_1(p)| > 0 \end{cases}, \quad \text{where } E_1(p) = \{x \in \Omega_R\colon p(x) = 1\}.$$

THEOREM 3.8 (Samko) *Let $\mathcal{K}(x) \in L^Q(B_R)$ and let $p(x), 1 \leq p(x) \leq P < \infty, x \in \Omega_R$ satisfy condition* (4) *in Ω_R. Then the operators K_ϵ are uniformly bounded from $L^{p(\cdot)}(\Omega)$ into $L^{p(\cdot)}(\Omega_R)$:*

$$\|K_\epsilon f\|_{L^{p(x)}(\Omega_R)} \leq c\|f\|_{L^{p(\cdot)}(\Omega)}$$

where c does not depend on ϵ. If $\int_{B_R} \mathcal{K}(y)\mathrm{d}y = 1$, then (9) *is an identity approximation in $L^{p(x)}(\Omega)$:*

$$\lim_{\epsilon \to 0} \|K_\epsilon f - f\|_{L^{p(x)}(\Omega_R)} = 0, \quad f(x) \in L^{p(x)}(\Omega).$$

A periodic analogue of Theorem 3.8 was earlier proved in the one-dimensional case in ref. [52].

COROLLARY 3.1 *Let*

$$f_\epsilon(x) = \frac{1}{\epsilon^n |B(0,1)|} \int_{y \in \Omega, |y-x|<\epsilon} f(y)\,\mathrm{d}y \tag{14}$$

be the Steklov mean of the function $f(y)$. Then

$$\lim_{\epsilon \to 0} \|f_\epsilon - f\|_{L^{p(x)}(\Omega)} = 0 \tag{15}$$

under the assumptions of Theorem 3.8 *on $p(x)$.*

Remark 1 The statement (15) is an analogue of mean continuity property for $L^{p(x)}$-spaces, but with respect to the averaged 'shift' operator (14). In the standard form, the mean continuity property $\lim_{h \to 0} \|f(x+h) - f(x)\|_p = 0$, generally speaking, is not valid for variable exponents $p(x)$ and, moreover, there exist functions $p(x)$ and $f(x) \in L^{p(x)}$ such that $f(x+h_k) \notin L^{p(x)}$ for some $h_k \to 0$; see ref. [30, Example 2.9 and Theorem 2.10].

In this connection, we also note that in the paper by Fiorenza [53], it was shown that the known estimation of $\|f(x+h) - f(x)\|_p$ via $|h| \cdot \|\nabla f\|_p$ for functions $f \in W^{1,p}$ admits a certain extension for variable exponents $p(x)$ in the case of a bounded cube Ω in $\mathbb{R}^n$; see details in ref. [53, Theorem 2.1].

Remark 2 Compare Theorem 3.8 with a similar statement given in Theorem 5.7; the latter is given under less restrictive assumptions on integrability of the kernel $k(x)$, but on the other hand Theorem 3.8 does not require that $k(x)$ must have a decreasing integrable radial dominant.

In this survey, mainly devoted to the operator theory in the spaces $L^{p(\cdot)}$, we do not dwell more on Sobolev spaces $W^{m,p}(\Omega)$, but observe that some basic properties of these spaces were first obtained in the paper by Kovácǐk and Rákosnǐk [30]. Recently, there were obtained some breakthrough results on Sobolev imbeddings with variable exponents. We do not go into details with the exact formulations of the results on Sobolev imbeddings, but refer the interested reader to the papers by Edmunds and Rákosnǐk [54, 55], Rákosnǐk [56], Diening [34, 57, 58], Fan *et al.* [59, 60].

4. Hardy–type operators in the spaces $L^{p(\cdot)}$

Let now $n = 1$, $\Omega = (0, \ell)$ with $0 < \ell < \infty$ and

$$H^\beta f(x) = x^{\beta-1}\int_0^x \frac{f(t)}{t^\beta}\,\mathrm{d}t, \quad H_*^\beta f(x) = x^\beta \int_x^\ell \frac{f(t)}{t^{\beta+1}}\,\mathrm{d}t \tag{16}$$

be the weighted Hardy-type operators and

$$\mathcal{H}^\beta f(x) = x^\beta \int_0^\ell \frac{f(t)}{t^\beta (t+x)}\,\mathrm{d}t. \tag{17}$$

the weighted Hankel-type operator; obviously $\mathcal{H}^\beta f(x) \le H^\beta f(x) + H_*^\beta f(x)$ on non-negative functions $f(x)$. It was natural to expect that the boundedness of these operators in the spaces $L^{p(\cdot)}(0, \ell)$ holds under the assumptions (3) and (4) if

$$-\frac{1}{p(0)} < \beta < 1 - \frac{1}{p(0)}. \tag{18}$$

This is the case as was shown in refs. [61, 62] and even more, it suffices to assume that conditions (3) and (4) holds in a neigbourhood of the point $x = 0$.

THEOREM 4.1 (Kokilashvili and Samko) *Let conditions* (3) *and* (4) *be satisfied on a neighbourhood* $[0, d]$ *of the origin,* $d > 0$. *Then the operators* H^β, H_*^β *and* $\mathcal{H}^\beta$ *are bounded in the space* $L^{p(\cdot)}(0, \ell)$ *under condition* (18).

Theorem 4.1 was proved in refs. [61, 62] in a more general setting showing that the operators H^β, H_*^β and $\mathcal{H}^\beta$ are bounded from $L^{p(\cdot)}(0, \ell)$ into $L^{s(\cdot)}(0, \ell)$ with an arbiitrary $s(x)$ such that $p(0) = s(0)$, stated as follows.

THEOREM 4.2 (Kokilashvili and Samko) *Let* $1 \le p(x) \le P < \infty$ *for* $x \in [0, \ell]$.

I. *Let conditions* (3) *and* (4) *be satisfied on a neighbourhood* $[0, d]$ *of the origin,* $d > 0$. *Under condition* (18), *the operators* H^β, H_*^β *and* $\mathcal{H}^\beta$ *are bounded from* $L^{p(\cdot)}(0, \ell)$ *into* $L^{s(\cdot)}(0, \ell)$ *with any* $s(x)$ *such that* $1 \le s(x) \le S < \infty$ *for* $0 \le x \le \ell$,

$$s(0) = p(0) \quad \textit{and} \quad |s(x) - p(x)| \le \frac{A}{\ln(1/x)}, \ \ 0 < x < \delta, \ \ \delta > 0. \tag{19}$$

II. *If* $p(0) \le p(x)$, $0 \le x \le d$, *for some* $d > 0$, *then the same statement on boundedness from* $L^{p(\cdot)}(\Omega)$ *into* $L^{s(\cdot)}(\Omega)$ *is true if the requirement of the validity of conditions* (3) *and* (4) *on* $[0, d]$ *is replaced by the weaker assumption that*

$$p(0) > 1 \quad \textit{and} \quad |s(x) - p(0)| < \frac{A}{\ln(1/x)}, \ \ 0 < x < \min\left(\ell, \frac{1}{2}\right). \tag{20}$$

5. Maximal operators in the spaces $L^{p(\cdot)}$

Let

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} |f(y)|\,\mathrm{d}y \tag{21}$$

be the maximal operator and

$$M^{\beta} f(x) = |x-x_0|^{\beta} \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} \frac{|f(y)|}{|y-x_0|^{\beta}}\,\mathrm{d}y, \tag{22}$$

its weighted version, $x_0 \in \overline{\Omega}$.

For the operator M to be bounded in the space $L^{p(\cdot)}$, it was expected that the function $p(x)$ must be continuous and even more, satisfy condition (4).

First it was proved that condition (4) is in fact necessary. In ref. [63], the following statement was proved.

THEOREM 5.1 (Pick and Ružička) *Let* $\Omega = (-1, 1) \subset \mathbb{R}^1$ *and* $1 < p_0 < \infty$. *Let* φ *be a positive increasing function on* $[0, 1]$ *with* $\varphi(0) = 0$ *and*

$$\lim_{x\to 0+} \varphi(x) \ln\left(\frac{1}{x}\right) = \infty.$$

Assume that $p(x) \le p_0$ *for* $x \in (-1, 0]$ *and* $p(x) = p_0 + \varphi(x)$ *for* $x \in [0, 1)$. *Then the maximal operator* M *is not bounded in the space* $L^{p(\cdot)(\Omega)}$.

The sufficiency of condition (4), provided by the next theorem, was proved by Diening [34, 51].

THEOREM 5.2 (Diening) *Let* Ω *be a bounded domain. Under conditions* (3) *and* (4), *the maximal operator* M *is bounded in the space* $L^{p(\cdot)}(\Omega)$.

Diening also showed that this statement is valid in the case $\Omega = \mathbb{R}^n$ if $p(x)$ is constant outside some ball; see refs. [34, 57, 58].

When Ω is an unbounded domain, for the exponents $p(x)$ not necessarily constant at infinity, the boundedness results for the maximal operator were independently obtained by Nekvinda [64, 65] and Cruz-Uribe *et al.* [41, 42]. Their results run as follows.

THEOREM 5.3 (Cruz-Uribe, Fiorenza and Neugebauer) *Let* $p(x)$ *satisfy conditions* (3), (4) *and* (6). *Then the maximal operator* M *is bounded in the space* $L^{p(\cdot)}(\Omega)$.

THEOREM 5.4 (Nekvinda) *Let* $\Omega = \mathbb{R}^n$ *and* $p(x)$ *meet assumptions* (3) *and* (4) *and let there exist a constant* $p_\infty > 1$ *such that the function* $\varphi(x) = |p(x) - p_\infty|$ *satisfies the condition*

$$\int_{\mathbb{R}^n} \varphi(x) C^{1/\varphi(x)}\,\mathrm{d}x < \infty \quad \textit{for some } C > 0. \tag{23}$$

Then the maximal operator M *is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^n)$.

We note that assumption (23) for $\varphi(x) = |p(x) - p_\infty|$ is valid in the case of any function $p(x)$ satisfying condition (6). Indeed, equation (6) implies equation (5) and it remains to observe that the function $\varphi_0(x) = 1/\ln(e + |x|)$ satisfies equation (23), with any $C \in (0, 1/n)$

(and φ_0 multiplied by a positive bounded function, again satisfies equation (23); see Lemma 2.10 from ref. [65]).

In refs. [41, 42], it was also shown that the condition $\inf_{x\in} p(x) > 1$ is necessary for the maximal operator to be bounded, at least within the frameworks of upper semicontinuous exponents $p: \Omega \to [1, \infty)$; see ref. [42], Theorem 1.7. It was also proved in ref. [42] that when $p(x)$ stabilizes to a constant p_∞ at infinity, then the logarithmic condition (6) is also necessary in a sense, at the least in the one-dimensional case as can be seen from the following theorem of the type of Theorem 5.1.

THEOREM 5.5 (Cruz-Uribe, Fiorenza and Neugebauer) *Let* $\Omega = \mathbb{R}^1$ *and* $p(x) = p_\infty + \varphi(x)$, *where* $1 < p_\infty < \infty$ *and* $\varphi(x) \equiv 0$ *for* $x \leq 0$ *and* $\varphi(x)$ *is decreasing when* $x \geq 1$ *with* $\lim_{x\to\infty} \varphi(x) = 0$ *and* $0 \leq \varphi(x) \leq p_\infty - 1$. *If* $\lim_{x\to\infty} \varphi(x) \ln x = \infty$, *then the maximal operator* M *is not bounded in the space* $L^{p(\cdot)}(\Omega)$.

The importance of the logarithmic condition $\sup_{|x|>2} |p(x) - p_\infty| \ln x < \infty$ at infinity is also underlined by a counter-example of Edmunds and Nekvinda [36]. For the space $L^{p(\cdot)}(\mathbb{R}^1_+)$, they showed that there exists a bounded Lipschitz function $p(x)$ on $\mathbb{R}^1_+$ (but not satisfying the earlier condition at infinity) such that not only the maximal operator $(Mf)(x) = \sup_{r>0}(1/r) \int_{\max\{0,x-r\}}^{x+r} |f(y)|\,\mathrm{d}y$, but even the average

$$M_r f(x) = \frac{1}{r} \int_x^{x+r} f(y)\,\mathrm{d}y \quad \text{with } |M_r f(x)| \leq (M|f|)(x)$$

is not bounded in the space $L^{p(\cdot)}(\mathbb{R}^1_+)$ for any fixed $r > 0$.

Weak type estimate for the maximal operator in the case of variable p was given in the paper by Cruz-Uribe *et al.* [41, 42]. Up to our knowledge, this is the only result on weak estimates in $L^{p(\cdot)}$-spaces. As can be seen from its formulation as follows, it does not require the exponent $p(x)$ to be even continuous.

THEOREM 5.6 (Cruz-Uribe, Fiorenza and Neugebauer) *Given an open set* Ω, *suppose that* $p(x): \Omega \to [1, \infty)$ *can be extended to* $\mathbb{R}^n$ *in such a way that it satisfies the property*

$$\frac{1}{p(x)} \leq \frac{C}{|B|} \int_B \frac{\mathrm{d}y}{p(y)} \quad \textit{for any ball } B \textit{ and for almost all } x \in B.$$

Then

$$|\{x \in \Omega: Mf(x) > t\}| \leq C \int_\Omega \left(\frac{|f(y)|}{t}\right)^{p(y)} \mathrm{d}y. \tag{24}$$

A very important observation made by Diening [51, Corollary 3.6] was that the Stein theorem on uniform boundedness of dilation convolution operators with radial integrable dominant remains valid for the variable exponents. Namely, the following statement holds.

THEOREM 5.7 (Diening) *Let* $p(x)$ *satisfy condition* (12) *and* $k(x)$ *an integrable function whose least decreasing majorant is integrable, that is,* $A := \int_{\mathbb{R}^n} \sup_{|y|\geq|x|} |k(y)|\,\mathrm{d}x < \infty$. *Then*

(i) $|\sup_{\varepsilon>0}(1/\varepsilon^n) \int_{\mathbb{R}^n} k((x-y)/\varepsilon) f(y)\,\mathrm{d}y| \leq 2A(Mf)(x)$ *for all* $f \in L^{p(\cdot)}(\mathbb{R}^n)$; *if also the maximal operator* M *is bounded in* $L^{p(\cdot)}(\mathbb{R}^n)$, *then*

(ii) $\| \sup_{\varepsilon>0}(1/\varepsilon^n) \int_{\mathbb{R}^n} k((x-y)/\varepsilon) f(y)\,\mathrm{d}y\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$; *if in addition* $\int_{\mathbb{R}^n} k(y)\,\mathrm{d}y = 1$, *then also*

(iii) $(1/\varepsilon^n) \int_{\mathbb{R}^n} k((x-y)/\varepsilon) f(y)\,\mathrm{d}y \to f$ *in* $L^{p(\cdot)}(\mathbb{R}^n)$ *and almost everywhere.*

For the weighted maximal operator M^{β}, a criterion of boundedness in $L^{p(\cdot)}(\Omega)$ in the case of bounded domains was obtained in refs. [61, 62, 66]. To formulate this result, in the necessity part we need the following restriction on the boundary

$$|\Omega_r(x_0)| \sim r^n, \quad \text{where } \Omega_r(x_0) = \{y \in \Omega: r < |y - x_0| < 2r\}, \tag{25}$$

in the case $x_0 \in \partial\Omega$.

The necessary and sufficient condition (26) in the following theorem on the exponent β of the weight $|x - x_0|^{\beta}$ fixed to the point x_0 is naturally related to the local value of the exponent $p(x)$ at the point x_0.

THEOREM 5.8 (Kokilashvili and Samko) *Let Ω be a bounded domain and $p(x)$ satisfy conditions* (3) *and* (4). *The operator M^{β} with $x_0 \in \Omega$ is bounded in $L^{p(x)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{26}$$

If $x_0 \in \partial\Omega$, condition (26) *is sufficient for the boundedness of M^{β}. If $x_0 \in \partial\Omega$ and assumption* (25) *is satisfied, then condition* (26) *is also necessary for the boundedness of M^{β}.*

6. Singular operators in the spaces $L^{p(\cdot)}$

6.1 *Boundedness results*

There is also an evident progress on boundedness of singular operators in the spaces $L^{p(\cdot)}$. Diening and Ružička [24, 25] considered the Calderon–Zygmund-type operators. Let

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} k(x, x-y) f(y)\,\mathrm{d}y \tag{27}$$

where the kernel $k(x, y)$ satisfies the assumptions:

$$|k(x, y)| \le A|x - y|^{-n}, \tag{28}$$

$$|k(x, y) - k(z, y| \le A\frac{|x - z|^{\delta}}{|x - z|^{\delta+n}}, \quad |k(y, x) - k(y, z| \le A\frac{|x - z|^{\delta}}{|x - z|^{\delta+n}} \tag{29}$$

with some $A > 0$ and $\delta > 0$. When the operator T extends to a bounded operator on $L^2(\mathbb{R}^n)$, it is called the Calderon–Zygmund-type operator. It is known that any Calderon–Zygmund-type operator is bounded in any space $L^p(\mathbb{R}^n)$, $1 < p < \infty$, $p = \text{const}$; see ref. [67].

Let also

$$T^{*}f(x) = \sup_{\varepsilon > 0} \int_{|x-y|>\varepsilon} k(x, x-y) f(y)\,\mathrm{d}y. \tag{30}$$

Diening and Ružička [24, 25] proved the boundedness of the operators of T and T^{*} in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ under the assumption that also the following conditions are satisfied:

$$k(x, z) \text{ is homogeneous in } z \text{ of degree } -n, \tag{31}$$

$$\sup_{x\in\mathbb{R}^n} \int_{S_{n-1}} |k(x, z)|^r \,\mathrm{d}S(z) < \infty \text{ for some } r > 1 \quad \text{and} \quad \int_{S_{n-1}} k(x, z)\,\mathrm{d}S(z) = 0. \tag{32}$$

We refer also to the preprint [68] where the results from refs. [24, 25], are extended to Calderon–Zygmund singular operators related to the half-space $\mathbb{R}^{n+1}_{+}$.

The result of Diening and Ružička [24, 25] combined with Theorems 5.3 and 5.4 may be formulated as follows.

THEOREM 6.1 (Diening and Ružička) *Let the kernel $k(x, z)$ satisfy the assumptions* (28), (29) *and* (31), (32). *Then the operators T and T^* are bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$, if $p(x)$ satisfies assumptions* (3) *and* (4) *and one of the conditions* (6) *and* (23).

COROLLARY 6.1 *The singular integral operator*

$$Sf(x) = \frac{1}{\pi} \int_{\mathbb{R}^1} \frac{f(t)}{t - x} \, \mathrm{d}t, \quad x \in \mathbb{R}^1, \tag{33}$$

is bounded in the space $L^{p(\cdot)}(\mathbb{R}^1)$, if $p(x)$ satisfies assumptions (3) *and* (4) *and one of the conditions* (5) *and* (23).

As is known, for application the weighted boundedness of singular operators is required. In the case of bounded domains, the weighted estimates with power weights for the operators T and T^* were proved in refs. [69, 70]. Let

$$\rho(x) = \prod_{k=1}^{m} |x - a_k|^{\beta_k}, \quad \text{where } a_k \in \Omega$$

and

$$L^{p(\cdot)}(\Omega, \rho) = \{f \colon \rho f \in L^{p(\cdot)}(\Omega)\}.$$

THEOREM 6.2 (Kokilashvili and Samko) *Let Ω be a bounded domain and $p(x)$ satisfy assumptions* (3) *and* (4). *The operators T and T^* are bounded in the space $L^{p(\cdot)}(\Omega, \rho)$, if*

$$-\frac{n}{p(a_k)} < \beta_k < \frac{n}{q(a_k)}, \quad k = 1, \ldots, m. \tag{34}$$

Because of applications to integral equations, similar results on curves are of special interest. Let

$$L^{p(\cdot)}(\Gamma, \rho) = \{f \colon \|f[t(s)]\rho(s)\|_{L^{p(s)}} < \infty\},$$

where

$$\rho(s) = \prod_{k=1}^{m} |t(s) - t(c_k)|^{\beta_k} \approx \prod_{k=1}^{m} |s - c_k|^{\beta_k}, \quad c_k \in [0, \ell], \ k = 1, 2, \ldots, m. \tag{35}$$

THEOREM 6.3 (Kokilashvili and Samko) *Let Γ be a Lyapunov curve or a curve of bounded rotation without cusps and let $p(t)$ meet conditions* (7) *and* (8). *The singular operator*

$$S_\Gamma f(t) = \frac{1}{\pi} \int_\Gamma \frac{f(\tau) \, \mathrm{d}\tau}{\tau - t} \tag{36}$$

is bounded in the space $L^{p(\cdot)}(\Gamma, \rho)$ with the weight function (35) *if and only if*

$$-\frac{1}{p(c_k)} < \beta_k < \frac{1}{q(c_k)}, \quad k = 1, 2, \ldots, m. \tag{37}$$

COROLLARY 6.2 *Let* $\Omega = [a, b]$, $\rho(x) = \prod_{k=1}^{m} |x - a_k|^{\beta_k}$, $a_k \in [a, b]$, $k = 1, \ldots, m$, *and* $p(x)$ *satisfy conditions* (3) *and* (4) *on* $[a, b]$. *Then the finite Hilbert transform is bounded in the space* $L^{p(\cdot)}([a, b], \rho)$, *if* $-1/p(a_k) < \beta_k < 1/q(a_k)$, $k = 1, 2, \ldots, m$.

Another version of weighted estimates for the singular operator was given in refs. [71, 72] for some modification of the space $L^{p(\cdot)}$. Let $f^*(t) = \sup\{s \geq 0 \colon m\{x \in \Omega \colon |f(x)| > s\} > t\}$ be the usual non-increasing rearrangement of a function f, m denoting the Lebesgue measure; $f^*(t) = 0$ for $t > |\Omega|$. Let also

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(y)\,\mathrm{d}y, \qquad f^*(t) \leq f^{**}(t).$$

Assuming that the function p satisfies assumption (3) on $[0, \ell]$, $\ell = |\Omega|$, by $\Lambda^{p(\cdot)}(\Omega)$, we denote the space of functions measurable on Ω for which

$$\|f\|_{\Lambda^{p(\cdot)}(\Omega)} := \|f^{**}\|_{L^{p(\cdot)}[0,\ell]} < \infty, \tag{38}$$

which was introduced in refs. [71, 72].

Let

$$Kf(x) = v.p. \int_{R^n} \frac{k(y)}{|y|^n} f(x - y)\,\mathrm{d}y, \qquad x \in \Omega, \tag{39}$$

be the Calderon–Zygmund operator with an odd kernel k homogeneous of degree 0 and satisfying the Dini condition on the unit sphere S^{n-1}

$$\int_0^2 \frac{\omega(\delta)}{\delta}\mathrm{d}\delta < \infty, \qquad \text{where } \omega(\delta) = \sup_{x,y \in S^{n-1}, |x-y| \leq \delta} |k(x) - k(y)|.$$

THEOREM 6.4 (Kokilashvili and Samko) *Let* $1 \leq p(t) \leq P < \infty$ *on* $[0, \ell]$, $\ell = |\Omega|$ *and let the conditions* $1 < p_0 \leq p(t) < P < \infty$ *and*

$$|p(t_1) - p(t_2)| \leq \frac{A}{\ln(1/|t_1 - t_2|)}, \quad |t_1 - t_2| \leq \frac{1}{2}$$

be satisfied in a neighbourhood $[0, d]$ *of the origin,* $d > 0$. *Then the operator* K *is bounded in* $\Lambda^{p(\cdot)}(\Omega)$.

In refs. [71, 72], there was also obtained a weighted version of Theorem 6.4. Let $\Lambda_w^{p(\cdot)}(\Omega)$ be the weighted space of functions for which

$$\|f\|_{\Lambda_w^{p(\cdot)}} = \|wf^{**}\|_{L^{p(\cdot)}} < \infty$$

In the case $w(t) = t^\beta$, $-1/p(0) < \beta < 1/q(0)$, we have $\|f\|_{\Lambda_w^{p(\cdot)}} \approx \|wf^*\|_{\Lambda^{p(\cdot)}}$.

THEOREM 6.5 (Kokilashvili and Samko) *Let* $p(t)$ *satisfy assumptions of Theorem* 6.4 *on* $[0, \ell]$, $\ell = |\Omega|$ *and let* $w(t) = t^\beta$. *The operator* K *is bounded in the space* $\Lambda_w^{p(\cdot)}(\Omega)$, *if*

$$-\frac{1}{p(0)} < \beta < \frac{1}{q(0)}.$$

Similar statements on boundedness of singular operators in the spaces $\Lambda^{p(\cdot)}(\Gamma)$ and $\Lambda_w^{p(\cdot)}(\Gamma)$ on Lyapunov curves Γ or curves with bounded rotation without cusps were also given in refs. [71, 72]; see Theorems 4.1 and 4.3 in ref. [71].

6.2 *An open problem*

The validity of Theorem 6.3 on 'bad' curves and with general weights remains an open problem. Theorem 6.3 states that *the logarithmic smoothness condition* guarantees the boundedness of the singular operator on Lyapunov curves or curves of bounded rotation without cusps. An open question is whether the boundedness of the singular integral operator may be proved only under the logarithmic smoothness condition on an arbitrary Carleson curve.

Or, can it be proved on Carleson curves if $p(t)$ is even infinitely differentiable, but variable. Or probably on the whole class of Carleson curves, the boundedness may be true only for constant p? All these questions are open.

6.3 *Compactness of some integral operators in the spaces $L^{p(\cdot)}$*

Let

$$Af(x) = \int_{\Omega} \frac{c(x, y)}{|x - y|^{n-\alpha(x)}} f(y)\, \mathrm{d}y \tag{40}$$

be an integral operator with a weak singularity ('of variable order'). In refs. [61, 62], there was obtained the following statement.

THEOREM 6.6 (Kokilashvili and Samko) *Let Ω be a bounded open set in $\mathbb{R}^n$ and $c(x, y)$, a measurable bounded function on $\Omega \times \Omega$. Under assumptions* (3) *and* (4) *on $p(x)$ and the condition* $\inf_{x\in\overline{\Omega}} \alpha(x) > 0$, *the operator* (40) *is compact in the space $L^{p(\cdot)}(\Omega)$.*

We refer also to the paper by Edmunds and Meskhi [73], where some compactness statements were obtained for the operators of one-dimensional fractional integration operators on an interval $\Omega = [0, 1]$.

We also observe that the property of the commutators

$$aS_\Gamma - S_\Gamma a$$

where a stands for the operator of multiplication by a function $a \in C(\Gamma)$, to be compact in $L^p(\Gamma)$, $1 < p < \infty$, remains valid for the case of variable p under the only assumption that the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma)$.

6.4 *Fredholmness of singular integral equations in the spaces $L^{p(\cdot)}(\Gamma)$*

The theory of singular integral equations is a rather old and highly developed topic. We refer to the well-known books by Gakhov [74] and Muskhelishvili [75] for the 'classical' period and to the books by Gohberg and Krupnik [76, 77] for the later 'operator theory' period; many aspects of the modern theory of singular integral operators may be found in Böttcher and Karlovich [78, 79].

Let

$$A\varphi := \mathcal{A}(t)\varphi(t) + \mathcal{B}(t)(S_\Gamma\varphi)(t), \quad t \in \Gamma \tag{41}$$

be the well-known singular integral operator on a closed curve Γ. In the case where the coefficients $\mathcal{A}$ and $\mathcal{B}$ are continuous on Γ, the Fredholm properties and the index of the operator N, do not depend on the choice of the space, the only requirement to the space, in fact, is that the singular operator S_Γ must be bounded in the space under the consideration. When the coefficients $\mathcal{A}$ and $\mathcal{B}$ are discontinuous, the theory is more interesting. We remind the well-known result on Fredholmness of the operators (41) in the spaces $L^p(\Gamma)$ with constant p

in the case of piece-wise continuous coefficients, which is due to Gohberg and Krupnik [80]; see also refs. [76, 77]. First, we rewrite the operator A in the standard form

$$(A\varphi)(t) = a(t)(P_+\varphi)(t) + b(t)(P_-\varphi)(t), \tag{42}$$

via projectors $P_\pm = (1/2)(I \pm S_\Gamma)$.

We remind that a function $a(t) \in \mathrm{PC}(\Gamma)$ with the discontinuity points $t_1, t_2, \ldots, t_n \in \Gamma$ is said to be p-nonsingular on Γ, if $\inf_{t\in\Gamma} |a(t)| > 0$ and at all the points of discontinuity of $a(t)$ the following condition is satisfied:

$$\arg \frac{a(t_k - 0)}{a(t_k + 0)} \neq \frac{2\pi}{p} (\mathrm{mod}\, 2\pi), \quad k = 1, 2, \ldots, n. \tag{43}$$

Under these conditions, the integer

$$\mathrm{ind}_p\, a = \frac{1}{2\pi} \int_\Gamma \mathrm{d} \arg a(t) - \sum_{k=1}^{n} \frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)}, \tag{44}$$

where the values of $(1/2\pi) \arg(a(t_k - 0)/a(t_k + 0))$ are chosen in the interval

$$-\frac{1}{q} < \frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)} < \frac{1}{p}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{45}$$

is called the p-index of the function a.

The result on Fredholmness of the operator (42) with PC-coefficients in the spaces $L^p(\Gamma)$ is given by the following theorem.

THEOREM 6.7 (Gohberg and Krupnik) *Let Γ be a closed Lyapunov curve and let $p(t) \in \mathcal{P}$. The operator $A = aP_+ + bP_-$ with $a, b \in \mathrm{PC}(\Gamma)$ is Fredholm in the space $L^{p(\cdot)}(\Gamma)$ if and only if* $\inf_{t\in\Gamma} |a(t)| \neq 0$, $\inf_{t\in\Gamma} |b(t)| \neq 0$ *and the function $a(t)/b(t)$ is p-nonsingular. Under these conditions* $\mathrm{Ind}_{L_{p(\cdot)}} A = -ind_p(a/b)$.

Now, for the spaces $L^{p(\cdot)}(\Gamma)$, the question is whether it is possible to obtain a 'localized' version of the above theorem, that is to connect the jumps of the coefficients at the points t_k with the values of $p(t)$ at the points t_k. The answer is positive and the following corresponding result gives necessary and sufficient conditions for the operator A to be Fredholm in the space $L^{p(\cdot)}(\Gamma)$ together with a formula for the index under some natural assumptions on $p(x)$. The obtained criterion shows that Fredholmness of the operator A in the space $L^{p(t)}(\Gamma)$ and its index depend on values of the function $p(t)$ at the discontinuity points of the coefficients $a(t)$ and $b(t)$, but do not depend on values of $p(t)$ at points of continuity.

In fact, the result on Fredholmness is a consequence of the fact that we have necessary and sufficient conditions for the operator S_Γ to be bounded in the space $L^{p(t)}(\Gamma)$ with the power weight. They were given in Theorem 6.3.

We reformulate the above definitions for a variable exponent.

We say that a function $a(t) \in \mathrm{PC}(\Gamma)$ with discontinuity points $t_1, t_2, \ldots, t_n$ is $p(\cdot)$-nonsingular, if $\inf_{t\in\Gamma} |a(t)| > 0$ and at all the points of discontinuity of $a(t)$ the following condition is satisfied:

$$\arg \frac{a(t_k - 0)}{a(t_k + 0)} \neq \frac{2\pi}{p(t_k)} (\mathrm{mod}\, 2\pi), \quad k = 1, 2, \ldots, n. \tag{46}$$

Under these conditions, the integer

$$\mathrm{ind}_{p(\cdot)} a = \frac{1}{2\pi} \int_\Gamma \mathrm{d} \arg a(t) - \sum_{k=1}^{n} \frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)}, \tag{47}$$

where the values of $(1/2\pi) \arg(a(t_k - 0)/a(t_k + 0))$ are chosen in the interval

$$-\frac{1}{q(t_k)} < \frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)} < \frac{1}{p(t_k)}, \tag{48}$$

where $1/p(t) + 1/q(t) \equiv 1$ is called the $p(\cdot)$-index of the function a.

The following extension of Theorem 6.7 to the case of variable $p(t)$ was given in refs. [45, 46].

THEOREM 6.8 (Kokilashvili and Samko) *Let Γ be a closed Lyapunov curve or a curve of bounded rotation without cusps and let $p(t) \in \mathcal{P}$. The operator $A = aP_+ + bP_-$ with $a, b \in$* $\mathrm{PC}(\Gamma)$ *is Fredholm in the space $L^{p(\cdot)}(\Gamma)$ if and only if* $\inf_{t\in\Gamma} |a(t)| \neq 0$, $\inf_{t\in\Gamma} |b(t)| \neq 0$ *and the function $a(t)/b(t)$ is $p(\cdot)$-nonsingular. Under these conditions*

$$Ind_{L_{p(\cdot)}} A = -ind_{p(\cdot)} \frac{a}{b}. \tag{49}$$

We note that Theorem 6.8 was obtained in refs. [45, 46] as a corollary to a similar statement on Fredholmness of singular integral operators in abstract Banach function spaces. We mention briefly this general approach later which is, in fact, an abstract Banach space reformulation of the Gohberg–Krupnik scheme.

Let $X = X(\Gamma)$ be any Banach space of functions on a closed simple Jordan rectifiable curve Γ satisfying the following assumptions: (i) $C(\Gamma) \subset X(\Gamma) \subset L_1(\Gamma)$, (ii) $\|af\|_X \leq \sup_{t\in\Gamma} |a(t)| \cdot \|f\|_X$ for any $a \in L_\infty(\Gamma)$, (iii) the operator S is bounded in $X(\Gamma)$ and (iv) $C^\infty(\Gamma)$ is dense in $X(\Gamma)$.

We introduce also the following two axioms with $X(\Gamma, |t - t_0|^\Gamma) = \{f : |t - t_0|^\Gamma f(t) \in X(\Gamma)\}$ in the second axiom.

AXIOM 1 *For the space $X(\Gamma)$, there exist two functions $\alpha(t)$ and $\beta(t)$,* $0 < \alpha(t) < 1, 0 < \beta(t) < 1$, *such that the operator $|t - t_0|^{\gamma(t_0)} S |t - t_0|^{-\gamma(t_0)} I$, $t_0 \in \Gamma$ is bounded in the space $X(\Gamma)$ for all $\gamma(t_0)$ such that $-\alpha(t_0) < \gamma(t_0) < 1 - \beta(t_0)$ and is unbounded in $X(\Gamma)$ if $\gamma(t_0) \notin (-\alpha(t_0), 1 - \beta(t_0))$.*

We call the functions $\alpha(t)$ and $\beta(t)$ index functions of the space $X(\Gamma)$. For the spaces $X(\Gamma) = L^{p(\cdot)}(\Gamma, \rho) = \{f : |t - t_0|^\mu f(t) \in L^{p(\cdot)}(\Gamma)\}$ which are of the first interest for us, we have $\alpha(t) = \beta(t) = 1/p(t) + \mu$, at the least on Lyapunov curves or curves of bounded rotations without cusps, *according to Theorem* 6.3.

AXIOM 2 *For any $\nu < 1 - \beta(t_0)$, the imbedding $X(\Gamma, |t - t_0|^\nu) \subset L^1(\Gamma)$ is valid and $C^\infty(\Gamma)$ is dense in $X(\Gamma, |t - t_0|^\nu)$, whatsoever $t_0 \in \Gamma$ is.*

We note that the idea of singling out the bounds for the weight functions (used in Axioms 1 and 2) as the base of construction of Fredholm criterion is well known in the theory of singular integral operators; see refs. [79, 81, 82].

For a function $a \in \mathrm{PC}(\Gamma)$, we put $\gamma(t) = 1/(2\pi i) \ln a(t-0)/a(t+0)$ with $\Re\Gamma(t_k) := 1/2\pi \arg a(t_k - 0)/a(t_k + 0)$ and say that $a \in \mathrm{PC}(\Gamma)$ is X-nonsingular, where $X = X(\Gamma)$, if $\inf_{t\in\Gamma} |a(t)| > 0$ and $\Re\gamma(t_k) \notin [\alpha(t_k), \beta(t_k)] (\mathrm{mod}\, 1)$ where $\alpha(t)$ and $\beta(t)$ are the index functions of the space X. The integer

$$\mathrm{ind}_X\, a = \frac{1}{2\pi} \int_\Gamma \mathrm{d} \arg a(t) - \sum_{k=1}^{n} \Re\gamma(t_k) \tag{50}$$

where $\Re\gamma(t_k)$ are chosen in the interval $\beta(t_k) - 1 < \Re\gamma(t_k) < \alpha(t_k)$ will be referred to as X-index of the function a.

THEOREM 6.9 *Let $X(\Gamma)$ be any Banach function space satisfying assumptions* (i)–(iv) *and Axioms* 1 *and* 2. *The operator $A = aP_+ + bP_-$ with $a, b \in \mathrm{PC}(\Gamma)$ is Fredholm in the space X if* (a) $\inf_{t\in\Gamma} |a(t)| \neq 0$, $\inf_{t\in\Gamma} |b(t)| \neq 0$ *and* (b) *$a(t)/b(t)$ is X-nonsingular. In this case,*

$$\mathrm{Ind}_X A = -\mathrm{ind}_X \frac{a}{b}. \tag{51}$$

Condition (a) *is also necessary for the operator A to be Fredholm in X. If the index functions $\alpha(t)$ and $\beta(t)$ of the space X coincide at the points t_k of discontinuity of the coefficients $a(t), b(t)$: $\alpha(t_k) = \beta(t_k),\ k = 1, 2, \ldots, n$, then condition* (b) *is necessary as well.*

A further generalization of Fredholmness results for singular integral operators in Banach function spaces, including the necessity of Fredholmness conditions for the weighted spaces $L^{p(\cdot)}(\Gamma, \rho)$ on 'bad' curves may be found in ref. [83].

7. Potential operators: the Sobolev theorem and weighted estimates

7.1 *On Sobolev theorem*

The boundedness of the Riesz-type potential operator

$$I^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha(x)}}\, \mathrm{d}y \tag{52}$$

from the space $L^{p(\cdot)}(\Omega)$ into the space $L^{q(\cdot)}(\Omega)$ with the limiting Sobolev exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \tag{53}$$

was an open problem for a long time. It still remains open for unbounded domains in the general case.

For many applications, it suffices to consider the case of constant α, but within the frameworks of variable exponents, it is natural to deal with the variable α in relation (53) as well. The order $\alpha(x)$ of the potential is not assumed to be continuous.

In ref. [33], in the case of bounded domains Ω, there was proved the following conditional result.

THEOREM 7.1 (Samko) *Let Ω be a bounded open set in $\mathbb{R}^n$ and $p(x)$ satisfy assumptions* (3) *and* (4) *and let*

$$\inf_{x\in\Omega} \alpha(x) > 0 \quad \text{and} \quad \sup_{x\in\Omega} p(x)\alpha(x) < n. \tag{54}$$

If the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$, then the Sobolev theorem

$$\|I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}$$

is valid.

After Diening [34, 51] proved the boundedness of the maximal operator over bounded domains (see Theorem 5.2), the validity of the Sobolev theorem for bounded domains became an unconditional statement.

For the whole space $\mathbb{R}^n$, the Sobolev theorem was proved by Diening [34, 57, 58, Theorem 3.8], under the condition that the exponent $p(x)$ is constant outside some ball of large radius.

THEOREM 7.2 (Diening) *Let $\Omega = \mathbb{R}^n$, $\alpha = \text{const}$, $0 < \alpha < n$, and let $p(x)$ satisfy conditions* (3) *and* (4) *and be constant outside some large ball $B(0, R)$. If $\sup_{x\in\mathbb{R}^n} p(x) \leq (n/\alpha)$, then*

$$\|I^{\alpha}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{55}$$

Another version of the Sobolev theorem for the space $\mathbb{R}^n$ was proved in refs. [84, 85] for the exponents $p(x)$ not necessarily constant in a neigbourhood of infinity, but with some 'extra' power weight fixed to infinity and under the assumption that $p(x)$ takes its minimal value at infinity.

THEOREM 7.3 (Kokilashvili and Samko) *Let $\Omega = \mathbb{R}^n$, $1 < p(\infty) \leq p(x) \leq P < \infty$ and $p(x)$ satisfy conditions* (3)–(5) *and let $\alpha(x)$ meet conditions* (54) *and also the condition $\sup_{x\in\mathbb{R}^n} p(\infty)\alpha(x) < n$. Then the following weighted Sobolev-type estimate is valid for the operator $I^{\alpha(\cdot)}$:*

$$\|(1+|x|)^{-\gamma(x)} I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{56}$$

where $1/q(x) = 1/p(x) - \alpha(x)/n$ and $\gamma(x) = A_\infty \alpha(x)[1 - (\alpha(x)/n)]$ with the constant A_∞ from equation (5).

Note that $\gamma(x) \leq (n/4)A_\infty$.

We observe that the proof of Theorem 7.3 is based on Theorem 5.3 and on a modification of the estimates for $\| \, |x - x_0|^{\beta(x_0)} \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(x_0, r))}$, as $r \to 0$ and $r \to \infty$, uniform in x_0 obtained in ref. [33].

7.2 *Weighted estimates*

Weighted estimates for the Riesz potential in spaces with variable exponent are also of great interest. Let

$$I_\beta^{\alpha(x)} f(x) = |x - x_0|^\beta \int_\Omega \frac{f(y)\,\mathrm{d}y}{|y - x_0|^\beta |x - y|^{n-\alpha(x)}}, \quad x_0 \in \overline{\Omega}. \tag{57}$$

In refs. [61, 62, 66], the following statement was obtained.

THEOREM 7.4 *Let Ω be an open bounded set, $p(x)$ satisfy conditions* (3) *and* (4) *and* $\inf_{x\in\Omega}\alpha(x) > 0$. *Then the operator $I_\beta^{\alpha(\cdot)}$ is bounded in $L^{p(\cdot)}(\Omega)$ if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{58}$$

The Hardy-type inequality given in the next theorem was proved in ref. [86].

THEOREM 7.5 (Samko) *Let Ω be a bounded open set in $\mathbb{R}^n$, $x_0 \in \overline{\Omega}$ and $p(x)$ satisfy conditions* (3) *and* (4). *Suppose that $\alpha(x)$ satisfies the same logarithmic condition as $p(x)$ in equation* (4) *and* $\inf_{x\in\Omega}\alpha(x) > 0$ *and $\alpha(x_0) < n$. Then the Hardy-type inequality is valid*

$$\left\| |x - x_0|^{\beta-\alpha} \int_\Omega \frac{f(y)\,\mathrm{d}y}{|y - x_0|^\beta |x - y|^{n-\alpha(x)}} \right\|_{L^{p(\cdot)}(\Omega)} \le c\|f\|_{L^{p(\cdot)}(\Omega)} \tag{59}$$

for all β in the interval

$$\alpha - \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{60}$$

In the one-dimensional case, for the Riemann–Liouville fractional integration operators

$$(I_{0+}^{\alpha(\cdot)} f)(x) = \int_0^x \frac{f(t)\,\mathrm{d}t}{(x - t)^{1-\alpha(x)}}, \quad (I_{-}^{\alpha(\cdot)} f)(x) = \int_x^\infty \frac{f(t)\,\mathrm{d}t}{(t - x)^{1-\alpha(x)}}, \quad x \in \mathbb{R}^1_+, \tag{61}$$

the Hardy-type inequality of the type (59) with $\beta = 0$ was proved in ref. [74]. After some reformulation, the results from ref. [73] may be given as follows.

THEOREM 7.6 (Edmunds and Meskhi) *Let $0 < \alpha(x) \le 1$ on* $[0, 1]$ *and $p(x)$ satisfy assumptions* (3) *and* (4) *on $\Omega = [0, 1]$. Then*

$$\|x^{-\alpha(x)}|2^{\alpha(x)} - 1|(I_{0+}^{\alpha(\cdot)} f)(x)\|_{L^{p(\cdot)}[0,1]} \le C\|f\|_{L^{p(\cdot)}[0,1]}. \tag{62}$$

THEOREM 7.7 (Edmunds and Meskhi) *Let $\alpha(x)\colon \mathbb{R}^1_+ \to (0, 1]$ be a non-decreasing function on $\mathbb{R}^1_+$ and $p(x)$ satisfy assumptions* (3–5) *on $\Omega = \mathbb{R}^1_+$. Then*

$$\|vI_{0+}^{\alpha(\cdot)} wf\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}, \tag{63}$$

where

$$v(x) = x^{-\alpha(x)}(1 + x)^{(2/p'(x))+1}[2^{\alpha(x)} - 1], \quad w(x) = (1 + x)^{(2p(x))-\alpha(x)-1}$$

In addition, if $\alpha(x)$ satisfies assumptions (4) *and* (5) *on $\mathbb{R}^1_+$, then also*

$$\|\rho I_{-}^{\alpha(\cdot)} rf\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)},$$

where

$$\rho(x) = (1 + x)^{(2/p'(x))-\alpha(x)-1}, \quad r(x) = x^{-\alpha(x)}(1 + x)^{(2/p(x))+1}[2^{\alpha(x)} - 1].$$

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