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WEIGHTED BOUNDEDNESS IN LEBESGUE SPACES WITH VARIABLE EXPONENTS OF CLASSICAL OPERATORS ON CARLESON CURVES

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1. Introduction

On an arbitrary Carleson curve $\Gamma = \{t \in \mathbb{C} : t = t(s),\, 0 \le s \le \ell \le \infty\}$ (finite or infinite) we consider maximal and singular operators and potential type operators. There are proved theorems on weighted boundedness of maximal and singular operators in the generalized Lebesgue spaces with variable exponent $p(\cdot)$ and power type weight and weighted Sobolev-type $p(\cdot) \to q(\cdot)$-theorems for potential operators on Γ.

Let $\nu(t) = s$ be the arc-length measure and let $\Gamma(t, r) := \Gamma \cap B(t, r)$, $t \in \Gamma$, $r > 0$, $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$, so that $\nu\{\Gamma(t, r)\} \le c_0 r$ for a Carleson curve, with $c > 0$ not depending on $t \in \Gamma$ and $r > 0$.

We consider along Carleson curves Γ the following operators within the frameworks of weighted spaces $L^{p(\cdot)}(\Gamma, w)$, $w(t) = \prod\limits_{k=1}^{n} |t - t_k|^{\beta_k}$, $t_k \in \Gamma$ with variable exponent $p(t)$:

$$Mf(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t, r)\}} \int\limits_{\Gamma(t,r)} |f(\tau)| d\nu(\tau) \tag{1.1}$$

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{f(\tau)}{\tau - t} d\nu(\tau), \qquad I^{\alpha(\cdot)} f(t) = \int\limits_{\Gamma} \frac{f(\tau)\, d\nu(\tau)}{|t - \tau|^{1-\alpha(t)}} \tag{1.2}$$

where it is supposed that

$$\alpha_- := \inf_{t\in\Gamma} \alpha(t) > 0, \quad \alpha_+ := \sup_{t\in\Gamma} \alpha(t) < 1. \tag{1.3}$$

2. Definitions

Let p be a measurable function on Γ such that $p : \Gamma \to (1, \infty)$. In what follows we assume that p satisfies the conditions

$$1 < p_- := \operatorname*{ess\,inf}_{t\in\Gamma} p(t) \le \operatorname*{ess\,sup}_{t\in\Gamma} p(t) =: p_+ < \infty, \tag{2.1}$$

$$\big|p(t) - p(\tau)\big| \le \frac{A}{\ln \frac{1}{|t-\tau|}}, \quad t \in \Gamma, \quad \tau \in \Gamma, \quad |t - \tau| \le \frac{1}{2}. \tag{2.2}$$

Definition 2.1. By $\mathcal{P} = \mathcal{P}(\Gamma)$ we denote the class of exponents p satisfying condition (2.1) and by $\mathbb{P} = \mathbb{P}(\Gamma)$ the class of those p for which the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma)$.

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The generalized Lebesgue space with variable exponent is defined via the modular

$$I_\Gamma^p(f) := \int\limits_\Gamma |f(t)|^{p(t)}\, d\nu(\tau)$$

by the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I_\Gamma^p\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

By $L^{p(\cdot)}(\Gamma, w)$ we denote the weighted Banach space of all measurable functions $f : \Gamma \to \mathbb{C}$ such that $\|f\|_{L^{p(\cdot)}(\Gamma,w)} := \|wf\|_{p(\cdot)} < \infty$. We denote $p'(t) = \frac{p(t)}{p(t)-1}$.

3. The main statements

We consider the power weights of the form $w(t) = \prod\limits_{k=1}^{n} |t - t_k|^{\beta_k}$, $t_k \in \Gamma$ in the case of finite curve and the weights, and $w(t) = |t - z_0|^\beta \prod\limits_{k=1}^{n} |t - t_k|^{\beta_k}$, $t_k \in \Gamma$, $z_0 \notin \Gamma$ in the case of infinite curve.

Theorem A. *Let* i) Γ *be a simple Carleson curve of a finite length;* ii) p *satisfy conditions* (2.1)–(2.2). *Then the maximal operator* M *is bounded in the space* $L^{p(\cdot)}(\Gamma, w)$, *if and only if* $-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}$, $k = 1, \ldots, n$.

Theorem B. *Let* i) Γ *be an infinite simple Carleson curve;* ii) p *satisfy conditions* (2.1)–(2.2) *and let there exist a circle* $B(0, R)$ *such that* $p(t) \equiv p_\infty = \text{const}$ *for* $t \in \Gamma\backslash(\Gamma \cap B(0, R))$. *Then the maximal operator* M *is bounded in the space* $L^{p(\cdot)}(\Gamma, w)$, *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}$$

and

$$-\frac{1}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'_\infty}.$$

The Euclidean space versions of Theorems A and B for variable exponents were proved in [9] and [4], respectively.

Theorem C. *Let* i) Γ *be a simple Carleson curve of a finite length;* ii) p *satisfy conditions* (2.1)–(2.2)*;* iii) $\alpha(t)$ *satisfy assumptions* (1.3) *and the condition* $\sup\limits_{t\in\Gamma} \alpha(t)p(t) < 1$. *Then the operator* $I^{\alpha(\cdot)}$ *is bounded from the space* $L^{p(\cdot)}(\Gamma)$ *into the space* $L^{q(\cdot)}(\Gamma)$ *with* $\frac{1}{q(t)} = \frac{1}{p(t)} - \alpha(t)$. *This statement remains valid for infinite Carleson curves if, in addition to conditions* i)–iii), $p \in \mathbb{P}$, *in particular, if* $p(t) = p_\infty = const$ *outside some circle* $B(0, R)$.

Theorem D. *Let* Γ *be a finite Carleson curve. Under assumptions* i)–iii) *of Theorem C, the operator* $I^{\alpha(\cdot)}$ *is bounded from the space* $L^{p(\cdot)}(\Gamma, w)$ *into the space* $L^{q(\cdot)}(\Gamma, w)$ *where* $\frac{1}{q(t)} = \frac{1}{p(t)} - \alpha(t)$, *if*

$$\alpha(t_k) - \frac{1}{p(t_k)} < \beta_k < 1 - \frac{1}{p(t_k)}, \quad k = 1, \ldots, n.$$

Theorem E. *Let* i) Γ *be a simple Carleson curve;* ii) p *satisfy conditions* (2.1)–(2.2), *and the following condition at infinity*

$$|p(t) - p(\tau)| \leq \frac{A_\infty}{\ln \frac{1}{\left|\frac{1}{t} - \frac{1}{\tau}\right|}}, \quad \left|\frac{1}{t} - \frac{1}{\tau}\right| \leq \frac{1}{2}, \quad |t| \geq L, \quad |\tau| \geq L \tag{3.1}$$

for some $L > 0$ in the case Γ is an infinite curve. Then the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, w)$, if and only if

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \quad k = 1, \dots, n$$

and also

$$-\frac{1}{p(\infty)} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'(\infty)}$$

in the case Γ is infinite.

For constant p Theorem D is due to G. David [3] in the non-weighted case, for the weighted case with constant p see [2] and [5]. For earlier results on the subject we refer to [6], Theorem 2.2. The statements of Theorem D and Theorem E for variable $p(\cdot)$ was proved in the case of finite Lyapunov curves or curves of bounded turning without cusps in [7] and [8] respectively.

Theorem F. *Let $a \in C(\Gamma)$ and in the case where Γ is an infinite curve starting and ending at infinity, let $a \in C(\dot{\Gamma})$, where $\dot{\Gamma}$ is the compactification of Γ by a single infinite point, that is, $a(t(-\infty)) = a(t(+\infty))$. Then under conditions of Theorem E, the operator*

$$(S_\Gamma aI - aS_\Gamma)f = \frac{1}{\pi i} \int_\Gamma \frac{a(\tau) - a(t)}{\tau - t} f(\tau) d\nu(\tau)$$

is compact in the space $L^{p(\cdot)}(\Gamma, w)$.

We observe that the proofs are essentially based on the following statements we preliminary prove.

Lemma 3.1. *Let $t_0 \in \Gamma$ and $0 \le \beta < 1$. Then*

$$J_\beta(t, \tau; r) := \frac{|t - t_0|^\beta}{\nu(\Gamma(t, r))} \int_{\Gamma(t,r)} \frac{d\nu(\tau)}{|\tau - t_0|^\beta} \le c < \infty \tag{3.2}$$

where $c > 0$ does not depend on $t, t_0 \in \Gamma$ and $r > 0$.

For the sharp maximal function

$$\mathcal{M}^\# f(t) = \sup_{r>0} \frac{1}{\nu(\Gamma(t, r))} \int_{\Gamma(t,r)} \left| f(\tau) - f_{\Gamma(t,r)} \right| d\nu(\tau) \tag{3.3}$$

where $f_{\Gamma(t,r)} = \frac{1}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} f(\tau)\, d\nu(\tau)$, there is valid the following extension to the case of variable exponent $p(\cdot)$ of the result known for Euclidean space.

Theorem 3.2. *Let Γ be an infinite Carleson curve. Let $p(t)$ satisfy conditions (2.1)–(2.2) and $p(t) = p_\infty$ outside some ball $B(t_0, R)$. Let $w(t) = |t - t_0|^\beta$, $t_0 \in \mathbb{C}$, where $-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)}$ and $-\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty}$, if $t_0 \in \Gamma$ and $-\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty}$, if $t_0 \notin \Gamma$. Then for $f \in L^{p(\cdot)}(\Gamma, w)$*

$$\|f\|_{L^{p(\cdot)}(\Gamma,w)} \le c \left\| \mathcal{M}^\# f \right\|_{L^{p(\cdot)}(\Gamma,w)}. \tag{3.4}$$

Theorem 3.3. *Let $p(t)$ satisfy conditions (2.1)–(2.2). If $0 \le \beta < \frac{1}{p'(t_0)}$, then*

$$\left[\frac{1}{r} \int_{\Gamma(t,r)} \Big(\frac{|t - t_0|}{|\tau - t_0|} \Big)^\beta \left| f(\tau) \right| d\nu(\tau) \right]^{p(t)} \le c \left(1 + \frac{1}{r} \int_{\Gamma(t,r)} \left| f(\tau) \right|^{p(\tau)} d\nu(\tau) \right) \tag{3.5}$$

for all $f \in L^{(p(\cdot)}(\Gamma)$ such that $\|f\|_{p(\cdot)} \le 1$, where $c = c(p, \beta)$ is a constant not depending on t, $t_0 \in \Gamma$ and $r > 0$.

Lemma 3.4. *Let Γ be a bounded Carleson curve of the length ℓ, $0<r<\ell$, t, $t_0\in\Gamma$, $\sigma>-1$ and a bounded measurable function $h(t)$ defined on Γ satisfy the conditions* $\sup\limits_{x\in\Gamma}|h(t)|:=H<\infty$, $\sup\limits_{t\in\Gamma}[h(t)+1]:=-d_0<0$ *and* $\sup\limits_{t\in\Gamma}[h(t)+1+\sigma]:=-d_1<0$. *Then*

$$A(t,t_0;r):=\int\limits_{\Gamma\setminus\Gamma(t,r)}\big|t-\tau\big|^{h(t)}\big|\tau-t_0\big|^{\sigma}\,d\nu(\tau)\le Cr^{h(t)+1}\big(r+|t-t_0|\big)^{\sigma},\quad t\in\Gamma. \tag{3.6}$$

Let $\chi_r(\rho)=\begin{cases}1, & \text{if } \rho>r\\ 0, & \text{if } \rho<r\end{cases}$, let $g_\delta(t,\tau,r)=|t-\tau|^{\delta(t)}\chi_r(|t-\tau|)$, (where $\delta(t)=\alpha(t)-1$) and $n_{\delta,\varkappa,p}(t,r)=\|g_\delta(t,\tau,r)\|_{L^{p(\cdot)}(\Gamma,|\tau-t_0|^{\varkappa(\tau)})}$.

Theorem 3.5. *Let Γ be a bounded Carleson curve, $t_0\in\Gamma$, let $p\in\mathbb{P}(\Gamma)$, $\varkappa\in L^\infty(\Gamma)$ and $\delta\in L^\infty(\Gamma)$ and let also $\varkappa(t)$ satisfy the logarithmic condition at the point t_0*

$$\big|\varkappa(\tau)-\varkappa(t_0)\big|\le\frac{A_2}{\ln\frac{1}{|\tau-t_0|}},\quad \tau\in\Gamma,\quad |\tau-t_0|\le\frac{1}{2} \tag{3.7}$$

and let $\varkappa(t_0)p(t_0)>-1$. If $\sup\limits_{t\in\Gamma}[\delta(t)p(t)+1]:=-d_0<0$, $\sup\limits_{t\in\Gamma}\big\{1+[\delta(t)+\varkappa(t)]p(t)\big\}:=-d_1<0$ *and* $\sup\limits_{t\in\Gamma}\big\{1+[\delta(t)+\varkappa(t_0)]p(t)\big\}:=-d_2<0$, *then*

$$n_{\delta,\varkappa,p}(t,r)\le Cr^{\delta(t)+\frac{1}{p(t)}}(r+|t-t_0|)^{\varkappa(t)}. \tag{3.8}$$

for all $t\in\Gamma$, $0<r<\ell$, where $C>0$ does not depend on t and r.

Proposition 3.6. *Let Γ be a simple Carleson curve. Then the following pointwise estimate is valid*

$$\mathcal{M}^{\#}\left(|S_\Gamma f|^s\right)(t)\le c[Mf(t)]^s,\quad 0<s<1, \tag{3.9}$$

where the constant $c>0$ may depend on Γ and s, but does not depend on $t\in\Gamma$ and f.

Proposition 3.6 for singular integrals in the Euclidean space was proved in [1].

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