Published in
"Izvestija VUZov. Severo-Kavkazskii region.
Estestvennie nauki 2005,
Special issue
"Pseudodifferential equations
and some problems of mathematical physics
dedicated to 70th birthday of Prof. I.B.Simonenko,
p. 152-158.

УДК 539.3

BOUNDEDNESS IN LEBESGUE SPACES WITH VARIABLE EXPONENT OF MAXIMAL, SINGULAR AND POTENTIAL OPERATORS

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We present results on the boundedness of the maximal operator with oscillating weights on domains in Euclidean space and maximal and singular operators with power weights on an arbitrary Carleson curve in the Lebesgue spaces with variable exponent. There are also given weighted Sobolev type theorems for potential operators on Carleson curves.

1. Introduction

Last years there was a strong rise of interest to the study of the classical operators of harmonic analysis - maximal, singular and potential operators in the spaces known as the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, with variable order $p(x)$ (the generalized Lebesgue spaces with variable exponent), or also as spaces with non-standard growth.

These spaces and Sobolev spaces based on them proved to be of importance in the study of differential equations with $p(x)$-Laplacian, variational problems and applications to mechanics of the continuum medium. In some problems of mechanics there arise variational problems with Lagrangians, for example, of the form $|\xi|^{\alpha(x)}$ when the character of non-linearity varies from point to point (Lagrangians of the plasticity theory, Lagrangians of the mechanics of the so-called rheological fluids, and others).

These applications gave rise to a rapidly developing field of harmonic analysis related to these spaces. The interest of many researchers, apart from applications, was also stirred up by the difficulties they met. These difficulties are, in particular caused by the fact that the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are not invariant with respect to translations and dilatations and convolutions $k * f$ do not obey the standard Young theorem $\|k*f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|k\|_{L^1(\mathbb{R}^n)}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Roughly speaking, a convolution may be a candidate for the boundedness, if its kernel nowhere has singularities except for the origin. Calderon-Zygmund singular operators are among them, as well as the maximal operator.

The state of affairs in this field up to the beginning of 2004 was summarized in the survey papers [27], [16]. Meanwhile in the subsequent two years 2004-2005 there were obtained further important results, concerning in particular weighted spaces and operators on Carleson curves.

In this paper we will discuss some of these results. We will mainly focus upon the following items:
1) boundedness of the maximal operator in the weighted spaces $L^{p(\cdot)}(\Omega, \rho)$ in the case of bounded domains $\Omega \in \mathbb{R}^n$ and a certain class of non-power weights characterized in terms of their Boyd-type indices;
2) boundedness of the maximal operator in the weighted spaces $L^{p(\cdot)}(\Omega, \rho)$ over finite or infinite Carleson curves in the complex plane in the case of power weights;
3) boundedness of the Cauchy singular operator in the same setting as in 2);
4) Sobolev-type theorem with variable exponents for potential operators on Carleson curves, finite or infinite;
5) generalization of 2)-4) to the case of homogeneous type spaces (HTS).

Sections 3.-7. of the paper correspond to the above items 1)-5).

2. Preliminaries

Let Ω be an open set in $\mathbb{R}^n$. By $L^{p(\cdot)}(\Omega)$ we denote the space of functions $f(x)$ on Ω with the finite norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$

We assume that $p(x)$ satisfies the conditions

$$1 < p_- \le p(x) \le p_+ < \infty, \quad x \in \Omega, \tag{2.2}$$

$$|p(x)-p(y)| \le \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \le \frac{1}{2}, \quad x,\, y \in \Omega. \tag{2.3}$$

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell \le \infty\}$ be a simple Carleson curve with arc-length measure $\nu(t) = s$, $\nu\{\Gamma(t,r)\} \le c_0 r$. In the sequel we denote

$$\Gamma(t,r) := \Gamma \cap B(t,r), \qquad t \in \Gamma, \quad r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$. By $L^{p(\cdot)}(\Gamma, w)$ we denote the weighted Banach space of all measurable functions $f : \Gamma \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma,w)} := \|wf\|_{L^{p(\cdot)}(\Gamma)}$$

$$= \inf\left\{\lambda > 0 : \int\limits_\Gamma \left|\frac{w(t)f(t)}{\lambda}\right|^{p(t)} d\nu(t) \le 1\right\} < \infty.$$

Similarly to (2.2) and (2.3) we assume that

$$1 < p_- := \operatorname*{ess\,inf}_{t\in\Gamma} p(t) \le \operatorname*{ess\,sup}_{t\in\Gamma} p(t) =: p_+ < \infty, \tag{2.4}$$

$$|p(t)-p(\tau)| \le \frac{A}{ln \frac{1}{|t-\tau|}}, \quad t, \tau \in \Gamma, \quad |t-\tau| \le \frac{1}{2}. \tag{2.5}$$

3. Maximal operators; the Euclidean case

3.1 Existing results.

Let

$$\mathcal{M}^\beta f(x) = \sup_{r>0} \frac{|x-x_0|^\beta}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} \frac{|f(y)|}{|y-x_0|^\beta}\, dy,$$

be be the weighted maximal operator, $x_0 \in \overline{\Omega}$, $\mathcal{M} = \mathcal{M}^0$.

The first important result on the boundedness of the maximal operator in the variable exponent spaces is due to L.Diening ([7] and [9]) who proved that for a bounded domain Ω the operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega)$ under conditions (2.2) and (2.3). He also showed that this statement is valid in the case $\Omega = \mathbb{R}^n$ if $p(x)$ is constant outside some ball, see [8]-[10]. When Ω is an unbounded domain, for the exponents $p(x)$ not necessarily constant at infinity, the boundedness results for the maximal operator were obtained in [5] and [23].

Theorem below with the criterion of the boundedness of the operator $\mathcal{M}^\beta$ in $L^{p(\cdot)}(\Omega)$ in the case of bounded domains was obtained in [20], where the following restriction on the boundary is used in the necessity part:

$$|\Omega_r(x_0)| \sim r^n, \quad x_0 \in \partial\Omega \tag{3.6}$$

where $\Omega_r(x_0) = \{y \in \Omega : \quad r < |y-x_0| < 2r\}$. The necessary and sufficient condition (3.7) in the following theorem on the exponent β of the weight $|x-x_0|^\beta$ fixed to the point x_0, is naturally related to the local value of the exponent $p(x)$ at the point x_0.

Theorem. Let Ω be a bounded domain and $p(x)$ satisfy conditions (2.2) and (2.3). In the case $x_0 \in \Omega$ the operator $\mathcal{M}^\beta$ is bounded in $L^{p(x)}(\Omega)$ if and only if

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}, \tag{3.7}$$

where $p'(x_0) = \frac{p(x_0)}{p(x_0)-1}$. In the case $x_0 \in \partial\Omega$, condition (3.7) is sufficient for the boundedness of $\mathcal{M}^\beta$ and also necessary if (3.6) is satisfied.

3.2 About more general weights

We show that it is possible to obtain the boundedness result for the weighted maximal operator

$$\mathcal{M}^w f(x) = \sup_{r>0} \frac{w(x)}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} \frac{|f(y)|}{w(y)}\, dy.$$

where the weights ρ more general than power weights may be admitted. We consider weights of the form

$$w(x) = \prod_{k=1}^m w_k(|x-x_k|), \quad x_k \in \overline{\Omega}$$

where $w_k(r)$ belong to the Zygmund-Bary-Stechkin (ZBS) class Φ_2 and the corresponding statement on the boundedness is given in terms of the Boyd-type indices of the $w_k(r)$.

A problem of more general weights remains open. An explicit description of weights for which the maximal operator is bounded in the spaces $L^{p(\cdot)}$ is a challenging problem. What should be the corresponding $A_{p(\cdot)}$-condition? It is natural to suppose that the Muckenhoupt condition written in the natural terms of the inverse Hölder inequality may be the corresponding characterization. Whether this is true or not, is an open question.

3.3 ZBS-type weights and the boundedness of maximal operator

The detailed proofs of the statements of this subsection will be given in [17], based on the properties of the functions in the ZBS-class developed in [24], [25] and [14]. Here we only expose the main ideas of proofs.

Let

$$W = \{w \in C([0,\ell]) : w(0) = 0,$$

$$w(x) > 0 \quad \text{for} \quad x > 0, \ w(x) \ \text{ is a.i.}\},$$

where a.i. = almost increasing. The numbers

$$m_w = \sup_{x>1} \frac{\ln\left(\liminf\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x}$$

and

$$M_w = \sup_{x>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x}$$

(see [24], [25]) are known as the lower and upper indices of the function $w(x)$ (they are of the type of Matuszewska-Orlicz indices, see [22], p. 20; the reference to the Boyd indices is also relevant). We have $0 \le m_w \le M_w \le \infty$ for $w \in W$.

Let $\gamma > 0$. The following class Φ^0_γ was introduced and studied in [2] (with integer γ); there are also known "two-parametrical"classes $\Phi^\beta_\gamma, 0 \le \beta < \gamma < \infty$, see [28], p. 253; we refer also to [14] where various properties of these classes may be found).

The Zygmund-Bary-Stechkin type class Φ^0_γ, $0 < \gamma < \infty$, is defined as $\Phi^\beta_\gamma := \mathcal{Z}^0 \cap \mathcal{Z}_\gamma$, where $\mathcal{Z}^0$ is the class of functions $w \in W$ satisfying the condition $\int_0^h \frac{w(x)}{x} dx \le cw(h)$ and $\mathcal{Z}_\gamma$ is the class of functions $w \in W$ satisfying the condition $\int_h^\ell \frac{w(x)}{x^{1+\gamma}} dx \le c\frac{w(h)}{h^\gamma}$, where $c = c(w) > 0$ does not depend on $h \in (0,\ell]$.

Theorem A. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $p(x)$ satisfy conditions (2.2), (2.3). The operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega,\rho)$ with the weight $\rho(x) = \prod_{k=1}^m w_k(|x - x_k|)$, $x_k \in \Omega$, where $w_k(r)$ are such functions that $r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi^0_n$, if

$$-\frac{n}{p(x_k)} < m_{w_k} \le M_{w_k} < \frac{n}{p'(x_k)}, \quad k = 1, 2, ..., m.$$

Omitting the details of the proof, we only enumerate the basic facts on which the proof is based.

a) Properties of functions $\omega \in \Phi_\gamma$ Theorem ([25] for $\gamma = 1$ and [14] for an arbitrary $\gamma > 0$.) A function $w \in W$ belongs to $\mathcal{Z}^0$ if and only if $m_w > 0$ and it belongs to $\mathcal{Z}_\gamma$, $\gamma > 0$, if and only if $M_w < \gamma$, so that

$$w \in \Phi^0_\gamma \Longleftrightarrow 0 < m_w \le M_w < \gamma.$$

Besides this, for $w \in \Phi^0_\gamma$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$c_1 t^{M_w+\varepsilon} \le w(t) \le c_2 t^{m_w-\varepsilon}, \qquad 0 \le t \le \ell.$$

The following properties are also valid

$$m_w = \sup\{\lambda \in (0,1) : \ t^{-\lambda} w(t) \quad \text{is a.i.}\},$$

$$M_w = \inf\{\mu \in (0,1) : \ t^{-\mu} w(t) \quad \text{is a.d.}\}.$$

b) A_p-Properties of the weights.

Lemma Let w be such that $r^a w(r) \in W$ for some $a \in \mathbb{R}$, let $\lambda \in \mathbb{R}^1$ and let Ω be a bounded domain in $\mathbb{R}^n$. Then $[w(|x - x_0|)]^\lambda \in A_p(\Omega)$ if

$$[w(r)]^{\lambda p} r^n, \ [w(r)]^{-\lambda q} r^n \in \mathcal{Z}^0. \tag{3.8}$$

Condition (3.8) is equivalent to the following inequalities

$$-\frac{n}{\lambda p} < m_w \le M_w < \frac{n}{\lambda p'} \qquad \text{when} \qquad \lambda > 0$$

and

$$-\frac{n}{|\lambda| p'} < m_w \le M_w < \frac{n}{|\lambda| p} \qquad \text{when} \qquad \lambda < 0.$$

c) Weighted averages of bounded functions. Let $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $x_0 \in \Omega$. We prove that the weighted averages

$$\frac{[w(|x - x_0|)]^{\lambda(x)}}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} \frac{|f(y)|}{[w(|y - x_0|)]^{\lambda(x)}} \, dy,$$

of bounded functions f, where $x_0 \in \Omega$, are also bounded functions if $\lambda(x) \geq 0$ and $\sup\limits_{\Omega} \lambda(x) < \frac{n}{M_\omega}$.

4. Maximal operator; the case of Carleson curves

Let

$$\mathcal{M}^\beta f(t) =$$

$$\sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}} \int\limits_{\Gamma(t,r)} \left(\frac{|t-t_0|}{|\tau-t_0|}\right)^\beta |f(\tau)| d\nu(\tau)$$

and $\mathcal{M}f(t) = \mathcal{M}^0 f(t)$. We consider power weights of the form

$$w(t) = \prod_{k=1}^m |t-t_k|^{\beta_k}, \qquad t_k \in \Gamma \qquad (4.9)$$

in the case of finite curve and the weights

$$w(t) = |t-z_0|^\beta \prod_{k=1}^m |t-t_k|^{\beta_k}, \qquad (4.10)$$

where $t_k \in \Gamma,\ z_0 \notin \Gamma$ in the case of infinite curve.

4.1 The main statements

Theorem B. Let
i) Γ be a simple Carleson curve of a finite length;
ii) p satisfy conditions (2.4)-(2.5).
Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(\Gamma, w)$ with weight (4.9), if and only if

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, ..., n.$$

Theorem C. Let
i) Γ be an infinite simple Carleson curve;
ii) p satisfy conditions (2.4)-(2.5) and let there exist a circle $B(0,R)$ such that $p(t) \equiv p_\infty = const$ for $t \in \Gamma \backslash (\Gamma \cap B(0,R))$.
Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(\Gamma, w)$, with weight (4.10), if and only if

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}$$

and

$$-\frac{1}{p_\infty} < \beta + \sum_{k=1}^n \beta_k < \frac{1}{p'_\infty}.$$

4.2 The ideas of the proof

The complete proofs of Theorems B and C will be given in another publication, here we dwell only on the principal facts on which the proofs are based.

a) On averages of $\frac{1}{p(t)}$. Let

$$\frac{1}{p_\gamma} = \frac{1}{\nu(\gamma)} \int\limits_\gamma \frac{d\nu(t)}{p(t)}, \quad \gamma \subset \Gamma$$

where $\gamma = \Gamma(t,r),\ t \in \Gamma,\ \ r > 0,$ and let $\chi_\gamma(\tau)$ be the characteristic function of γ. The following statement is valid (which was proved in [10], Lemma 3.4, for balls in the Euclidean space, the proof for arcs γ on Carleson curves remains the same).

Let $p(t)$ satisfy condition (2.4) and the maximal operator $\mathcal{M}$ be bounded in $L^{p(\cdot)}(\Gamma)$. Then there exists a constant $C > 0$ such that

$$\|\chi_\gamma\|_{p(\cdot)} \leq C[\nu(\gamma)]^{\frac{1}{p_\gamma}} \quad \text{for all} \quad \gamma = \Gamma(t,r) \subset \Gamma.$$

b) On weighted mean of a constant function. Next, we observe that for an arbitrary Carleson curve it proves to be possible to get the following estimate

$$\sup_{\substack{t,t_0 \in \Gamma \\ r>0}} \frac{|t-t_0|^\beta}{\nu(\Gamma(t,r))} \int\limits_{\Gamma(t,r)} \frac{d\nu(\tau)}{|\tau-t_0|^\beta} < \infty, \ \ 0 \leq \beta < 1.$$

c) On the sharp maximal function. For the sharp maximal function

$$\mathcal{M}^\# f(t) =$$

$$\sup_{r>0} \frac{1}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} |f(\tau) - f_{\Gamma(t,r)}|\, d\nu(\tau)$$

where $f_{\Gamma(t,r)} = \frac{1}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} f(\tau)\ d\nu(\tau)$, the following extension to the case of variable exponent $p(\cdot)$ of the result known for Euclidean space is valid.

Let Γ be an infinite Carleson curve. Let $p(t)$ satisfy conditions (2.4)-(2.5) and $p(t) = p_\infty$ outside some ball $B(t_0, R)$. Let $w(t) = |t-t_0|^\beta,\ t_0 \in$ C, where

$$-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)} \quad \text{and} \quad -\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty}$$

if $t_0 \in \Gamma$ and only $-\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty}$ if $t_0 \notin \Gamma$. Then for $f \in L^{p(\cdot)}(\Gamma, w)$

$$\|f\|_{L^{p(\cdot)}(\Gamma,w)} \leq c \left\|\mathcal{M}^\# f\right\|_{L^{p(\cdot)}(\Gamma,w)}. \qquad (4.11)$$

d) A pointwise estimate for the weighted means. Let

$$\mathcal{M}_r^\beta f(t) = \frac{1}{r} \int\limits_{\Gamma(t,r)} \left(\frac{|t-t_0|}{|\tau - t_0|} \right)^\beta |f(\tau)|\, d\nu(\tau).$$

The statement below may be proved following mainly the ideas of the proof in [20] where the Euclidean case was treated.

Let $p(t)$ satisfy conditions (2.4)-(2.5). If $0 \leq \beta < \frac{1}{p'(t_0)}$, then

$$\left[\mathcal{M}_r^\beta f(t)\right]^{p(t)} \leq c \left(1 + \frac{1}{r} \int\limits_{\Gamma(t,r)} |f(\tau)|^{p(\tau)}\, d\nu(\tau) \right)$$

for all $f \in L^{(p(\cdot)}(\Gamma)$ such that $\|f\|_{p(\cdot)} \leq 1$, where $c = c(p,\beta)$ is a constant not depending on $t, t_0 \in \Gamma$ and $r > 0$.

5. Boundedness of the Cauchy singular integral operator on Carleson curves

5.1 The main statements

Let

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau.$$

Theorem D. Let
i) Γ be a simple Carleson curve ;
ii) p satisfy conditions (2.4)-(2.5), and the following condition at infinity

$$|p(t) - p(\tau)| \leq \frac{A_\infty}{ln \frac{1}{\left|\frac{1}{t} - \frac{1}{\tau}\right|}}, \qquad \left| \frac{1}{t} - \frac{1}{\tau} \right| \leq \frac{1}{2},$$

for $|t| \geq L$, $|\tau| \geq L$ with some $L > 0$ in the case Γ is an infinite curve ;
Then the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, w)$ with weight (4.9) or (4.10), if and only if

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, ..., m, \quad (5.12)$$

and also

$$-\frac{1}{p(\infty)} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'(\infty)} \qquad (5.13)$$

in the case Γ is infinite.

For constant p Theorem D is due to G.David [6] in the non-weighted case, for the weighted case with constant p see [3]. For earlier results on the subject we refer to [15], Theorem 2.2. The statement of Theorem D for variable $p(\cdot)$ was proved in [19] in the case of finite Lyapunov curves or curves of bounded rotation without cusps.

Theorem E. Let assumptions i)-ii) of Theorem E be satisfied, and let $a \in C(\Gamma)$. In the case where Γ is an infinite curve starting and ending at infinity, we assume that $a \in C(\dot{\Gamma})$, where $\dot{\Gamma}$ is the compactification of Γ by a single infinite point, that is, $a(t(-\infty)) = a(t(+\infty))$. Then under conditions (5.12)-(5.13). the operator

$$(S_\Gamma aI - aS_\Gamma)f = \frac{1}{\pi i} \int\limits_\Gamma \frac{a(\tau) - a(t)}{\tau - t} f(\tau) d\nu(\tau)$$

is compact in the space $L^{p(\cdot)}(\Gamma, w)$ with weight (4.9)-(4.10).

The detailed proof of Theorem D will be given in [17]. Note that in [17] it is also proved that for the operator S_Γ to be bounded in $L^{p(\cdot)}(\Gamma)$, it is necessary that Γ is a Carleson curve. Namely, the following result is proved there.

Let Γ be a finite rectifiable curve. Let $p : \Gamma \to [1, \infty)$ be a bounded continuous function. If the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma)$, then the curve Γ has the property $\sup\limits_{t\in\Gamma, r>0} \frac{\Gamma(t,r)}{r^{1-\varepsilon}} < \infty$ for every $\varepsilon > 0$. If $p(t)$ satisfies the log-condition (2.5), then the above property holds with $\varepsilon = 0$, that is, Γ is a Carleson curve.

5.2 Ideas of the proof.

a) Sharp maximal function of $|S_\Gamma f|^s$. We prove the following statement which was earlier proved for the Euclidean case in [1].

Proposition Let Γ be a simple Carleson curve. Then the following pointwise estimate is valid

$$\mathcal{M}^\# \left(|S_\Gamma f|^s\right)(t) \leq c[\mathcal{M}f(t)]^s, \qquad 0 < s < 1. \qquad (5.14)$$

The proof of the above Proposition is based on the following Kolmogorov-type theorem ([21], [6], [13]): Let Γ be a Carleson curve of a finite length. Then for any $s \in (0,1)$

$$\left(\frac{1}{\nu(\Gamma)}\int_{\Gamma}|S_{\Gamma}f(t)|^{s}d\nu(t)\right)^{\frac{1}{s}} \leq \frac{C}{\nu(\Gamma)}\int_{\Gamma}|f(t)|d\nu(t), \tag{5.15}$$

which is a consequence of the fact that the singular operator on Carleson curves has weak (1,1)-type: $\nu\{t \in \Gamma : |S_{\Gamma}f(t)| > \lambda\} \leq \frac{c}{\lambda}\int_{\Gamma}|f(t)|d\nu(t)$, the latter being proved in [6], and on the following technical lemma.

Lemma Let Γ be a simple Carleson curve, $z_0 \in \Gamma$ and $\gamma_r = \Gamma(z_0, r)$ and

$$H_{r,z_0}(t) = \frac{1}{[\nu(\gamma_r)]^2}\int_{\gamma_r}\int_{\gamma_r}\left|\frac{1}{z-t} - \frac{1}{\tau-t}\right| d\nu(z)d\nu(\tau).$$

Then for any locally integrable function f the pointwise estimate holds

$$\sup_{r>0}\int_{t\in\Gamma:|t-z_0|>2r}|f(t)|H_{r,z_0}(t)d\nu(t) \leq C\mathcal{M}f(z_0)$$

where $C > 0$ does not depend on f and z_0.

b). The case of infinite curve and p constant at infinity. This case is derived from (5.14), (4.11) and Theorem C, since $\|S_{\Gamma}f\|_{L^{p(\cdot)}(\Gamma,w)} = \||S_{\Gamma}f|^{s}\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,w)}$ for constant $s \in (0,1)$ and we may take s as close to 1 as we wish.

c) The case of finite curve and p constant on some arc. It may be shown that this case can be reduced to the previous one by the corresponding fractional linear map.

d) The general case of finite curve. This case may be covered by application of the Riesz interpolation theorem known for the variable exponent spaces, interpolating between the two cases when $p(t)$ is constant on one arc γ_1 and another one γ_2, $\gamma_1 \cap \gamma_2 = \emptyset$.

The general case of infinite curve. This case may be reduced to the previous one, as in c) by a fractional linear map. It is important to note that under both the mappings the transformed exponent is also log-continuous on the transformed curve.

6. Sobolev-type theorem for potential operators on Carleson curves

Let

$$I^{\alpha(\cdot)}f(t) = \int_{\Gamma}\frac{f(\tau)\,d\nu(\tau)}{|t-\tau|^{1-\alpha(t)}}.$$

Theorem F. Let

i) Γ be a simple Carleson curve of a finite length;

ii) p satisfy conditions (2.4)-(2.5);

iii) $\inf_{t\in\Gamma}\alpha(t) > 0$ and $\sup_{t\in\Gamma}\alpha(t)p(t) < 1$.

Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Gamma)$ into $L^{q(\cdot)}(\Gamma)$ with $\frac{1}{q(t)} = \frac{1}{p(t)} - \alpha(t)$. This statement remains valid for infinite Carleson curves if, in addition to conditions i)-iii), $p(t) = p_{\infty} =$ *const* outside some circle $B(0,R)$.

The next theorem is a weighted generalization of Theorem F for finite curves.

Theorem G. Under assumptions i)-iii) of Theorem F, the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Gamma, w)$ into the space $L^{q(\cdot)}(\Gamma, w)$ where $\frac{1}{q(t)} = \frac{1}{p(t)} - \alpha(t)$, and w is the weight (4.9) if

$$\alpha(t_k) - \frac{1}{p(t_k)} < \beta_k < 1 - \frac{1}{p(t_k)}, \qquad k = 1, ..., n.$$

Observe that Euclidean space version of Theorem G was proved in [26] for the case of bounded domains $\Omega \subset \mathbb{R}^n$ and in [29] for the case of the whole space $\mathbb{R}^n$.

The proof of Theorems F and G is based on the known Hedberg's approach and the estimation of the following variable norm:

$$\left\||t-\tau|^{\alpha(t)-1}\chi_r(|t-\tau|)\right\|_{L^{p(\cdot)}\left(\Gamma,|\tau-t_0|^{-\beta p'(\tau)}\right)}$$
$$\leq Cr^{\alpha(t)-1+\frac{1}{p(t)}}(r+|t-t_0|)^{-\beta p'(t)}$$

where $t, t_0 \in \Gamma, 0 < r < \ell < \infty$ and $\chi_r(\rho) = 1$ if $\rho > r$ and $\chi_r(\rho) = 0$ otherwise, the proof of the latter requiring the most efforts.

7. The case of HTS

Let (X, μ) be a homogeneous type metric space with quasi-distance $d(x, y)$ and measure μ and $B(x, r)$ a ball in X of radius $r > 0$. (We refer for instance to [11], [12] for the theory of HTS). We assume that the following condition is satisfied:

$$c_1 r^s \leq \mu B(x,r) \leq c_2 r^s, \qquad s > 0$$

where c_1 and c_2 do not depend on $r > 0$ and $x \in X$. We consider the weighted spaces $L^{p(\cdot)}(X, \mu, w)$ defined by the modular

$$\int_X |w(x)f(x)|^{p(x)}d\mu(x) < \infty$$

and admit weights of the form

$$w(x) = \prod_{k=1}^{m} [d(x, a_k)]^{\beta_k}, \tag{7.16}$$

when $\mu(X) > \infty$ and

$$w(x) = [1 + d(a_0, x)]^{\beta} \prod_{k=1}^{m} [d(a_k, x)]^{\beta_k},$$

when $\mu(X) = \infty$, with $a_k \in X, k = 0, 1, ..., m$. The function $p(x)$ is assumed to satisfy the standard conditions

$$1 < p_- \le p(x) \le p_+ < \infty, \quad x \in X, \tag{7.17}$$

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{d(x,y)}}, \quad d(x, y) \le \frac{1}{2}, \quad x, y \in X. \tag{7.18}$$

7.1 The maximal operator.

Let

$$\mathcal{M} f(x) = \sup_{r>0} \frac{1}{\mu B(x, r)} \int\limits_{B(x,r)} |f(y)| d\mu(y)$$

The following generalizations of Theorems A and C are valid.

Theorem A′. Let X be a metric space with $\mu(X) < \infty$ and let $p(x)$ satisfy conditions (7.17), (7.18). The operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X, \mu, w)$ with weight (7.16), where $w_k(r)$ are such functions that $r^{\frac{s}{p(a_k)}} w_k(r) \in \Phi_s^0$, if

$$-\frac{s}{p(a_k)} < m_{w_k} \le M_{w_k} < \frac{s}{p'(a_k)}, \quad k = 1, 2, ..., m.$$

Theorem C′. Let
i) X be a HTS with $\mu(X) = \infty$;
ii) p satisfy conditions (7.17)-(7.18) and let $p(x) \equiv p_\infty = const$ at infinity, that is, for $x \in X \backslash B(x_0, R)$ for some $x_0 \in X$.
Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(X, w)$, with weight (7.16), if and only if

$$-\frac{s}{p(a_k)} < \beta_k < \frac{s}{p'(a_k)}$$

and

$$-\frac{s}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{s}{p'_\infty}.$$

7.2 The singular operator.

Let

$$Tf(x) = \lim_{\varepsilon \to 0} \int\limits_{d(x,y)>\varepsilon} k(x, y) f(y) \, d\mu(y)$$

be a generalized Calderon-Zygmund operator, where

$$|k(x, y)| \le A[d(x, y)]^{-s}, \tag{7.19}$$

$$|k(x, y) - k(z, y| \le A \frac{[d(x, z)]^{\delta}}{[d(x, z)]^{\delta+s}}, \tag{7.20}$$

$$|k(y, x) - k(y, z| \le A \frac{[d(x, z)]^{\delta}}{[d(x, z)]^{\delta+s}} \tag{7.21}$$

with some $A > 0$ and $\delta > 0$ (such operators, in case $X = \mathbb{R}^n$, if bounded in $L^2(X)$, are bounded in $L^p(\mathbb{R}^n), 1 < p < \infty$, see [4]).

Theorem D′. Let p satisfy conditions (7.17)-(7.18) and $p(x) \equiv p_\infty = const$ outside some large ball in the case $\mu(X) = \infty$. Let assumptions (7.19)- (7.21) be satisfied and the operator T be bounded in the space $L^2(X, \mu)$. Then the singular operator T is bounded in the space $L^{p(\cdot)}(\Gamma, w)$, if and only if $-\frac{s}{p(a_k)} < \beta_k < \frac{s}{p'(a_k)}$, $k = 1, ..., m$, and also $-\frac{s}{p_\infty} < \beta + \sum\limits_{k=1}^{n} \beta_k < \frac{s}{p'_\infty}$ in the case $\mu(X) = \infty$.

7.3 The potential operator.

Let

$$I^{\alpha(\cdot)} f(x) = \int\limits_X \frac{f(y) \, d\mu(y)}{[d(x, y)]^{s - \alpha(x)}}.$$

The following statements are valid.

Theorem F′. Let $\mu(X) < \infty$, p satisfy conditions (7.17)-(7.18), $\inf\limits_{x \in X} \alpha(x) > 0$ and $\sup\limits_{x \in X} \alpha(x) p(x) < s$. Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(X)$ into $L^{q(\cdot)}(X)$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{s}$. This statement remains valid in the case $\mu(X) = \infty$, if $p(x) = const$ outside some large ball.

Theorem G′. Let $\mu(X) < \infty$. Under assumptions of Theorem F′ on $p(x)$ and $\alpha(x)$, the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(X, w)$ into the space $L^{q(\cdot)}(X, w)$ where with weight (7.16) if

$$\alpha(a_k) - \frac{s}{p(a_k)} < \beta_k < 1 - \frac{s}{p(a_k)}, \qquad k = 1, ..., m.$$

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