Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 8, NUMBER 1 (2005) ISSN 1311-0454

BEST CONSTANT IN THE WEIGHTED HARDY INEQUALITY: THE SPATIAL AND SPHERICAL VERSION

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Dedicated to Acad. Bogoljub Stanković,
on the occasion of his 80-th anniversary

Abstract

The sharp constant is obtained for the Hardy-Stein-Weiss inequality for fractional Riesz potential operator in the space $L^p(R^n, \rho)$ with the power weight $\rho = |x|^\beta$. As a corollary, the sharp constant is found for a similar weighted inequality for fractional powers of the Beltrami-Laplace operator on the unit sphere.

Mathematics Subject Classification: 26D10
Key Words and Phrases: Hardy inequality, Rellich inequality, fractional powers, Riesz potentials, Beltrami-Laplace operator, stereographic projection

1. Introduction

The inequality of the type

$$\int_{\mathbb{R}^n} |f(x)|^2 |x|^\mu \, dx \leq c \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} f(x)|^2 |x|^{\mu+2\alpha} \, dx, \qquad \alpha > 0, \tag{1.1}$$

where $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional power of the minus Laplace operator, is known as the Hardy-Rellich-type inequality.

In (1.1) the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ on "nice" functions may be treated via Fourier transforms: $(-\Delta)^{\frac{\alpha}{2}} f = F^{-1}|\xi|^{\alpha} F f$; on not so nice functions it may be treated as the hypersingular integral (see [11], Ch. 3).

In [2] there was calculated the best constant in inequality (1.1) or in the inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} f(x)|^2 |x|^{2(\alpha+\eta)-n}\, dx \le c \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\beta}{2}} f(x)|^2 |x|^{2(\beta+\eta)-n}\, dx \tag{1.2}$$

which is, in fact, an equivalent form of (1.1), under the appropriate assumptions on the parameters involved. The best constant for the case $\mu = 0$ in (1.1) was calculated in [15].

Meanwhile, the best constant may be calculated, and in an effective form, for a more general non-Hilbert version of inequality (1.1):

$$\int_{\mathbb{R}^n} |f(x)|^p |x|^{\mu}\, dx \le C^p \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} f(x)|^p |x|^{\gamma}\, dx \tag{1.3}$$

with

$$\gamma = \mu + \alpha p, \tag{1.4}$$

see Theorem 2.1. In the case when $\frac{\alpha}{2}$ is an integer, $\frac{\alpha}{2} = 1, 2, 3, ...$, the sharp constant for (1.3) was calculated in [1]. We also note the best constant for an inequality of the type (1.3) corresponding to the case $\alpha = 1$ but with different L_p- and L_2-norms on the left and right hand sides, was studied in [4].

An integral form of (1.3) may be given in terms of the Riesz potential operator:

$$\int_{\mathbb{R}^n} |x|^{\mu}\, |I^{\alpha}\varphi(x)|^p\, dx \le C^p \int_{\mathbb{R}^n} |x|^{\gamma} |\varphi(x)|^p\, dx, \tag{1.5}$$

where the Riesz potential operator I^{α} is defined for all $\alpha > 0$ as

$$I^{\alpha}\varphi(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{\varphi(y)\, dy}{|x-y|^{n-\alpha}}, \qquad \alpha > 0, \alpha \ne n + 2k, \qquad k \in \mathbb{N}, \tag{1.6}$$

where $\gamma_n(\alpha) = \frac{2^{\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$, see for instance [11], p.37. The validity of inequality (1.5), known as "doubly weighted" Stein-Weiss inequality, see [7], was proved in [13] under the natural assumptions on the parameters:

$$1 \le p < \infty, \qquad \alpha > 0, \qquad \alpha p - n < \gamma < n(p-1), \qquad \mu = \gamma - \alpha p, \tag{1.7}$$

which are indeed necessary for inequality (1.5) to be valid, as it follows from Theorem 3.1 ; the relation $\mu = \gamma - \alpha p$ being a consequence of the homogeneity of the kernel and weights, easily obtained by dilatation arguments, see [12], Subsection 1.2 of Ch. 5. We note also that the condition $\alpha p - n < \gamma < n(p-1)$ implies that necessarily $0 < \alpha < n$.

We show that it is possible to easily calculate the best constant in (1.5), for arbitrary $p > 1$ as a simple ratio of gamma-functions. The crucial points of the calculation are: 1) the exact knowledge of the norm of an integral operator in $L^p(\mathbb{R}^n)$ with kernel homogeneous of degree $-n$ and invariant with respect to rotations (see Theorem 3.1), and 2) the Catalan formula for integrals over sphere of functions depending on scalar product, see (3.5).

Note that the sharp constant for the Sobolev theorem, that is, for the equality of the type (1.5) with $\beta = \gamma = 0$, but with the norm in the space $L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ on the left hand side, was calculated by E. Lieb [7] in the cases $p = q'$ or $p = 2$ or $q = 2$.

Remark 1.1. The weighted Hardy inequality, corresponding to the case $\alpha > n$ was proved by T. Kurokawa [6] who showed that

$$\int_{\mathbb{R}^n} |x|^{-\alpha p} \left| (I^\alpha f)(x) - \sum_{|j| \le m} \frac{(D^j I^\alpha f)(0)}{j!} x^j \right|^p dx \le C \|f\|_p^p, \tag{1.8}$$

where $1 < p < \infty$, $m = \left[\alpha - \frac{n}{p}\right]$, $\alpha - \frac{n}{p} \notin \mathbb{N}_0$; in case $\alpha - \frac{n}{p}$ is a non-negative integer, in (1.8) there appears the logarithmic factor on the left hand side.

The sharp constant calculated in [15] corresponds to the non-weighted case in (1.8) and $p = 2$.

Remark 1.2. Inequalities (1.1)-(1.3) are valid for "nice" functions f, for example, for f in the Lizorkin subspace of the Schwarts space, see Subsection 3.3, which is dense in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, see Theorem 3.2. Inequality (1.5) holds for any $\varphi \in L^p(\mathbb{R}^n)$ under assumptions (1.7). Therefore, inequalities (1.1)-(1.3) hold on the range $I^\alpha[L^p(\mathbb{R}^n, \rho)]$ of the Riesz potential operator, where $\rho(x) = |x|^{\frac{\mu}{p}}$, see the characterization of this range (including the case $p \ge \frac{n}{\alpha}$) in [11], Section 1 of Ch. 7 in the non-weighted case $\mu = 0$ and Section 3 of Ch. 7 for the weighted case. This range is in fact the domain of definition of $(-\Delta)^{\frac{\alpha}{2}}$ in $L^p(\mathbb{R}^n, \rho)$, it may be characterized in terms of the weighted L_p-convergence of the hypersingular integrals of the form

$$\int_{|y|>\varepsilon} \frac{\Delta_y^\ell f(x)}{|y|^{n+\alpha}}\, dy, \qquad \ell > \alpha,$$

where $\Delta_y^\ell f(x) = \sum_{k=0} \ell(-1)^k \binom{\ell}{k} f(x-kt)$, see for instance Theorems 7.9 and 7.36 in [11].

We also consider the Hardy-Rellich inequality in weighted L^p-norms for the fractional powers $(-\delta)^{\frac{\alpha}{2}}$ of the spherical Beltrami-Laplace operator:

$$\int_{S^{n-1}} \rho(\sigma)|f(\sigma)|^p\, d\sigma \leq C_1^p \int_{S^{n-1}} \rho_1(\sigma) \left| f(\sigma) + (-\delta)^{\frac{\alpha}{2}} f(\sigma)\right|^p\, d\sigma, \qquad (1.9)$$

where the power weight functions

$$\rho(\sigma) = |\sigma-\sigma_0|^\mu \cdot |\sigma+\sigma_0|^\nu \quad \text{and} \quad \rho_1(\sigma) = |\sigma-\sigma_0|^{\mu+\alpha p} \cdot |\sigma+\sigma_0|^{\nu+\alpha p},\ \sigma \in S^{n-1} \qquad (1.10)$$

are fixed to an arbitrary pair of the opposite poles $\pm\sigma_0 \in S^{n-1}$, and

$$0 < \alpha < n-1, \qquad 1 < p < \infty, \qquad (1.11)$$

$$1-n < \mu < (n-1)(p-1) - \alpha p, \quad \mu + \nu = (n-1)(p-2) - \alpha p. \quad (1.12)$$

One may also replace $I + (-\delta)^{\frac{\alpha}{2}}$ on the right-hand side of (1.9) by $(I-\delta)^{\frac{\alpha}{2}}$; in the case of functions f with $f_0 := \int_{S^{n-1}} f(\sigma)\, d\sigma = 0$ it is possible also to take just $(-\delta)^{\frac{\alpha}{2}}$.

Similarly to inequality (1.5), its spherical analogue may be considered:

$$\int_{S^{n-1}} \rho(\sigma)|I_S^\alpha \varphi(\sigma)|^p\, d\sigma \leq C_2^p \int_{S^{n-1}} \rho_1(\sigma)\, |\varphi(\sigma)|^p\, d\sigma \qquad (1.13)$$

under conditions (1.11)-(1.12) and the assumption that the integral on the right-hand side exists. Here

$$I_S^\alpha \varphi(\xi) = \frac{1}{\gamma_{n-1}(\alpha)} \int_{S^{n-1}} \frac{\varphi(\sigma)}{|\xi-\sigma|^{n-1-\alpha}}\, d\sigma, \qquad \xi \in S^{n-1}, \qquad (1.14)$$

where $\alpha > 0,\ \alpha - n + 1 \notin \mathbb{N}$, is the spherical Riesz-type potential operator, see [11], p.151.

In the case of inequality (1.13) we also calculate the best constant C_2.

N o t a t i o n:
S^{n-1} is the unit sphere in $\mathbb{R}^n$; $|S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is its area;
$\omega(x)$ is a rotation in $\mathbb{R}^n$: $|\omega(x)| = |x|$;
$e_1 = (1, 0, ..., 0)$, $e_n = (0, ..., 0, 1)$, $x \cdot y = x_1 y_1 + \cdots + x_n y_n$;
$\frac{1}{p} + \frac{1}{p'} = 1$.

2. Statements of the main results

THEOREM 2.1. *Let* $\alpha > 0$, $\alpha - n \notin \mathbb{N}$. *Inequality (1.3) is valid on all the functions for which the right-hand side is finite if and only if conditions (1.7) are satisfied. Under these conditions the best constant in (1.3) is given by*

$$C = 2^{-\alpha} \frac{\Gamma\left(\frac{n(p-1)-\gamma}{2p}\right)\Gamma\left(\frac{n+\gamma-\alpha p}{2p}\right)}{\Gamma\left(\frac{n+\gamma}{2p}\right)\Gamma\left(\frac{n(p-1)+\alpha p-\gamma}{2p}\right)}. \tag{2.1}$$

COROLLARY . *In the case* $p = 2$ *the best constant for (1.1) is given by*

$$c = \left[2^{-\alpha} \frac{\Gamma\left(\frac{n+\mu}{4}\right)\Gamma\left(\frac{n-\mu}{4} - \frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\mu}{4}\right)\Gamma\left(\frac{n+\mu}{4} + \frac{\alpha}{2}\right)}\right]^2 \tag{2.2}$$

under the assumptions $0 < \alpha < n$, $-n < \mu < n - 2\alpha$, *and for (1.2) by*

$$c = \left[2^{\alpha-\beta} \frac{\Gamma\left(\frac{\alpha+\eta}{2}\right)\Gamma\left(\frac{n-\beta-\mu}{2}\right)}{\Gamma\left(\frac{\beta+\eta}{2}\right)\Gamma\left(\frac{n-\alpha-\eta}{2}\right)}\right]^2, \tag{2.3}$$

under the assumptions $\beta \geq \alpha$, $-\alpha < \eta < n - \beta$.

We observe that the condition $-n < \mu < n - 2\alpha$ on the weight (or the condition $-\alpha < \eta < n - \beta$ in case of (2.3)) is necessary for the validity of inequality (1.5) in the case $p = 2$.

THEOREM 2.2. *Inequalities (1.9) and (1.13) are valid provided that* $0 < \alpha < n - 1$ *and*

$$-(n-1) < \mu < (n-1)(p-1) - \alpha p. \tag{2.4}$$

The best constant in (1.13) is equal to

$$C = 2^{-2\alpha} \frac{\Gamma\left(\frac{(n-1)(p-1)-\mu-\alpha p}{2p}\right)\Gamma\left(\frac{n-1+\mu}{2p}\right)}{\Gamma\left(\frac{n-1+\mu+\alpha p}{2p}\right)\Gamma\left(\frac{(n-1)(p-1)-\mu}{2p}\right)}. \tag{2.5}$$

Corollary . *In the case $p = 2$ inequality (1.13) takes the form*

$$\int_{S^{n-1}} \frac{|\sigma-\sigma_0|^{n-1-2\theta}}{|\sigma+\sigma_0|^{n-1-2\theta+2\alpha}} |I_S^\alpha \varphi(\sigma)|^2 \, d\sigma$$

$$\leq C_2^2 \int_{S^{n-1}} \frac{|\sigma-\sigma_0|^{n-1-2\theta+2\alpha}}{|\sigma+\sigma_0|^{n-1-2\theta}} |\varphi(\sigma)|^2 \, d\sigma, \qquad (2.6)$$

where $\alpha < \theta < n-1$ and the best constant is equal to

$$C_2 = 2^{-2\alpha} \frac{\Gamma\left(\frac{\theta-\alpha}{2}\right)\Gamma\left(\frac{n-1-\theta}{2}\right)}{\Gamma\left(\frac{\theta+\alpha}{2}\right)\Gamma\left(\frac{\theta}{2}\right)} \, . \qquad (2.7)$$

3. Preliminaries

3.1. Integral operators with kernels homogeneous of degree $-n$

Let $k(x,y)$ satisfy the homogeneity condition

$$k(\lambda x, \lambda y) = \lambda^{-n} k(x,y), \qquad \lambda > 0, \quad x, y \in \mathbb{R}^n \qquad (3.1)$$

and the condition of the invariance

$$k[\omega(x), \omega(y)] = k(x,y), \qquad x, y \in \mathbb{R}^n \qquad (3.2)$$

for any rotation $\omega(x)$ in $\mathbb{R}^n$. Let

$$\kappa_p = \int_{\mathbb{R}^n} |k(\sigma, y)| \frac{dy}{|y|^{\frac{n}{p}}}, \qquad \sigma \in S^{n-1}. \qquad (3.3)$$

Because of the invariance condition (3.2), the integral in (3.3) does not depend on the choice of $\sigma \in S^{n-1}$ (see details in [5]), so one may choose $\sigma = e_1 = (1, 0, ..., 0)$ in (3.3).

The following statement is known, see [5], p.70, Th. 6.4.

Theorem 3.1. *Under assumptions (3.1)-(3.2) and the condition $\kappa_p < \infty$, the operator*

$$\mathbf{K} f(x) = \int_{\mathbb{R}^n} k(x,y) \, dy \qquad (3.4)$$

is bounded in the space $L_p(\mathbb{R}^n), 1 \leq p \leq \infty$ and $\|\mathbf{K}\| \leq \kappa_p$. If $k(x,y) \geq 0$, then the condition $\kappa_p < \infty$ is necessary for the boundedness and $\|\mathbf{K}\| = \kappa_p$.

3.2. The Catalan formula and some preliminaries from special functions

The following formula

$$\int_{S^{n-1}} f(x \cdot \sigma)\, d\sigma = |S^{n-2}| \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} f(|x|t)\, dt, \qquad x \in \mathbb{R}^n \tag{3.5}$$

is known as the Catalan formula, see [5], p.13, [3], N 4.644.

We remind the following formulas for the gamma-function:

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right), \quad x \in \mathbb{R}^1_+, \tag{3.6}$$

$$\Gamma(x+k) = \Gamma(x)(x)_k, \quad (x)_k = x(x+1)\cdots(x+k-1), \quad x \in \mathbb{R}^1_+. \tag{3.7}$$

The binomial coefficients are given by

$$\binom{\nu}{k} = \frac{(-1)^k \Gamma(k-\nu)}{k!\Gamma(-\nu)}. \tag{3.8}$$

We shall also need the hypergeometric function

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}, \qquad |z| < 1. \tag{3.9}$$

Observe that

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \qquad \gamma > \max\{\alpha, \beta, \alpha+\beta\}. \tag{3.10}$$

3.3. The Lizorkin space Φ

The Lizorkin space Φ (see for instance, [11], p.39) is a subspace of the Schwartz space $\mathcal{S}$ which consists of functions orthogonal to all the polynomials:

$$\int_{\mathbb{R}^n} x^j f(x\, dx = 0, \quad j = (j_1, j_2, ..., j_n), \quad |j| = 0, 1, 2, \dots. \tag{3.11}$$

Obviously, the Fourier transforms of functions $f \in \Phi$ vanish at the origin together with all their partial derivatives.

The subspace Φ is invariant with respect to the operator I^α and its (left) inverse $(-\Delta)^{\frac{\alpha}{2}}$:

$$(-\Delta)^{\frac{\alpha}{2}}(\Phi) = I^\alpha(\Phi) = \Phi.$$

THEOREM 3.2. *The space Φ is dense in the weighted space $L^p(\mathbb{R}^n, \rho)$, $1 < p < \infty$, with an arbitrary Muckenhoupt weight $\rho(x)$.*

The proof of Theorem 3.2 may be found in ([8]; [11], p.41, for the non-weighted case and [11], p.195 in the weighted case.

3.4. The Beltrami-Laplace operator and stereographic projection

Let

$$\delta f = |x|\Delta f\left(\frac{x}{|x|}\right), \qquad \frac{x}{|x|} \in S^{n-1}$$

be the Beltrami-Laplace operator, defined on functions $f(\sigma)$ on the unit sphere S^{n-1}. It has eigenvalues $-m(m+n-2),\ m = 1, 2, 3, \dots$:

$$\delta Y_m = -m(m+n-2)Y_m$$

for any spherical harmonic Y_m, so that the fractional power $(-\delta)^{\frac{\alpha}{2}}$ on "nice" functions is defined by

$$(-\delta)^{\frac{\alpha}{2}} f = \sum_{m=0}^{\infty} [m(m+n-2)]^{\frac{\alpha}{2}} Y_m(f, x), \qquad x \in S^{n-1},$$

where

$$Y_m(f, x) = \frac{d_n(m)}{|S^{n-1}|} \int_{S^{n-1}} f(\sigma) P_m(x \cdot \sigma)\, d\sigma$$

is the m-th harmonic component of $f(x), x \in S^{n-1},\ P_m(t)$ being the Legendre polynomial and $d_n(m) = (n+2m-2)\frac{(n+m-3)!}{m!(n-2)!}$, see details on spherical harmonics and Fourier-Laplace series in [12], [13], [10], [11], p.145.

Following S. Mikhlin [9], p.35-36, we shall use the stereographic projection of the sphere S^{n-1} onto the space $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n :\ x_n = 0\}$ which is generated by the change of variables in $\mathbb{R}^n$:

$$\xi = s(x) = \{s_1(x), s_2(x), \dots, s_n(x)\} \tag{3.12}$$

with $s_k(x) = \frac{2x_k}{1+|x|^2},\ k = 1, 2, \dots, n-1$ and $s_n(x) = \frac{|x|^2-1}{|x|^2+1}$. Here $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Let also $\sigma = s(y),\ y \in \mathbb{R}^n$. The following is valid:

$$|\xi - \sigma| = \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}},\ d\sigma = \frac{2^{n-1}\,dy}{(1+|y|^2)^{n-1}},\ dy = \frac{2^{n-1}\,d\sigma}{|\sigma - e_n|^{2(n-1)}}, \tag{3.13}$$

see [9], p.35-36. Also

$$|x| = \sqrt{\frac{1+\xi_n}{1-\xi_n}} = \frac{|\xi + e_n|}{|\xi - e_n|}, \quad \sqrt{1+|x|^2} = \sqrt{\frac{2}{1-\xi_n}} = \frac{2}{|\xi - e_n|}. \tag{3.14}$$

4. Proof of Theorem 2.1

Since the best constant in (1.3) is the same as in (1.5), we base ourselves on inequality (1.5) and rewrite it in the form

$$\int_{\mathbb{R}^n} |x|^\mu \left| I^\alpha \left(\frac{u(\cdot))}{|\cdot|^{\frac{\gamma}{p}}} \right) \right|^p dx \le c \int_{\mathbb{R}^n} |u(x)|^p\, dx, \tag{4.1}$$

where $u(x) = |x|^{\frac{\gamma}{p}} \varphi(x) \in L^p(\mathbb{R}^n)$. Therefore, the best constant in (1.5), or which is the same, in (4.1) is the norm of the integral operator

$$\mathbf{K}u(x) = \int_{\mathbb{R}^n} k(x,y)u(y)\,dy, \qquad k(x,y) = \frac{1}{\gamma_n(\alpha)} \cdot \frac{|x|^{\frac{\mu}{p}}}{|y|^{\frac{\mu}{p}+\alpha}|x-y|^{n-\alpha}}.$$

The kernel $k(x,y)$ is homogeneous of degree $-n$, invariant with respect to rotations and non-negative. Therefore, by Theorem 3.1,

$$C = \|\mathbf{K}\|_{L^p \to L^p} = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{dy}{|y|^{\frac{\mu+n}{p}+\alpha}|y-e_1|^{n-\alpha}}. \tag{4.2}$$

Thus, the problem of the best constant in (1.5) is reduced to calculation of the integral in (4.2). We denote for brevity

$$J(a,b) = \int_{\mathbb{R}^n} \frac{dy}{|y|^a|y-e_1|^b}, \qquad 0<a<n,\ 0<b<n,\ a+b>n. \tag{4.3}$$

Obviously, $J(a,b) = J(b,a)$.

Lemma 4.1. *Under the conditions* $0<a<n,\ 0<b<n,\ a+b>n$

$$J(a,b) = \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{n-a}{2}\right)\Gamma\left(\frac{n-b}{2}\right)\Gamma\left(\frac{a+b-n}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(n-\frac{a+b}{2}\right)}. \tag{4.4}$$

P r o o f. Passing to polar coordinates in (4.3), we have

$$J(a,b)=\int_0^\infty \rho^{n-1-a}d\rho\int_{S^{n-1}}\frac{d\sigma}{(\rho^2-2\rho\sigma_1+1)^{\frac{b}{2}}}$$

$$=\int_0^\infty \frac{\rho^{n-1-a}}{(\rho^2+1)^{\frac{b}{2}}}d\rho\int_{S^{n-1}}(1-r\sigma_1)^{-\frac{b}{2}}\,d\sigma,$$

where $r=\frac{2\rho}{1+\rho^2}\le 1$. Making use of the Catalan formula (3.5), we obtain

$$J(a,b)=|S^{n-2}|\int_0^\infty \frac{\rho^{n-1-a}}{(\rho^2+1)^{\frac{b}{2}}}d\rho\int_{-1}^1(1-t^2)^{\frac{n-3}{2}}(1-rt)^{-\frac{b}{2}}\,d\sigma$$

$$=|S^{n-2}|\int_{-1}^1(1-t^2)^{\frac{n-3}{2}}V(t)\,dt, \tag{4.5}$$

where

$$V(t)=\int_0^\infty \frac{\rho^{n-1-a}}{(\rho^2+1)^{\frac{b}{2}}}(1-rt)^{-\frac{b}{2}}\,d\rho.$$

To calculate the integral $V(t)$, we expand $(1-rt)^{-\frac{b}{2}}$ into the binomial series $(1-rt)^{-\frac{b}{2}}=\sum_{k=0}^\infty\binom{-b/2}{k}(-rt)^k$ and get

$$V(t)=\sum_{k=0}^\infty\binom{-b/2}{k}(-2t)^k\int_0^\infty\frac{\rho^{n-1-a+k}}{(\rho^2+1)^{\frac{b}{2}+k}}\,d\rho. \tag{4.6}$$

The integral

$$I_k(t)=\int_0^\infty\frac{\rho^{n-1-a+k}}{(\rho^2+1)^{\frac{b}{2}+k}}\,d\rho=\frac{1}{2}B\left(\frac{n-a+k}{2},\frac{a+b-n+k}{2}\right) \tag{4.7}$$

is easily calculated via the change $\rho^2+1=\frac{1}{s}$. Therefore, from (4.6) with (3.8) and (4.7) taken into account, we arrive at

$$V(t)=\frac{1}{2}\sum_{k=0}^\infty\frac{\Gamma\left(\frac{n-a+k}{2}\right)\Gamma\left(\frac{a+b-n+k}{2}\right)}{\Gamma\left(\frac{b}{2}\right)\Gamma\left(k+1\right)}(2t)^k.$$

We substitute this into (4.5) and observe that the terms with odd values of k vanish so that

$$J(a,b)=\frac{|S^{n-2}|}{\Gamma\left(\frac{b}{2}\right)}\sum_{k=0}^\infty 2^{2k}\frac{\Gamma\left(k+\frac{n-a}{2}\right)\Gamma\left(k+\frac{a+b-n}{2}\right)}{\Gamma\left(2k+1\right)}\int_0^1 t^{2k}(1-t^2)^{\frac{n-3}{2}}\,dt.$$

Making use of the duplication formula (3.6) for $\Gamma(2k+1)$ and substituting $t = \sqrt{s}$, after easy calculations we arrive at the series

$$J(a,b) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{b}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma(k+\nu_1)\Gamma(k+\nu_2)}{k!\Gamma\left(k+\frac{n}{2}\right)}$$

where we denoted $\nu_1 = \frac{n-a}{2}$ and $\nu_2 = \frac{a+b-n}{2}$ for brevity. By (3.7) this reduces to

$$J(a,b) = \frac{\pi^{\frac{n}{2}}\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{\infty} \frac{(\nu_1)_k(\nu_1)_k}{\left(\frac{n}{2}\right)} \frac{1}{k!}$$

$$= \frac{\pi^{\frac{n}{2}}\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{n}{2}\right)} F\left(\nu_1, \nu_2; \frac{n}{2}; 1\right)$$

and we obtain (4.4) thanks to (3.10). ■

The lemma having been proved, it remains to use (4.4) in (4.2), which yields

$$\|K\|_{L^p \to L^p} = 2^{-\alpha} \frac{\Gamma\left(\frac{n}{2p'} - \frac{\gamma}{2p}\right)\Gamma\left(\frac{n+\gamma}{2p} - \frac{\alpha}{2}\right)}{\Gamma\left(\frac{n+\gamma}{2p}\right)\Gamma\left(\frac{n}{2p'} + \frac{\alpha p-\gamma}{2p}\right)}$$

and we arrive at (2.1).

Finally, we observe that (2.2) is contained in (2.1) with $p = 2$, while in order to obtain (2.3) from (2.1), it suffices to denote $(-\Delta)^{\frac{\alpha}{2}} f(x) = \varphi(x)$ and replace α by $\beta - \alpha$.

5. Proof of Theorem 2.2

It is natural to reduce (1.13) to (1.5) via the stereographic projection (3.12). It is interesting to observe that E. Lieb [7], on the contrary, reduced the consideration on the Euclidean space to the case of the sphere via the stereographic projection, in order to construct the maximizing functions for the inequalities in $\mathbb{R}^n$. We proceed in the opposite direction, finding more convenient to use the homogeneity property of the potential kernel in $\mathbb{R}^n$ and then to derive the required information for the Beltrami-Laplace operator on the unit sphere as a corollary to the results for $\mathbb{R}^n$. (We mention also the paper [14] where the stereographic projection was used to transfer some properties of the spatial Riesz potentials to the case of spherical Riesz potentials).

It is known that the spherical Riesz potential (1.14) is connected with the spatial Riesz potential over $\mathbb{R}^{n-1}$ in terms of the stereographic projection:

$$\int_{S^{n-1}} \frac{f(\sigma)}{|\xi-\sigma|^{n-1-\alpha}}\, d\sigma$$

$$= 2^\alpha (1+|x|^2)^{\frac{n-1-\alpha}{2}} \int_{\mathbb{R}^{n-1}} \frac{f[s(y)]\, dy}{|x-y|^{n-1-\alpha}(1+|y|^2)^{\frac{n-1+\alpha}{2}}},$$

where $\xi \in S^{n-1},\ x = s^{-1}(\xi) \in \mathbb{R}^{n-1}$, see [11], p.153; inversely,

$$\int_{\mathbb{R}^{n-1}} \frac{\varphi(y)\, dy}{|x-y|^{n-1-\alpha}} = 2^\alpha |\xi - e_n|^{n-1-\alpha} \int_{S^{n-1}} \frac{\varphi_*(\sigma)}{|\xi-\sigma|^{n-1-\alpha}}\, d\sigma, \tag{5.1}$$

where for given $x \in \mathbb{R}^{n-1}$ one has $\xi = s(x) \in S^{n-1}$ and

$$\varphi_*(\sigma) = \frac{\varphi[s^{-1}(\sigma)]}{|\sigma - e_n|^{n-1+\alpha}} = \left(\frac{\sqrt{1+|y|^2}}{2}\right)^{n-1+\alpha} \varphi(y), \quad y = s^{-1}(\sigma). \tag{5.2}$$

In notations (5.2) and (1.14), relation (5.1) has the form

$$I^\alpha \varphi(x) = 2^\alpha |\xi - e_n|^{n-1-\alpha} I_S^\alpha \varphi_*(\xi), \quad \xi = s(x),\ x \in \mathbb{R}^{n-1},\ \xi \in S^{n-1}, \tag{5.3}$$

where $I^\alpha\varphi(x)$ on the left-hand side stands for the Riesz potential (1.6) over $\mathbb{R}^{n-1}$. The following isometry is also directly checked in view of (3.13) and (3.14):

$$\int_{\mathbb{R}^{n-1}} |y|^\mu |\varphi(y)|^p\, dy = 2^{n-1} \int_{S^{n-1}} \rho(\sigma) |\varphi_*(\sigma)|^p\, d\sigma, \tag{5.4}$$

where

$$\rho(\sigma) = |\sigma + e_n|^\mu |\sigma + e_n|^\nu, \qquad \nu = (n-1)(p-2) + \alpha p - \mu.$$

We rewrite the Hardy-Stein-Weiss inequality (1.5) for the dimension $n-1$:

$$\int_{\mathbb{R}^{n-1}} |x|^\mu\, |I^\alpha \varphi(x)|^p\ dx \le c \int_{\mathbb{R}^{n-1}} |x|^\gamma |\varphi(x)|^p\, dx, \tag{5.5}$$

$$-n-1 < \mu < (n-1)(p-1) - \alpha p, \qquad \gamma - \mu + \alpha p$$

and substitute (5.2)-(5.4) into (5.5). After easy calculations we get

$$\int_{S^{n-1}} |\sigma + e_n|^\mu |\sigma - e_n|^{(n-1)(p-2)-\alpha p - \mu}\, |I_S^\alpha \varphi_*(\sigma)|^p\ d\sigma$$

$$\leq 2^{-\alpha p} C^p \int_{S^{n-1}} |\sigma + e_n|^{\mu+\alpha p} |\sigma - e_n|^{(n-1)(p-2)-\mu} |\varphi_*(\sigma)|^p \, d\sigma \tag{5.6}$$

which is the desired inequality (1.13) in the case of the pole $\sigma = -e_n$. The passage to an arbitrary pole $\sigma \in S^{n-1}$ may be done by means of the corresponding rotation, taking into account that the Riesz potential I_S^α is rotation invariant.

Since the best constant for (1.5) is given in (2.1), from (5.6) we see that the best constant for (1.13) is $c_2 = 2^{-\alpha} c$, which is given in (2.5).

It remains to show the validity of (1.9), or which is the same, the validity of (1.13) with I_S^α replaced by the operator $\left[I + (-\delta)^{\frac{\alpha}{2}}\right]^{-1}$. The latter is the spherical convolution operator with the Fourier-Laplace multiplier (eigenvalues):

$$\lambda_m = \frac{1}{[m(m+n-2)]^{\frac{\alpha}{2}}} = \sum_{j=0}^{N} \frac{c_j}{m^{\alpha+j}} + O\left(\frac{1}{m^{\alpha+N}}\right), \qquad c_o \neq 0,$$

as $m \to \infty$, where $N = 1, 2, 3, ...$ is arbitrary. Such operators are spherical convolution operators with kernel $k(x \cdot \sigma)$ dominated by the Riesz kernel, see [11], Lemma 6.21. Therefore, the pointwise estimate is valid:

$$\left| \left[I + (-\delta)^{\frac{\alpha}{2}}\right]^{-1} f(x) \right| \leq c(I_S^\alpha |f|)(x), \qquad x \in S^{n-1}$$

which yields (1.9).

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Received: November 7, 2004

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