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Integral Equations and Operator Theory

Local Fredholm spectrums and Fredholm properties of singular integral operators on Carleson curves acting on weighted Hölder spaces

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Abstract. We study the local Fredholm spectrums and global Fredholm properties for singular integral operators on composed Carleson curves with discontinuous coefficients acting on weighted Hölder spaces. We consider the curves, coefficients, and weights which are slowly oscillating at the nodes of the curve.

Application of pseudodifferential operators technique allows us to explain the influence of oscillation of curves, coefficients, and weights on the appearance of massive local Fredholm spectrums.

We obtain a criterion of Fredholmness and index formula for operators under consideration.

Contents

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1. Introduction

In the book [1] (see also references therein) A. Böttcher and Yu. Karlovich considered the algebra of operators acting in $L^p(\Gamma, w)$, where $1 < p < \infty$ and w is a Muckenhoupt weight, generated by the operator S_Γ of singular integration along a composed Carleson curve Γ and operators of multiplication by piece-wise continuous functions. Several new phenomena were discovered by them: the circular arcs and horns which typically arise in the spectral theory of Gohberg, Krupnik, and Spitkovsky, [9], [22] are converted into logarithmic double spirals and spiral horns. Notice that the approach of Böttcher/Yu. Karlovich is based on the Wiener-Hopf factorization, and theory of sub-multiplicative functions.

However, there is another approach to the problem based on the local principle and the Mellin pseudodifferential operators technique. Such an approach was applied in the papers [17], [19], [18], [20], [2], [4], [3], [1] for the study of singular integral operators acting on weighted L^p-spaces on composed Carleson curves. It has been shown that the local representative of a singular integral operator at singular points can be realized as a Mellin pseudodifferential operator with symbol containing all characteristics of the operator: characteristics of oscillations and rotations of the curve, oscillations of the weight and coefficients, the exponent p of L^p-space.

Here we apply this approach to the investigation of Fredholm properties of singular integral operators

$$Au(t) = a(t)u(t) + b(t)S_\Gamma u(t), \qquad t \in \Gamma$$

as operators acting on weighted Hölder spaces $\Lambda^{s,w}(\Gamma), 0 < s < 1$. We suppose that the coefficients a, b are smooth functions on Γ outside nodes t_k of the curve Γ, and they have slowly oscillating discontinuities at the nodes. We also suppose that the curve Γ and the weight w have slowly oscillating characteristics at every node t_k. We explain the appearance of massive local Fredholm spectrum at the nodes by the influence of oscillation of the curves, weights, and coefficients at the node. We show that the local Fredholm spectrum at the node is a union of logarithmic double spirals whose shapes depend on the behavior of coefficients a, b, curve Γ, and weight w at the node, and the exponent s of the considered Hölder space.

Note that Fredholm theory of singular integral operators with piece-wise continuous coefficients on composed Lyapunov curves acting on Hölder spaces with power weights was constructed by R. Duduchava in early seventy [5], [6], [7], see also the book of I. Gohberg and N. Krupnik [8]. Fredholm property of singular integral operators with piece-wise continuous coefficients on closed Lyapunov curves acting on *generalized* Hölder spaces with general weights was studied by N. Samko (see [26], [27], and reference therein).

The paper is organized as follows. In Section 1 we give an auxiliary material concerning pseudodifferential operators on $\mathbb{R}$ acting on Hölder spaces. The main result of this section is a criterion of local Fredholmness of pseudodifferential operators at the point $+\infty$. This result has a fundamental significance in what follows. In Section 2 the results of Section 1 are reformulated for Mellin pseudodifferential operators acting on the Hölder spaces $\tilde{\Lambda}^s(\mathbb{R}_+)$ with respect to the multiplicative structure of the group $\mathbb{R}_+$. Section 3 is the main in the paper and contains the following results:

1) a theorem on boundedness of the singular integral operator on composed Carleson curves acting on Hölder spaces with general weights. The proof of this theorem is based on the theorem on the boundedness of Mellin pseudodifferential operators in the spaces $\tilde{\Lambda}^s(\mathbb{R}_+), 0 < s < 1$, and an admissible partition of unity on the curve;

2) a criterion of local Fredholmness of singular integral operator with piece-wise slowly oscillating coefficients acting on weighted Hölder spaces and the description of their local Fredholm spectrums;

3) a criterion of Fredholmness of singular integral operators with piece-wise slowly oscillating coefficient at the nodes on weighted Hölder spaces. For the proof of sufficiency of the criterion we use the corresponding results for the local Fredholmness of singular integral operators, and "gluing" the global regularizers from the local ones by means of the admissible partition of unity on the curve Γ. The necessity of conditions of this theorem follows from necessity of conditions for the singular integral operator to be locally Fredholm operator.

From the local principles of Simonenko [23] and Allen- Douglas [21] it follows that the global Fredholmness of singular integral operators acting on weighted L^p-spaces is equivalent to their local Fredholmness at every point of the curve Γ. In the case of operators acting on weighted Hölder spaces the local principle is known also (see for instance [14], [28]), but it is more convenient for us to give an explicit construction of regularizers.

N o t a t i o n :

$\lvert a\rvert_{r,t}$	is the symbol norm, see (1);
$\mathcal{B}(X)$	is a Banach algebra of all bounded linear operators acting on a Banach space X;
$C_b^\infty(\mathbb{R})$,	see (2.14);
F	is the set of all the nodes of the curve Γ, see Subsection 4.1;
$\mathcal{K}(X)$	is a two-sided ideal in $\mathcal{B}(X)$ of all compact operators;
$Op(a)$,	see (2.2);
$Op_M(a)$,	see (3.5);
$OPS_{1,0}^m$,	see Definition 2.2;
$OPS_{1,0}^0(\widetilde{\mathbb{R}})$,	see Subsection 2.4;
$OP\mathcal{S}_{1,0}^0$	see Subsection 3.2 ;
$OPSO_\pm, OPSO$	are the classes of operators with symbols from $SO_\pm, SO$, respectively;
$\mathcal{OPSO}_0$,	see Subsection 3.2;
$OPSO_+(\widetilde{\mathbb{R}})$	see Subsection 2.4;
$\widetilde{\mathbb{R}}$	is the two point compactification of $\mathbb{R}$;
$S_{1,0}^m$	is the Hörmander class of symbols, see Definition 2.1;
$S_{1,0}^0(\widetilde{\mathbb{R}})$,	see Subsection 2.4;
$\mathcal{S}_{1,0}^0$,	see Subsection 3.2;
$\mathcal{SO}_0$,	see Subsection 3.2;
$SO_\pm$	is the class of symbols from $S_{1,0}^0$ slowly oscillating at $\pm\infty$, see Definition 2.4 ; $SO = SO_+ \cap SO_-$;
SO^∞,	see (2.21);
$sp_{+\infty}(A : X \to X)$,	see (2.42);
$V_h u(x) = u(x+h)$	is the translation operator;
$\Lambda^s(E)$	is the "translation type" Hölder class, see Definition 2.6;
$\Lambda_c^s(\mathbb{R})$	see (2.27);
$\Lambda_{x_0}^s(E)$	is the subspace of functions $u \in \Lambda^s(E)$ such that $\lim\limits_{E\ni x\to x_0} u(x) = 0$, where x_0 is a limit point of E;
$\tilde{\Lambda}^s(\mathbb{R}_+)$	is the "dilatation type" Hölder class, see Definition 3.1;
$\tilde{\Lambda}_-^s(\mathbb{R}_+, \mathbb{C}^N)$,	see (3.8);
$\Lambda^{s,w}(\Gamma)$,	see Subsection 4.2;
$\langle\xi\rangle = \sqrt{1+\lvert\xi\rvert^2}$	;
Φ,	see (2.17).

2. Local invertibility of pseudodifferential operators on weighted Hölder spaces

2.1. Calculus of pseudodifferential operators

The goal of this subsection is to set up some notations and summarize (without proof) some facts on pseudodifferential operators. Standard references are [13], [11], [24], [29], [30].

We restrict ourselves to the case of dimension one, because our aim is applications of pseudodifferential operators to the theory of one-dimensional singular integral operators.

Definition 2.1. *We say that a function a belongs to the L.Hörmander class* $S^m_{1,0}$ *where* $m > 0,\ m \in \mathbb{R}$, *if* $a \in C^\infty(\mathbb{R}\times\mathbb{R})$, *and for all* $r, t \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$

$$|a|_{r,t} = \sum_{\substack{\alpha,\beta\in\mathbb{N}_0\\ \alpha\le r,\beta\le t}} \sup_{\mathbb{R}\times\mathbb{R}} \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \langle\xi\rangle^{\alpha-m} < \infty, \tag{2.1}$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

As usual, we associate with a symbol a the pseudodifferential operator defined on the Schwartz space $S(\mathbb{R})$ by the formula

$$Op(a)u(x) = a(x,D)u(x) = (2\pi)^{-1}\int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a(x,\xi)u(y)e^{i(x-y)\xi}dy. \tag{2.2}$$

Definition 2.2. *We denote by* $OPS^m_{1,0}$ *the class of pseudodifferential operators with symbols in* $S^m_{1,0}$.

It is well-known that $A \in OPS^m_{1,0}$ is a bounded operator on $S(\mathbb{R})$. We say that A^t is a formally adjoint operator for A if

$$\int_{\mathbb{R}} (Au)(x)\overline{v(x)}dx = \int_{\mathbb{R}} u(x)\overline{(A^t v)(x)}dx \tag{2.3}$$

for all $u, v \in S(\mathbb{R})$. Then, $A^t = Op(a^t) \in OPS^m_{1,0}$, and we can define the action of $A \in OPS^m_{1,0}$ on $S'(\mathbb{R})$ by the formula

$$(Au)(v) = u(A^t v),$$

where $u \in S'(\mathbb{R})$, $v \in S(\mathbb{R})$.

Proposition 2.3. *Let* $A = Op(a) \in OPS^m_{1,0}(\mathbb{R})$. *Then there exists a Schwartz kernel* $k_a \in C^\infty(\mathbb{R})\otimes S'(\mathbb{R})$ *of the operator* A *such that*

$$Au(x) = (k_a(x,z), u(x-z)), u \in S'(\mathbb{R}), \tag{2.4}$$

where

$$k_a(x,z) = F^{-1}_{\xi\to z} a(x,\xi).$$

($F^{-1}_{\xi\to z}$ *is the inverse Fourier transform in the sense of distributions.)*

The kernel $k_a(x,z)$ *is in* $C^\infty(\mathbb{R}\times\mathbb{R}\backslash\{0\})$ *and satisfies*

$$\left|\partial_x^\beta \partial_z^\alpha k_a(x,z)\right| \le C_{\alpha,\beta,N}\,|z|^{-1-m-\alpha-N}, \quad x\in\mathbb{R},\ z\in\mathbb{R}\backslash\{0\} \tag{2.5}$$

for all the multi-indices α, β, *and all* $N \ge 0$ *such that* $1+m+|\alpha|+N > 0$.

Below we set up some facts on calculus of pseudodifferential operators with slowly oscillating symbols following [16], see also [15], Chap. 4.

Definition 2.4. *A symbol a is called slowly oscillating at the points $\pm\infty$ if $a \in S^0_{1,0}(\mathbb{R})$, and*

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha\beta}(x) \left\langle \xi \right\rangle^{-\alpha}, \tag{2.6}$$

where $\lim_{x\to\pm\infty} C_{\alpha\beta}(x) = 0$ for every $\alpha \ge 0$ and $\beta > 0$. We denote by $SO_\pm$ the class of slowly oscillating at the point $\pm\infty$ symbols, and set $SO = SO_+ \cap SO_-$ We use the notations $OPSO_\pm$, and $OPSO$ for the class of operators with symbols in $SO_\pm$ and SO, respectively.

Proposition 2.5. *Let $A = Op(a) \in OPSO_\pm, B = Op(b) \in OPSO_\pm$. Then $AB \in OPSO_\pm$, and*

$$AB = Op(ab) + Op(t(x,\xi)),$$

where $t(x,\xi)$ is such that

$$\left|\partial_x^\beta \partial_\xi^\alpha t(x,\xi)\right| \le C_{\alpha\beta}(x) \left\langle \xi \right\rangle^{-1-\alpha}, \tag{2.7}$$

and

$$\lim_{x\to\pm\infty} C_{\alpha\beta}(x) = 0 \tag{2.8}$$

for all α, β.

2.2. Hölder spaces

Definition 2.6. *Let $E \subset \mathbb{R}$, and $0 < s < 1$, we define $\Lambda^s(E)$ as the subspace of $C(E)$ consisting of those bounded functions u which satisfy in E the Hölder condition of order s, that is, there exists a constant c such that $|u(x) - u(y)| \le c\,|x-y|^s$ for all $x, y \in E$.*

Equipped with the norm

$$\|u\|_{\Lambda^s(E)} = \|u\|_{L^\infty(E)} + \sup_{x,y\in E,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^s}, \tag{2.9}$$

$\Lambda^s(E)$ is a Banach space.

Proposition 2.7. *Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(x) = 1$ if $|x| \le 1$, $\varphi(x) = 0$ if $|x| \ge 2$, and $0 \le \varphi(x) \le 1$ if $1 \le |x| \le 2$, and $\varphi_R(x) = \varphi(x/R)$. Then for every $s \in (0,1)$*

$$\lim_{R\to\infty} \|\varphi_R u\|_{\Lambda^s(\mathbb{R})} = \|u\|_{\Lambda^s(\mathbb{R})}. \tag{2.10}$$

Proof. Proof follows directly from the definition of norm (2.9). □

Proposition 2.8. *(see [32], p. 52). Let $s \le r$. Then $\Lambda^r(E) \subseteq \Lambda^s(E)$. If $s < r$, and E is an open set with a compact closure, then the imbedding $\Lambda^r(E) \subset \Lambda^s(E)$ is compact.*

We denote by $\Lambda^s_{x_0}(E)$ the subspace of $\Lambda^s(E)$ consisting of functions u such that $\lim_{E\ni x\to x_0} u(x) = 0$, where x_0 is a limit point of E.

Proposition 2.9. *Let $u \in \Lambda^s_{x_0}(E)$. Then for every $\delta \in (0, +\infty]$ there exists a constant $C_\delta > 0$ such that*

$$\|u\|_{L^\infty(E)} \leq C_\delta \sup_{x,y\in E, 0<|x-y|<\delta} \frac{|u(x)-u(y)|}{|x-y|^s}. \tag{2.11}$$

Proof. Suppose that inequality (2.11) does not hold for some $\delta > 0$. Then there exists a sequence $u_n \in \Lambda^s_{x_0}(E)$ such that $\|u_n\|_{L^\infty(E)} = 1$, and

$$\sup_{\substack{x,y\in E,\\ 0<|x-y|<\delta}} \frac{|u_n(x)-u_n(y)|}{|x-y|^s} < \frac{1}{n}. \tag{2.12}$$

Since $\|u_n\|_{L^\infty(E)} = 1$, the sequence u_n is uniformly bounded and by (2.12) it is equicontinuous. Then, there exists a subsequence u_{n_k} convergent to a limit function u uniformly on every compact set in E. This implies that $\|u\|_{L^\infty(E)} = 1$. Passing to the limit in (2.12) we obtain that

$$\sup_{\substack{x,y\in E\\ 0<|x-y|<\delta}} \frac{|u(x)-u(y)|}{|x-y|^s} = 0.$$

Hence u is a constant function on E. It implies that $u(x) = 0$ for every point $x \in E$, because $\lim_{E\ni x\to x_0} u(x) = 0$. This is a contradiction to $\|u\|_{L^\infty(E)} = 1$. □

Proposition 2.10. *On the space $\Lambda^s_{x_0}(E)$, the norm defined in (2.9) and the norm*

$$\|u\|_{\Lambda^s_\delta(E)} = \sup_{x,y\in E, 0<|x-y|<\delta} \frac{|u(x)-u(y)|}{|x-y|^s},$$

where $\delta \in (0, \infty]$, are equivalent.

Proof. It is evident that

$$\|u\|_{\Lambda^s_\delta(E)} \leq \|u\|_{\Lambda^s(E)}.$$

Further, by (2.11)

$$\begin{aligned} \|u\|_{\Lambda^s(E)} &\leq \sup_{x,y\in E, 0<|x-y|<\delta} \frac{|u(x)-u(y)|}{|x-y|^s} + \left(1 + \frac{2}{\delta^s}\right) \|u\|_{L^\infty(E)} \\ &\leq \left(1 + \left(1 + \frac{2}{\delta^s}\right) C_\delta\right) \|u\|_{\Lambda^s_\delta((E)}. \end{aligned}$$

□

2.3. Boundedness and compactness of pseudodifferential operators on Hölder spaces

Let X be a Banach space, $\mathcal{B}(X)$ a Banach algebra of all bounded linear operators acting on X, and $\mathcal{K}(X)$ a two-sided ideal in $\mathcal{B}(X)$ of all compact operators.

In what follows if X is a function space and a is a function we denote by aI the operator of multiplication by this function. If B is a linear operator we will write aB instead of aIB.

We will define the action of pseudodifferential operators on Hölder spaces $\Lambda^s(\mathbb{R})$ by formula (2.4).

Proposition 2.11. *A pseudodifferential operator $A = Op(a) \in OPS^{\varepsilon}_{1,0}$ is bounded from $\Lambda^s(\mathbb{R})$ in $\Lambda^{s-\varepsilon}(\mathbb{R})$ for every $s \in (0,1)$ and ε such that $s-\varepsilon \in (0,1)$. Moreover, there exist $C > 0$ independent of a such that*

$$\|Op(a)u\|_{\Lambda^{s-\varepsilon}(\mathbb{R})} \leq C\,|a|_{l_1,l_2}\,\|u\|_{\Lambda^s(\mathbb{R})}\,, \tag{2.13}$$

where $l_1 > 2, l_2 > 2$.

The boundedness of pseudodifferential operators on the Hölder spaces has been proved in [25], p. 253-257, see also [31], p.37-38 without estimate (2.13). But a careful analysis of those proofs allows us to obtain estimate (2.13).

We denote by $C_b^\infty(\mathbb{R})$ the class of functions in $C^\infty(\mathbb{R})$ bounded with all their derivatives, and with the topology defined by the seminorms

$$|a|_k = \sum_{j\leq k} \sup_{x\in\mathbb{R}} \left|a^{(j)}(x)\right|. \tag{2.14}$$

If $A, B \in \mathcal{B}(X)$, then we denote by $[A;B] = AB - BA$ the commutator of A, B.

Proposition 2.12. *Let $A = Op(a) \in OPS^0_{1,0}, \varphi \in C_b^\infty(\mathbb{R}), \varphi_R(x) = \varphi(x/R)$. Then*

$$\|[\varphi_R I, A]\|_{\mathcal{B}(\Lambda^s(\mathbb{R}))} \leq \frac{C}{R}, \quad R > 0 \tag{2.15}$$

where $C > 0$ does not depend on R.

Proof. It follows from the formulas of compositions for pseudodifferential operators that

$$[\varphi_R I, A] = Op(b_R),$$

where

$$|b_R|_{k,t} \leq CR^{-1}\,|a|_{k+2,t+2}\,. \tag{2.16}$$

Estimate (2.16) and Proposition 2.11 yield estimate (2.15). □

Let

$$\Phi = \{\varphi \in \Lambda^s(\mathbb{R}) : \ \varphi(x) \equiv 0 \quad \text{for} \quad x \leq b = b_\varphi \in \mathbb{R}\} \tag{2.17}$$

and

$$\Lambda^s_-(\mathbb{R}) = \overline{\Phi}, \tag{2.18}$$

the closure being taken with respect to the norm of $\Lambda^s(\mathbb{R})$. It is clear that $\Lambda^s_-(\mathbb{R})$ is a closed subspace of

$$\Lambda^s_{-\infty}(\mathbb{R}) = \left\{u \in \Lambda^s_-(\mathbb{R}) : \lim_{x\to-\infty} u(x) = 0\right\}.$$

Proposition 2.13. *A pseudodifferential operator $A = Op(a) \in OPS^{\varepsilon}_{1,0}$ is bounded from $\Lambda^s_-(\mathbb{R})$ in $\Lambda^{s-\varepsilon}_-(\mathbb{R})$ for every $s \in (0,1)$ and ε such that $s-\varepsilon \in (0,1)$ with estimate (2.13).*

The proof of this proposition easily follows from Propositions 2.11 and 2.12.

Proposition 2.14. *Let* $T = Op(t) \in OPS^{-\varepsilon}_{1,0}$, *and*

$$\left|\partial_x^\beta \partial_\xi^\alpha t(x,\xi)\right| \le C_{\alpha\beta}(x) \langle\xi\rangle^{-\varepsilon-\alpha}, \varepsilon > 0, \tag{2.19}$$

where

$$\lim_{x\to+\infty} C_{\alpha\beta}(x) = 0 \tag{2.20}$$

for all $\alpha, \beta \in \mathbb{N}_0$. *Let* $\chi \in C_b^\infty(\mathbb{R})$, *and* $\chi(x) = 0$ *for* $x < R$.

(i) Then $T\chi I$ *and* χT *are compact operators on* $\Lambda^s(\mathbb{R})$ $(\Lambda^s_-(\mathbb{R}))$ *for every* $\chi \in C_b^\infty(\mathbb{R})$, *such that* $\chi(x) = 0$ *for* $x < R$.

(ii) If $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ *for all* $\alpha, \beta \in \mathbb{N}_0$, *then* T *is a compact operator on* $\Lambda^s(\mathbb{R}), \left(\Lambda^s_-(\mathbb{R})\right)$.

Proof. We prove that χT is a compact operator. Let $\varphi \in C_0^\infty(\mathbb{R}), \varphi(x) = 1$ if $|x| \le 1$, and $\varphi(x) = 0$ if $|x| \ge 2, \varphi_R(x) = \varphi(x/R)$, $\psi_R = 1 - \varphi_R$. Then

$$\begin{aligned} &\|\chi T - \varphi_R \chi T\|_{\mathcal{B}(\Lambda^s(\mathbb{R}))} \\ \le\ &\|\psi_R \chi T\|_{\mathcal{B}(\Lambda^s(\mathbb{R}))} \le C\,|\psi_R \chi t|_{2k_1,2k_2} \end{aligned}$$

where $C\,(>0)$, and $2k_1 > 1, 2k_2 > 1$ are independent of t and R. Estimates (2.19), (2.20) imply that

$$\lim_{R\to\infty} |\psi_R \chi t|_{2k_1,2k_2} = 0.$$

Let us prove that $\varphi_R \chi T : \Lambda^s(\mathbb{R}) \to \Lambda^s(\mathbb{R})$ is a compact operator. Indeed, *supp* $\varphi_R \chi T u \subset B_{2R} = \{x \in \mathbb{R} : |x| < 2R\}$ for every function $u \in \Lambda^s(\mathbb{R})$. Hence, $\varphi_R \chi T$ maps bounded sets in $\Lambda^s(\mathbb{R})$ in bounded sets in $\Lambda^{s+\varepsilon}(B_{2R})$. By Proposition 2.8, the space $\Lambda^{s+\varepsilon}(B_{2R})$ is compactly imbedded into $\Lambda^s(B_{2R})$. In the same way we prove compactness of $T\chi$, and statement (ii).

The proof for the spaces $\Lambda^s_-(\mathbb{R})$ is similar. □

We denote by $SO^\infty(\mathbb{R})$ the class of slowly oscillating functions, that is, the functions in $C_b^\infty(\mathbb{R})$ which satisfy the condition

$$\lim_{x\to\infty} b'(x) = 0. \tag{2.21}$$

Note that this condition implies that

$$\lim_{x\to\infty} b^{(j)}(x) = 0, j \in \mathbb{N}.$$

Proposition 2.15. *Let* $A = Op(a) \in OPSO$, $b \in SO^\infty(\mathbb{R})$. *Then the commutator*

$$[A, bI] = AbI - bA$$

is a compact operator on $\Lambda^s(\mathbb{R})$ $(\Lambda^s_-(\mathbb{R}))$.

The proof easily follows from Propositions 2.5 and 2.14.

2.4. Local Fredholmness of pseudodifferential operators

Let $\chi \in C_b^\infty(\mathbb{R})$, and $\chi(x) = 0$ if $x \leq 1$, and $\chi(x) = 1$ if $x \geq 2$, $0 \leq \chi(x) \leq 1$. We set $\chi_R(x) = \chi(x/R), R > 0$. Let ψ_R be a function with the properties similar to the properties of χ_R, and $\psi_R\chi_R = \chi_R$.

Definition 2.16. *An operator $A \in \mathcal{B}(\Lambda^s(\mathbb{R}))$ is called a locally Fredholm operator at the point $+\infty$, if there exist operators $\mathcal{L}, \mathcal{R} \in \mathcal{B}(\Lambda^s(\mathbb{R}))$ and $R_0 > 0$ such that for $R > R_0$*

$$\mathcal{L}\psi_R A\chi_R I = \chi_R I + T'_R, \chi_R A\psi_R\mathcal{R} = \chi_R I + T''_R, \tag{2.22}$$

where $T'_R, T''_R \in \mathcal{K}(\Lambda^s(\mathbb{R}))$.

Equalities (2.22) can be written as follows

$$\psi_R\mathcal{L}\psi_R A\chi_R I = \chi_R I + \psi_R T'_R\psi_R I, \chi_R A\psi_R\mathcal{R}\psi_R = \chi_R I + \psi_R T''_R\psi_R I. \tag{2.23}$$

Note if $u \in \Lambda^s(R)$ then the functions $\psi_R u, \chi_R u$ belong to $\Lambda^s_-(R)$. It implies that A is a locally Fredholm operator on $\Lambda^s(R)$ at the point $+\infty$ if and only A is a locally Fredholm operator on $\Lambda^s_-(R)$ at this point.

We denote by $\widetilde{\mathbb{R}}$ the two point compactification of $\mathbb{R}$ homeomorphic to the segment $[-1, 1]$, and by $S^0_{1,0}(\widetilde{\mathbb{R}})$ the class of symbols in $S^0_{1,0}$ admitting extensions on $\widetilde{\mathbb{R}}$. The corresponding class of pseudodifferential operators is denoted by $OPS^0_{1,0}(\widetilde{\mathbb{R}})$.

Proposition 2.17. *Let $A = Op(a) \in OPS^0_{1,0}(\widetilde{\mathbb{R}})$, and a sequence $h_n \to \infty$. Then there exists a subsequence h_{n_k} and the function $a_h(x, \xi) = \lim_{k\to\infty} a(x + h_{n_k}, \xi)$ such that for every function $\varphi \in C_0^\infty(\mathbb{R}^N)$*

$$\lim_{k\to\infty} \left\| \left(V_{-h_{n_k}} AV_{h_{n_k}} - Op(a_h) \right) \varphi I \right\|_{\mathcal{B}(\Lambda^s(\mathbb{R}))} = 0. \tag{2.24}$$

Proof. Let $A = Op(a) \in OPS^0_{1,0}(\widetilde{\mathbb{R}})$ and $V_h u(x) = u(x - h)$ be the translation operator. For a sequence $h_n \to \infty$ we have $V_{-h_n}AV_{h_n} = Op(a(x + h_n, \xi)$, where the functional sequence $a(x + h_n, \xi)$ is uniformly bounded and equicontinuous on compact sets $K \times \widetilde{\mathbb{R}}$, where K is a compact set in $\mathbb{R}$. Applying Arcella-Ascoli's Theorem we obtain that there exists a subsequence h_{n_k} such that

$$a(x + h_{n_k}, \xi) \to a_h(x, \xi)$$

uniformly on every compact sets $K \times \widetilde{\mathbb{R}}$, that is,

$$\lim_{k\to\infty} \sup_{K\times\mathbb{R}^N} |a(x + h_{n_k}, \xi) - a_h(x, \xi)| = 0. \tag{2.25}$$

By the well-known inequality

$$\sup_X \left| \frac{\partial u(x)}{\partial x_j} \right| \leq C \sqrt{\sup_X |u(x)|} \sqrt{\sup_X \left| \frac{\partial^2 u(x)}{\partial x_j^2} \right|},$$

where X is a set in $\mathbb{R}$, we obtain that

$$\lim_{k\to\infty}\sup_{K\times\mathbb{R}}\left|\partial_\xi^\alpha\partial_x^\beta a(x+h_{n_k},\xi)-\partial_\xi^\alpha\partial_x^\beta a_h(x,\xi)\right|\langle\xi\rangle^\alpha=0. \tag{2.26}$$

Formula (2.26) implies that the limit symbol $a_h(x,\xi)$ is in $S^0_{1,0}(\mathbb{R})$. Moreover, estimate (2.26) and Proposition 2.11 yield (3.4). □

Let

$$\Lambda^s_c(\mathbb{R}) = \begin{array}{l}\text{the closure in } \Lambda^s(\mathbb{R}) \text{ of the set of functions}\\ \text{in } \Lambda^s(\mathbb{R}) \text{ with compact supports.}\end{array} \tag{2.27}$$

Corollary 2.18. *Let $A = Op(a) \in OPS^0_{1,0}(\widetilde{\mathbb{R}})$, and a_h be denoted by (2.26). Then*

$$s-\lim_{k\to\infty}(V_{-h_{n_k}}AV_{h_{n_k}}:\Lambda^s_c(\mathbb{R})\to\Lambda^s(\mathbb{R}))=Op(a_h). \tag{2.28}$$

Proof. It suffices to prove (2.28) for $u\in\Lambda^s(\mathbb{R})$ with compact support. Let $\varphi_R u = u$. Then

$$\begin{aligned}&\lim_{k\to\infty}\left\|\left(V_{-h_{n_k}}AV_{h_{n_k}}-Op(a_h)\right)u\right\|_{\Lambda^s(\mathbb{R})}\\ \leq\ &\lim_{k\to\infty}\left\|\left(V_{-h_{n_k}}AV_{h_{n_k}}u-Op(a_h)\right)\varphi_R I\right\|_{\mathcal{B}(\Lambda^s(\mathbb{R}))}\|u\|_{\Lambda^s(\mathbb{R})}=0.\end{aligned}$$

Hence 2.24 implies 2.28. □

We set $OPSO_+(\widetilde{\mathbb{R}}) = OPS^0_{1,0}(\widetilde{\mathbb{R}})\bigcap OPSO_+$. Note that if $a\in SO_+(\widetilde{\mathbb{R}})$ and a_h is a limit symbol defined by (2.26), then a_h is a function depending only on $\xi\in\mathbb{R}$: $a_h(x,\xi)=a_h(\xi)$ ([15], Chap. 4.4).

Theorem 2.19. *An operator $A = Op(a)\in OPSO_+(\widetilde{\mathbb{R}})$ acting on $\Lambda^s(\mathbb{R})$ $\left(\Lambda^s_-(\mathbb{R})\right)$ is a locally Fredholm operator at the point $+\infty$, if and only if*

$$\lim_{r\to\infty}\inf_{x>r,\xi\in\mathbb{R}}|a(x,\xi)|>0. \tag{2.29}$$

Proof. **1.** Let $\psi\in C_b^\infty(\mathbb{R})$, and $\psi(x)=0$ if $x\leq 1/2$, and $\psi(x)=1$ if $x\geq 1$, $0\leq\psi(x)\leq 1$, $\psi_R(x)=\psi(x/R), R>0$, and $\psi_R\chi_R=\chi_R$. Let condition (2.29) be fulfilled. Then there exist $R_0>0$ such that $b_{R_0}(x,\xi)=\psi_{R_0}(x)a^{-1}(x,\xi)\in SO_+$. Let $B_{R_0}=Op(b_{R_0})$. Then

$$B_{R_0}A=\psi_{R_0}+T',$$

where $T'=Op(t')$ with t' satisfying estimates (2.7) and (2.8). Then,

$$B_{R_0}A\chi_R I=\chi_R I+T'\chi_R I,$$

where $T'\chi_R I$ is a compact operator by Proposition 2.14. Moreover,

$$B_{R_0}A\chi_R I=B_{R_0}\psi_R A\chi_R I+B_{R_0}\left[\psi_R I,A\right]\chi_R I,$$

where $[\psi_R I,A]$ is a compact operator by Proposition 2.15. Hence,

$$B_{R_0}\psi_R A\chi_R I=\chi_R I+T',$$

where T' is a compact operator.

In the same way we obtain that

$$\chi_R A \psi_R B_{R_0} = \chi_R I + T'',$$

where T'' is a compact operator.

2. Let an operator $Op(a) \in OPSO_+(\widetilde{\mathbb{R}})$ be locally Fredholm at the point $+\infty$. Show that condition (2.29) is fulfilled. Notice that

$$s - \lim_{R\to\infty} (\psi_R I : \Lambda_c^s(\mathbb{R}) \to \Lambda^s(\mathbb{R})) = 0.$$

Let $A : \Lambda^s(\mathbb{R}) \to \Lambda^s(\mathbb{R})$ be a locally Fredholm operator at the point $+\infty$. Then the following a priory estimate holds

$$\|A\chi_{R_0} u\|_{\Lambda^s(\mathbb{R})} \geq C(\|\chi_{R_0} u\|_{\Lambda^s(\mathbb{R})} - \|Tu\|_{\Lambda^s(\mathbb{R})}) \tag{2.30}$$

where T is a compact operator, and $R_0 > 0$ is sufficiently large.

Let ω have the properties similar to the properties of χ, and $\omega_R \chi_{R_0} = \omega_R$. Then

$$\|A\omega_R u\|_{\Lambda^s(\mathbb{R})} \geq C\left(\|\omega_R u\|_{\Lambda^s(\mathbb{R})} - \|T\omega_R u\|_{\Lambda^s(\mathbb{R})}\right).$$

We can consider T as a compact operator from $\Lambda_c^s(\mathbb{R})$ in $\Lambda^s(\mathbb{R})$. Hence,

$$\lim_{R\to\infty} \|T\omega_R I\|_{\Lambda_c^s(\mathbb{R})\to\Lambda^s(\mathbb{R})} = 0. \tag{2.31}$$

Formulas (2.30), (2.31) yield that there exist R_0 such that for $R > R_0$

$$\|A\omega_R u\|_{\Lambda^s(\mathbb{R})} \geq C/2\, \|\omega_R u\|_{\Lambda^s(\mathbb{R})} \tag{2.32}$$

for every function $u \in \Lambda_c^s(\mathbb{R})$. Let a sequence $h_m \in \mathbb{R}$ tend to $+\infty$ and a function u have a compact support. Then for fixed $R > 0$ there exists $m \geq m_0$ such that $\omega_R V_{h_m} u = V_{h_m} u$. Thus, for $m \geq m_0$

$$\|V_{-h_m} A\omega_R V_{h_m} u\|_{\Lambda^s(\mathbb{R})} = \|V_{-h_m} A V_{h_m} u\|_{\Lambda^s(\mathbb{R})} \geq C/2\, \|u\|_{\Lambda^s(\mathbb{R})}.$$

Proposition 2.17 yields that for a compactly supported function $u\,(\in \Lambda^s(\mathbb{R}))$

$$\|Op(a_h)u\|_{\Lambda^s(\mathbb{R})} \geq C/2\, \|u\|_{\Lambda^s(\mathbb{R})}. \tag{2.33}$$

Let $u \in \Lambda^s(\mathbb{R})$ be an arbitrary function. Then (2.33), Propositions 2.12 and 2.14 imply that

$$\begin{aligned} \|\varphi_R Op(a_h)u\|_{\Lambda^s(\mathbb{R})} &\geq \|Op(a_h)\varphi_R u\|_{\Lambda^s(\mathbb{R})} + O(1/R) \\ &\geq C/2\, \|\varphi_R u\|_{\Lambda^s(\mathbb{R})} + O(1/R). \end{aligned} \tag{2.34}$$

In light of Proposition 2.7

$$\lim_{R\to\infty} \|\varphi_R u\|_{\Lambda^s(\mathbb{R})} = \|u\|_{\Lambda^s(\mathbb{R})}. \tag{2.35}$$

Passing to the limit in (2.34) as $R \to \infty$, and applying (2.35) we obtain the estimate

$$\|Op(a_h)u\|_{\Lambda^s(\mathbb{R})} \geq C/2\, \|u\|_{\Lambda^s(\mathbb{R})} \tag{2.36}$$

for every function $u \in \Lambda^s(\mathbb{R}), s \in (0,1)$.

Note that $a_h(x,\xi) = a_h(\xi)$ since $a(x,\xi) \in SO_+(\mathbb{R})$. Thus, (2.36) implies that

$$\|a_h(D)u\|_{\Lambda^s(\mathbb{R})} \geq C/2 \, \|u\|_{\Lambda^s(\mathbb{R})} \tag{2.37}$$

for every function $u \in \Lambda^s(\mathbb{R}), s \in (0,1)$. Set in (2.37) $u = e_\xi = e^{ix\xi}$. It is evident that $e_\xi \in \Lambda^s(\mathbb{R})$ for every $s \in (0,1)$, and $a_h(D)e_\xi(x) = a_h(\xi)e_\xi(x)$. Thus, (2.37) implies that

$$\inf_{\xi\in\mathbb{R}} |a_h(\xi)| \geq C/2 > 0, \tag{2.38}$$

where

$$\lim_{n\to\infty} \sup_{K\times\mathbb{R}} |a(x + h_n, \xi) - a_h(\xi)| = 0 \tag{2.39}$$

for every compact set $K \subset \mathbb{R}$. Let us show that indeed (2.38) implies (2.29). Suppose that (2.38) holds, but (2.29) does not hold. Then there exists a sequence (h_n, p_n), $h_n \to +\infty$ such that

$$\lim_{n\to\infty} a(h_n, p_n) = 0. \tag{2.40}$$

Let the sequence h_n be such that limit (2.39) exists. Then it follows from (2.39), (2.38) there exists $N \in \mathbb{N}$ such that for all $n > N$

$$|a(h_n, p_n)| \geq C/4 > 0. \tag{2.41}$$

Inequality (2.41) contradicts to (2.40). □

Let $A : \Lambda^s(\mathbb{R}) \to \Lambda^s(\mathbb{R})$. We say that $\lambda \in \mathbb{C}$ is a point of the local Fredholm spectrum of A at the point $+\infty$ if the operator $A - \lambda I$ is not a locally Fredholm operator at the point $+\infty$. We denote the local Fredholm spectrum at the point $+\infty$ as

$$sp_{+\infty}(A : \Lambda^s(\mathbb{R}) \to \Lambda^s(\mathbb{R})). \tag{2.42}$$

In the same way we define the local Fredholm spectrum for $A : \Lambda^s_-(\mathbb{R}) \to \Lambda^s_-(\mathbb{R})$.

Theorem 2.19 has the following corollary.

Theorem 2.20. *Let $A = Op(a) \in OPSO_+(\widetilde{\mathbb{R}})$. Then*

$$\begin{aligned} sp_{+\infty}(A \ &: \ \Lambda^s(\mathbb{R}) \to \Lambda^s(\mathbb{R})) = sp_{+\infty}(A : \Lambda^s_-(\mathbb{R}) \to \Lambda^s_-(\mathbb{R})) \\ &= \bigcup_{h\in\Omega_{+\infty}(a)} \left\{\lambda \in \mathbb{C} : \lambda = a_h(\xi), \xi \in \widetilde{\mathbb{R}}\right\}, \end{aligned}$$

where $\Omega_{+\infty}(a)$ is a set of all sequences $h_n \to +\infty$ such that the limits $\lim_{n\to\infty} a(h_n, \xi) = a_h(\xi)$ exist.

3. Local Fredholmness of Mellin pseudodifferential operator on Hölder spaces

3.1. Multiplicative Hölder spaces on $\mathbb{R}_+$

We consider here the Hölder spaces on $\mathbb{R}_+$ with respect to the multiplicative structure of the group $\mathbb{R}_+$.

Definition 3.1. *By $\tilde{\Lambda}^s(\mathbb{R}_+)$, where $0 < s < 1$, we denote the class of bounded continuous function on $\mathbb{R}_+$ satisfying the conditions:*

$$\begin{aligned}\|u\|_{\tilde{\Lambda}^s(\mathbb{R}_+)} &= \|u\|_{L^\infty(\mathbb{R}_+)} + \sup_{t\in\mathbb{R}_+,\lambda\in\mathbb{R}_+\setminus\{1\}} \frac{|u(\lambda t)-u(t)|}{|\log\lambda|^s} = \\ & \|u\|_{L^\infty(\mathbb{R}_+)} + \sup_{t,\tau\in\mathbb{R}_+,t\neq\tau} \frac{|u(t)-u(\tau)|}{\left|\log\frac{t}{\tau}\right|^s} < \infty.\end{aligned} \tag{3.1}$$

Note that the mapping $\eta : \mathbb{R}_+ \to \mathbb{R}, \eta(r) = -\log r$ generates the isomorphisms $\eta^* : \Lambda^s(\mathbb{R}) \to \tilde{\Lambda}^s(\mathbb{R}_+)$. We set $\tilde{\Lambda}^s_-(\mathbb{R}_+) = \eta^*\left(\Lambda^s_-(\mathbb{R})\right)$.

Proposition 3.2. *The norm (3.1) on $\tilde{\Lambda}^s_-(\mathbb{R}_+)$ is equivalent to the norm*

$$\|u\|_{\tilde{\Lambda}^s_-(\mathbb{R}_+)} = \sup_{\substack{t\in\mathbb{R}_+,\varepsilon\neq0 \\ \frac{1}{e}-1<\varepsilon<e-1}} \frac{|u((1+\varepsilon)t)-u(t)|}{|\varepsilon|^s}. \tag{3.2}$$

Proof. Following the proof of Proposition 2.9 one can show that norm (3.1) on $\tilde{\Lambda}^s_-(\mathbb{R}_+)$ is equivalent to the norm

$$\sup_{\substack{t\in\mathbb{R}_+,\lambda\neq1 \\ \frac{1}{e}<\lambda<e}} \frac{|u(\lambda t)-u(t)|}{|\log\lambda|^s}. \tag{3.3}$$

Set $\lambda = 1+\varepsilon$ in (3.3). Then, $\varepsilon \in (\frac{1}{e}-1, 0)\cup(0, e-1)$. Hence,

$$0 < \alpha = \inf_{\varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)} \frac{\log(1+\varepsilon)}{\varepsilon} \leq \sup_{\varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1).} \frac{\log(1+\varepsilon)}{\varepsilon} = \beta < \infty.$$

Then, we obtain that norm (3.3) is equivalent to the norm

$$\|u\|_{\tilde{\Lambda}^s_-(\mathbb{R}_+)} = \sup_{t\in\mathbb{R}_+,\varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)} \frac{|u((1+\varepsilon)t)-u(t)|}{|\varepsilon|^s}. \tag{3.4}$$

The next proposition gives a connection between the space

$$\Lambda^s_0(\mathbb{R}_+) = \left\{u \in \Lambda^s_-(\mathbb{R}_+) : \lim_{x\to0} u(x) = 0\right\} = \Lambda^s_-(\mathbb{R}_+).$$

and the space $\tilde{\Lambda}^s_-(\mathbb{R}_+)$. □

Proposition 3.3. *The operator $u \to x^s u$ is an isomorphism from $\tilde{\Lambda}^s_-(\mathbb{R}_+)$ on $\Lambda^s_0(\mathbb{R}_+)$.*

Proof. Let $u \in \tilde{\Lambda}^s_-(\mathbb{R}_+)$. If we set below $h = \varepsilon x$, then

$$\begin{aligned}
\|x^s u\|_{\Lambda^s_-(\mathbb{R}_+)} &= \sup_{x,x+h\in\mathbb{R}_+,h\neq 0} \frac{|(x+h)^s u(x+h) - x^s u(x)|}{|h|^s} \\
&= \sup_{x,(1+\varepsilon)x\in\mathbb{R}_+,\varepsilon\neq 0} \frac{|(1+\varepsilon)^s u((1+\varepsilon)\,x) - u(x)|}{|\varepsilon|^s} \leq \\
& C \sup_{x,(1+\varepsilon)x\in\mathbb{R}_+,\varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)} \frac{|(1+\varepsilon)^s u((1+\varepsilon)\,x) - u(x)|}{|\varepsilon|^s} \\
&\leq C \sup_{\varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)} \frac{|(1+\varepsilon)^s - 1|}{|\varepsilon|^s} \|u\|_{L^\infty(\mathbb{R}_+)} \\
& + C \sup_{\substack{x,(1+\varepsilon)x\in\mathbb{R}_+ \\ \varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)}} \frac{|u((1+\varepsilon)\,x) - u(x)|}{|\varepsilon|^s} \leq C\,\|u\|_{\tilde{\Lambda}^s_-(\mathbb{R}_+)}\,.
\end{aligned}$$

Hence, $u \to x^s u$ is a bounded operator from $\tilde{\Lambda}^s_-(\mathbb{R}_+)$ on $\Lambda^s_-(\mathbb{R}_+)$. Since the function $u \in \tilde{\Lambda}^s_-(\mathbb{R}_+)$ is bounded, we have $\lim_{x\to 0} x^s u(x) = 0$, so that $x^s u \in \Lambda^s_0(\mathbb{R}_+)$.

Let us prove the boundedness of the inverse operator from $\Lambda^s_0(\mathbb{R}_+)$ onto $\tilde{\Lambda}^s_-(\mathbb{R}_+)$. We have

$$\begin{aligned}
\|u\|_{\tilde{\Lambda}^s_-(\mathbb{R}_+)} &= \sup_{\substack{t,(1+\varepsilon)t\in\mathbb{R}_+ \\ \varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)}} \frac{|u((1+\varepsilon)t) - u(t)|}{|\varepsilon|^s} \\
&= \sup_{\substack{t,\tau=(1+\varepsilon)t\in\mathbb{R}_+, \\ \varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)}} \frac{|t^s u(\tau) - t^s u(t)|}{|t-\tau|^s} \\
&\leq e^s \sup_{\substack{t,\tau\in\in\mathbb{R}_+,\tau=(1+\varepsilon)t \\ \varepsilon\in(\frac{1}{e}-1,0)\cup(0,e-1)}} \frac{|\tau^s u(\tau) - t^s u(t)|}{|t-\tau|^s} \leq e^s\,\|t^s u\|_{\Lambda^s_-(\mathbb{R}_+)}\,.
\end{aligned}$$

This implies that

$$\left\|t^{-s}u\right\|_{\tilde{\Lambda}^s_-(\mathbb{R}_+)} \leq e^s\,\|u\|_{\Lambda^s_-(\mathbb{R}_+)}\,.$$

□

3.2. Mellin pseudodifferential operators

As a modification of Definition 2.1, we say that a complex-valued function a defined on $\mathbb{R}_+ \times \mathbb{R}$ *belongs to the class* $\mathcal{S}^0_{1,0}$ if $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and satisfies the estimates

$$|a|_{r,t} = \sum_{\alpha\leq r,\beta\leq t} \sup_{\mathbf{R}_+\times\mathbf{R}} \left|(r\partial_r)^\beta \partial^\alpha_\lambda a(r,\lambda)\right| \langle\lambda\rangle^\alpha < \infty$$

for all α, $\beta \in \mathbb{N}_0 = \{0,1,2,3....\}$, where $\langle\lambda\rangle = (1+|\lambda|^2)^{1/2}$

Definition 3.4. *Let $a \in \mathcal{S}^0_{1,0}$. The operator*

$$(Op_M(a)u)(r) = (2\pi)^{-1} \int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_+} a(r,\lambda) \left(\frac{r}{\rho}\right)^{i\lambda} u(\rho) \frac{d\rho}{\rho} \tag{3.5}$$

where $u \in C_0^\infty(\mathbb{R}_+)$, is called the Mellin pseudodifferential operator with symbol a.

The class of operators of the form (3.5) with $a \in \mathcal{S}^0_{1,0}$ is denoted by $OP\mathcal{S}^0_{1,0}$.

The Mellin pseudodifferential operators in the class $OP\mathcal{S}^0_{1,0}$ are the transplantation on $\mathbb{R}_+$ of pseudodifferential operators in the class $OPS^m_{1,0}$, by means of the mapping $\phi : \mathbb{R}_+ \to \mathbb{R}, \phi(r) = -\log r$.

Let us summarize some properties of Mellin pseudodifferential operators which follow from the corresponding properties of pseudodifferential operators on $\mathbb{R}$.

By $\mathbb{S}(\mathbb{R}_+)$ we denote the class of functions φ on $\mathbb{R}_+$ such that $\varphi(\exp x) \in S(\mathbb{R})$. From the boundedness of usual pseudodifferential operators in $OPS^0_{1,0}$ on $S(\mathbb{R})$ it follows that an operator $A \in OP\mathcal{S}^0_{1,0}$ is a bounded operator on $\mathbb{S}(\mathbb{R}_+)$. An operator A^t is called formally adjoint to the operator A if

$$\int_{\mathbb{R}_+} (Au)(r)\bar{v}(x)\frac{dr}{r} = \int_{\mathbb{R}_+} u(r)\overline{(A^t v)(r)}\frac{dr}{r} \tag{3.6}$$

for arbitrary functions $u, v \in \mathbb{S}(\mathbb{R}_+)$. Let $A = Op_M(a) \in OP\mathcal{S}^0_{1,0}$. Then the formally adjoint operator $A^t \in OP\mathcal{S}^0_{1,0}$. Thus formula (3.6) allows us to consider pseudodifferential operators on the space of distributions $\mathbb{S}'(\mathbb{R}_+)$, and consequently on the space of Hölder functions.

We say that a symbol $a (\in \mathcal{S}^0_{1,0})$ is slowly oscillating at the point 0, if

$$\left|(r\partial_r)^\beta \partial_\lambda^\alpha a(r,\lambda)\right| \le C_{\alpha\beta}(r) \langle\lambda\rangle^{-\alpha},$$

where

$$\lim_{r\to 0} C_{\alpha\beta}(r) = 0,$$

for all $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$.

We denote by $\mathcal{SO}_0$ the class of slowly oscillating at the point 0 symbols, and by $\mathcal{OPSO}_0$ the corresponding class of pseudodifferential operators.

The next propositions are reformulations for Mellin pseudodifferential operators of the corresponding propositions of Section 2.

Proposition 3.5. *Let $A = Op_M(a) \in \mathcal{SO}_0, B = Op_M(b) \in \mathcal{SO}_0$. Then*

$$AB = Op_M(ab) + Op_M(t),$$

where $Op_M(t) \in OP\mathcal{S}^{-1}_{1,0}$, and

$$\left|(r\partial_r)^\beta \partial_\lambda^\alpha t(r,\lambda)\right| \le C_{\alpha\beta}(r) \langle\lambda\rangle^{-\alpha-1},$$

where

$$\lim_{r\to 0} C_{\alpha\beta}(r) = 0,$$

for all $\alpha, \beta \in \mathbb{N}_0$.

Proposition 3.6. *Let* $A = Op_M(a) \in OP\mathcal{S}^0_{1,0}$. *Then* A *is bounded on* $\tilde{\Lambda}^s(\mathbb{R}_+)$ $\left(\tilde{\Lambda}^s_-(\mathbb{R}_+)\right)$ *and*

$$\|Op_M(a)\|_{\mathcal{B}(\tilde{\Lambda}^s(\mathbb{R}_+))} \leq C\, |a|_{l_1,l_2}\,, l_1 > 2, l_2 > 1.$$

Let χ be a function introduced in 2.4, $\tilde{\chi}_R(r) = \chi_R(-\log r), r \in \mathbb{R}_+$.We denote by $\mathcal{SO}^\infty(\mathbb{R}_+)$ the class of functions $\tilde{b} \in C^\infty(\mathbb{R}_+)$ such that $\tilde{b}(r) = b(-\log r)$, where $b \in SO^\infty(\mathbb{R})$.

Proposition 3.7. *Let* $A = Op_M(a) \in \mathcal{OPSO}_0$, $\tilde{b} \in \mathcal{SO}^\infty(\mathbb{R}_+)$. *Then the commutator* $\left[a, \tilde{\chi}_R\tilde{b}I\right]$ *is a compact operator on* $\tilde{\Lambda}^s(\mathbb{R}_+)$ $\left(\tilde{\Lambda}^s_-(\mathbb{R}_+)\right)$.

Definition 3.8. *An operator* $A : \tilde{\Lambda}^s(\mathbb{R}_+) \to \tilde{\Lambda}^s(\mathbb{R}_+)$ *is called a locally Fredholm operator at the point* 0 *if there exist operators* $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}\left(\tilde{\Lambda}^s(\mathbb{R}_+)\right)$ *such that*

$$\mathcal{L}_R\tilde{\psi}_R A\tilde{\chi}_R I = \tilde{\chi}_R I + T'_R, \tilde{\chi}_R A\tilde{\psi}_R\mathcal{R}_R = \tilde{\chi}_R I + T''_R,$$

where T'_R, T''_R *are compact operators on* $\tilde{\Lambda}^s(\mathbb{R}_+)$.

A point $\zeta \in \mathbb{C}$ *is called a point of local Fredholm spectrum at the point* 0 *if the operator* $A - \zeta I : \tilde{\Lambda}^s(\mathbb{R}_+) \to \tilde{\Lambda}^s(\mathbb{R}_+)$ *is not a locally Fredholm operator at the point* 0. *We denote by* $sp_0(A)$ *the local Fredholm spectrum of* A *at the point* 0.

We denote by $\mathcal{S}^0_{1,0}(\widetilde{\mathbb{R}})$ the class of symbols $a \in \mathcal{S}^0_{1,0}$ such that a is extended to a continuous function on $\mathbb{R}_+ \times \widetilde{\mathbb{R}}$.

The next theorems are reformulations of the results of Subsection 2.4 with respect to the Mellin pseudodifferential operators.

Theorem 3.9. *Let* $A = Op_M(a(r,\lambda)) \in \mathcal{OPSO}_0(\widetilde{\mathbb{R}}) = \mathcal{OPS}^0_{1,0}(\widetilde{\mathbb{R}}) \cap \mathcal{OPSO}_0$. *Then* $A : \tilde{\Lambda}^s_-(\mathbb{R}_+) \to \tilde{\Lambda}^s_-(\mathbb{R}_+)$ *is a locally Fredholm operator at the point* 0, *if and only if*

$$\lim_{\varrho\to+0} \inf_{\substack{0<r<\varrho \\ \lambda\in\mathbb{R}}} |a(r,\lambda)| > 0. \tag{3.7}$$

Moreover,

$$sp_0(A : \tilde{\Lambda}^s_-(\mathbb{R}_+) \to \tilde{\Lambda}^s_-(\mathbb{R}_+) = \bigcup_{h\in\Omega_0(a)} \left\{\zeta \in \mathbb{C} : \zeta = a_h(\lambda), \lambda \in \widetilde{\mathbb{R}}\right\},$$

where $\Omega_0(a)$ *is a set of all the sequences* $h_n \to +0$ *such that the limits* $\lim_{n\to\infty} a(h_n,\lambda) = a_h(\lambda)$ *exist.*

We will also make use of Mellin pseudodifferential operators in Hölder spaces of vector-valued functions. By $\tilde{\Lambda}^s_-(\mathbb{R}_+, \mathbb{C}^N)$ we denote the space of vector-valued functions $u = (u_1, ..., u_N)$, where $u_j \in \tilde{\Lambda}^s_-(\mathbb{R}_+)$ with the norm

$$\|u\|_{\Lambda^s_-(\mathbb{R}_+, \mathbb{C}^N)} = \max_{1 \le j \le N} \|u_j\|_{\Lambda^s_-(\mathbb{R}_+)}, \tag{3.8}$$

and by $\mathcal{OPSO}_0(\widetilde{\mathbb{R}}, \mathbb{C}^{N\times N})$ the class of matrix-valued pseudodifferential operators $Op_M(a(r,\lambda)) = Op_M((a_{ij}(r,\lambda))^N_{i,j=1})$, where $a_{ij}(r,\lambda) \in \mathcal{OPSO}_0(\widetilde{\mathbb{R}})$.

Theorem 3.10. *Let $A = Op_M(a(r,\lambda)) \in \mathcal{OPSO}_0(\widetilde{\mathbb{R}}, \mathbb{C}^{N\times N})$. Then $A : \tilde{\Lambda}^s_-(\mathbb{R}_+, \mathbb{C}^N) \to \tilde{\Lambda}^s_-(\mathbb{R}_+, \mathbb{C}^N)$ is a locally Fredholm operator at the point 0, if and only if*

$$\lim_{\varrho \to +0} \inf_{0<r<\varrho, \lambda \in \mathbb{R}} |\det a(r,\lambda)| > 0. \tag{3.9}$$

Moreover,

$$sp_0\left(A : \tilde{\Lambda}^s_-(\mathbb{R}_+, \mathbb{C}^N) \to \tilde{\Lambda}^s_-(\mathbb{R}_+, \mathbb{C}^N)\right) = \bigcup_{h \in \Omega_0(a)} \bigcup_{\lambda \in \widetilde{\mathbb{R}}} sp\ \left(a_h(\lambda) : \mathbb{C}^N \to \mathbb{C}^N\right),$$

where $\Omega_0(a)$ is the set of all the sequences $h_n \to +0$ such that the limits $\lim_{n\to\infty} a(h_n, \lambda) = a_h(\lambda)$ exist.

4. Singular integral operators

4.1. Curves, weights, and coefficients

A set $\gamma \subset \mathbb{C}$ is called a *simple smooth arc* if there exists a homeomorphism $\varphi : [0,1] \to \gamma$ such that $\varphi(r) \in C^\infty(0,1]$ and $\varphi'(r) \neq 0$ for all $r \in (0,1)$. The points $\varphi(0)$ and $\varphi(1)$ are called the *endpoints* of γ. We refer to a set $\Gamma \subset \mathbb{C}$ as a *composed curve* if $\Gamma = \bigcup_{k=1}^K \Gamma_k$ where $\Gamma_1, \dots, \Gamma_K$ are oriented and rectifiable simple smooth arcs each pair of which has at most endpoints in common. A *node* of Γ is a point which is an endpoint of at least one of the arcs $\Gamma_1, \dots, \Gamma_K$. The set of all the nodes of Γ will be denoted by F.

A C^∞-function $f : (0,\varepsilon] \to \mathbb{C}$ is said to be *slowly oscillating at the origin* if

$$\sup_{r \in (0,\varepsilon)} \left| \left(r\frac{d}{dr} \right)^k f(r) \right| < \infty \quad , \quad k \in \mathbb{N}_0 \tag{4.1}$$

and

$$\lim_{r\to 0} |rf'(r)| = 0 \tag{4.2}$$

We remark that (4.1) and (4.2) imply that actually

$$\lim_{r\to 0} \left| \left(r\frac{d}{dr} \right)^k f(r) \right| = 0, \quad k \in \mathbb{N}_0.$$

To have an example, notice that if $f(r) = g(\log(-\log r)), 0 < r < 1$, where $g \in C^\infty_b(\mathbb{R})$, then f is slowly oscillating at the origin.

Suppose Γ is a composed curve and $t_k \in F$.

Definition 4.1. *We say that Γ is slowly oscillating at the point t_k, if there exists an $\varepsilon > 0$ such that the portion $\Gamma(t_k, \varepsilon) := \{\tau \in \Gamma : |\tau - t_k| < \varepsilon\}$ has the form*

$$\Gamma(t_k, \varepsilon) = \{t_k\} \cup \gamma_1^k \cup \ldots \cup \gamma_{n_k}^k$$

where the arcs γ_j^k are defined by

$$\gamma_j^k = \left\{t : \ t = t_k + re^{i\omega_j^k(r)}, \ r \in (0, \varepsilon)\right\}, \quad j = 1, \ldots, n_k, \tag{4.3}$$

and the functions $\omega_j^k(r)$ may have the form

$$\omega_j^k(r) = \theta^k(r) + \theta_j^k(r)$$

where θ^k and $\theta_1^k, \ldots, \theta_n^k$ are real-valued C^∞-functions such that
i) the functions $\delta_k(r) = r\frac{d\theta^k(r)}{dr}$ and $\delta_{kj}(r) = r\frac{d\theta_j^k(r)}{dr}$ are slowly oscillating at $r = 0$,
ii) there exists constants m_j^k and M_j^k such that for all $r \in (0, \varepsilon)$

$$0 \le m_1^k < \theta_1^k(r) < M_1^k < m_2^k < \theta_2^k(r) < M_2^k < \ldots < m_n^k < \theta_n^k(r) < M_n^k < 2\pi. \tag{4.4}$$

Under assumption $i)$, the functions θ_j^k $(j = 1, \ldots, n)$ are also slowly oscillating at $r = 0$.

For example, the functions

$$\omega_j^k(r) = \theta^k \log r + \theta_j^k; \qquad \theta^k, \ \theta_j^k \in \mathbb{R}, \quad (j = 1, \ldots, n_k)$$

with $0 \le \theta_1^k < \theta_2^k < \ldots < \theta_n^k < 2\pi$, satisfy all the assumptions of Definition 4.1. The curve γ_j^k is a logarithmic spiral in this case, and $\Gamma(t_k, \varepsilon)$ is a star of logarithmic spirals at the node t_k.

A composed curve which is slowly oscillating at each of its nodes will be referred to as a *slowly oscillating composed curve.*

Let $w : \Gamma \to [0, +\infty]$ be a function which takes values in $(0, +\infty)$ on $\Gamma \setminus F$ and is C^∞ on $\Gamma \setminus F$. We call w a *slowly oscillating weight at $t_k \in F$* if, under the above notation, w is of the form

$$w(t_k + re^{i\omega_j^k(r)}) = e^{v^k(r)}, \ \ r \in (0, \varepsilon), \ j \in \{1, \ldots, n_k\},$$

where

$$\varkappa_k(r) = r\frac{dv^k(r)}{dr}$$

is slowly oscillating at $r = 0$. For instance, the weight w arising from

$$v(r) = f(\log(-\log r)) \log r, \ \ r \in (0, \varepsilon)$$

with a bounded function $f \in C_b^\infty(\mathbb{R})$ is slowly oscillating at t_k; in this case we have

$$\liminf_{r\to 0} rv'(r) = \liminf_{x\to+\infty}(f(x) + f'(x)), \quad \limsup_{r\to 0} rv'(r) = \limsup_{x\to+\infty}(f(x) + f'(x)).$$

Finally, a function $a : \Gamma \to \mathbf{C}$ is said to be *piecewise slowly oscillating on* Γ, $a \in PSO(\Gamma)$, if a is C^∞ on $\Gamma \setminus F$ and if for each node $t_k \in F$ we have

$$a(t_k + re^{i\omega_j^k(r)}) = a_{t_k,j}(r),\ r \in (0,\varepsilon],\ j \in \{1,\dots,n\}$$

where $a_{t_0,1}(r),\dots,a_{t_0,n}(r)$ are slowly oscillating at $r = 0$.

4.2. Boundedness of SIO on Hölder spaces

We say that $u \in \Lambda^s(\Gamma),\ 0 < s < 1$, if there exists a neighborhood F_ε of F such that $u \in \Lambda^s(\Gamma\backslash F_\varepsilon)$, that is, u is continuous on $\Gamma\backslash F_\varepsilon$,

$$\|u\|_{\Lambda^s(\Gamma\backslash F_\varepsilon)} = \|u\|_{L^\infty(\Gamma\backslash F_\varepsilon)} + \sup_{t,\tau\in\Gamma\backslash F_\varepsilon} \frac{|u(t)-u(\tau)|}{|t-\tau|^s} < \infty,$$

and $u(t_k + re^{i(\theta^k(r)+\theta_j^k(r))}) = u_k^j(r) \in \Lambda^s(0,\varepsilon)$, for every $k = 1,...,K$, and $j = 1,...,n_k$. A norm in $\Lambda^s(\Gamma)$ is introduced in the evident way.

By $\Lambda^{s,w}(\Gamma)$, where w is a weight introduced in Subsection 4.1, we denote the weighted Hölder space of functions such that $wu \in \Lambda^s(\Gamma)$. A norm in $\Lambda^{s,w}(\Gamma)$ is introduced as

$$\|u\|_{\Lambda^{s,w}(\Gamma)} = \|wu\|_{\Lambda^s(\Gamma)}\,.$$

Let

$$A = aI + bS_\Gamma,$$

where $a, b \in PSO(\Gamma)$, and

$$S_\Gamma u(t) = \lim_{\varepsilon\to 0}\frac{1}{\pi i}\int\limits_{\tau\in\Gamma:|t-\tau|\ge\varepsilon}\frac{u(\tau)d\tau}{\tau - t}, \qquad t\in\Gamma$$

is a *SIO* on Γ.

Let $t_k \in F$, and $\varphi_k \in C^\infty(\Gamma)$ be such that $\varphi_k(\tau) = 1$ for τ in a neighborhood U_k of the node t_k and $\varphi_k(\tau) = 0$ outside a neighborhood $U_k' \supset \overline{U_k}$, and $0 \le \varphi_k(\tau) \le 1$. Let ψ_k have the properties similar to those of φ_k, and $\psi_k\varphi_k = \varphi_k$.

Put $\varepsilon_k^j = 1$ if t_k is the starting point of the oriented arc γ_k^j and let $\varepsilon_k^j = -1$ if t_k is the end point of the oriented arc γ_k^j.

Define

$$\nu : [0, 2\pi) \times (\mathbb{C} \setminus i\mathbb{Z}) \to \mathbb{C}$$

by

$$\nu(\delta, z) = \begin{cases} \coth(\pi z),\ \delta = 0, \\ \frac{e^{(\pi-\delta)z}}{\sinh(\pi z)},\ \delta \in (0, 2\pi). \end{cases}$$

For $j, l \in \{1,\dots,n_k\}$, let

$$S_{jl}^k : \mathbb{R}_+ \times (\mathbb{C} \setminus i\mathbb{Z}) \to \mathbb{C},$$

be the functions

$$S_{jl}^k(r,z) = \begin{cases} \varepsilon_k^l \nu(2\pi + \theta_j^k(r) - \theta_l^k(r), z) & , \quad j < l, \\ \varepsilon_k \nu(0, z) & , \quad j = l, \\ \varepsilon_k \nu(\theta_j^k(r) - \theta_l^k(r), z) & , \quad j > l. \end{cases} \tag{4.5}$$

Let us introduce the map

$$(\Phi_{t_k} f)(r) = column\left(\frac{w(r)}{r^s} f\left(t_k + re^{i\theta^k(r)+i\theta_1^k(r)}\right), \cdots, \frac{w(r)}{r^s} f\left(t_k + re^{i\theta^k(r)+i\theta_{n_k}^k(r)}\right)\right).$$

It follows from Proposition 3.3 that

$$\Phi_{t_k} : \Lambda_0^s(\Gamma(0,\varepsilon), w) \to \Phi_{t_k}(\Lambda_0^s(\Gamma(0,\varepsilon), w)) \subset \tilde{\Lambda}_-^s(\mathbb{R}_+, \mathbb{C}^{n_k})$$

is a Banach space isomorphism.

Proposition 4.2. *Let $s \in (0,1)$, and the following condition hold:*

$$s < \liminf_{r\to+0} \varkappa_k(r) \le \limsup_{r\to+0} \varkappa_k(r) < 1 + s \tag{4.6}$$

for every $k = 1, ..., K$. Then the operator

$$\Phi_{t_k} \varphi_k S_\Gamma \psi_k \Phi_{t_k}^{-1}$$

is a Mellin pseudodifferential operator with the symbol $s_k(r,\lambda) = \left(s_k^{jl}(r,\lambda)\right)_{j,l=1}^{n_k} \in \mathcal{OPSO}_0(\widetilde{\mathbb{R}}, \mathbb{C}^{n_k})$ defined by

$$s_k^{jl}(r,\lambda) = \varepsilon_l \tilde{\varphi}_k^j(r) S_{jl}^k\left(r, \frac{\lambda + i(\varkappa_k(r) - s)}{1 + i\delta_k(r)}\right)\tilde{\psi}_k^l(r) + t_{jl}^k(r,\lambda) \tag{4.7}$$

where $\tilde{\varphi}_k^j(r) = \varphi_k(t_k + re^{i(\theta^k(r)+\theta_l^k(r))})$, $\tilde{\psi}_k^l(r) = \varphi_k(t_k + re^{i(\theta^k(r)+\theta_l^k(r))})$, and $t_{jl}^k(r,\lambda) \in \mathcal{S}_{1,0}^{-1}$ and

$$\lim_{r\to 0} (r\partial_r)^\beta \partial_\lambda^\alpha t_{jl}^k(r,\lambda) \langle\lambda\rangle^\alpha = 0$$

for every $\alpha, \beta \in \mathbb{N}_0$.

Proof. See papers [3], [19], and also book [15], Chap. 4.6. □

Corollary 4.3. *The operator $\Phi_{t_k}\varphi_k S_\Gamma \psi_k \Phi_{t_k}^{-1}$ is bounded on the space $\tilde{\Lambda}_-^s(\mathbb{R}_+), 0 < s < 1$.*

Proof. Indeed, condition (4.6) implies that $S_{jl}^k\left(r, \frac{\lambda+i(\varkappa_k(r)-s)}{1+i\delta_k(r)}\right) \in \mathcal{SO}_0(\widetilde{\mathbb{R}})$. Thus $Op_M(s_k(r,\lambda))$ is a bounded operator on $\tilde{\Lambda}_-^s(\mathbb{R}_+), 0 < s < 1$, by Proposition 3.6. □

Theorem 4.4. *Let $s \in (0,1)$, and the following condition hold:*

$$s < \liminf_{r\to+0} \varkappa_k(r) \le \limsup_{r\to+0} \varkappa_k(r) < 1 + s \tag{4.8}$$

for every $k = 1, ..., K$. Then $A : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ is a bounded operator.

Proof. Let $\varphi_k \in C^\infty(\Gamma)$ and $\varphi_k(\tau) = 1$ if $\tau \in U_k$ a neighborhood of the node t_k and $\varphi_k(\tau) = 0$ outside a neighborhood $U_k' \supset \overline{U_k}$, and $0 \le \varphi_k(\tau) \le 1$. Let ψ_k

have properties similar to the properties of φ_k, and $\psi_k\varphi_k = \varphi_k$. We set $\varphi_0 = 1 - \sum_{k=1}^{K}\varphi_k$. Then

$$A = \sum_{k=0}^{K}\varphi_k A\psi_k I + \sum_{k=0}^{K}\varphi_k A(1-\psi_k)I.$$

Note that a function $u \in \Lambda_0^s(\Gamma)$ and the weight w satisfy the following estimates in a neighborhood of the mode t_k

$$|u(t)| \leq C\,\|u\|_{\Lambda^s(\Gamma)}\,|t-t_k|^s\,, \tag{4.9}$$

and

$$c\,|t-t_k|^{\liminf\limits_{r\to+0}\varkappa_k(r)} \leq |w(t)| \leq C\,|t-t_k|^{\limsup\limits_{r\to+0}\varkappa_k(r)}. \tag{4.10}$$

It follows from (4.8), (4.9) and (4.10) that

$$\left\|w^{-1}u\right\|_{L^1(\Gamma)} \leq C\,\|u\|_{\Lambda^s(\Gamma)}\,,$$

if $u \in \Lambda_0^s(\Gamma)$. Since $supp\,(1-\psi_k)\cap supp\,\varphi_k = \emptyset$, the operators $\varphi_k wAw^{-1}(1-\psi_k)I$ are operators with $C^\infty-$kernels. Hence,

$$\left\|w\varphi_k A(1-\psi_k)w^{-1}u\right\|_{\Lambda^s(\Gamma)} \leq C\,\left\|w^{-1}u\right\|_{L^1(\Gamma)} \leq C\,\|u\|_{\Lambda^s(\Gamma)}\,.$$

Note that condition (4.8) implies that $\lim\limits_{t\to t_k} w(t) = 0$. Hence

$$\lim_{t\to t_k}\left(\varphi_k wAw^{-1}(1-\psi_k)u\right)(t) = 0$$

for every function $u \in \Lambda^s(\Gamma)$. Thus the operator $\sum_{k=0}^{K}\varphi_k A(1-\psi_k)I$ is bounded on $\Lambda_0^{s,w}(\Gamma)$. From the well-known classical results (see for instance [12]) it follows that the operator $\varphi_0 A\psi_0 I$ is bounded on $\Lambda_0^{s,w}(\Gamma)$ because the supports of φ_0, ψ_0 do not contain the nodes.

Hence, we reduced the proof of Theorem 4.4 to the problem of boundedness of operators $\varphi_k A\psi_k I : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$. Applying Proposition 4.2, we obtain that $\varphi_k A\psi_k I : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ is bounded, if and only if the operator $\mathcal{S}_k = \Phi_{t_k}\varphi_k S_\Gamma\psi_k\Phi_{t_k}^{-1'} : \tilde{\Lambda}_-^s(\mathbb{R}_+) \to \tilde{\Lambda}_-^s(\mathbb{R}_+)$ is bounded, but boundedness of $\mathcal{S}_k$ follows from Proposition 3.6. □

4.3. Fredholm properties of singular integral operators on composed Carleson curves

We say that an operator $A : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ is a locally Fredholm operator at the point $t \in \Gamma$ if there exists functions $\varphi_t, \psi_t \in C^\infty(\Gamma), (\varphi_t\psi_t = \varphi_t)$, equal to one in a neighborhood of t, and an operator $L^t, R^t : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ such that

$$L^t\psi_t A\varphi_t I = \varphi_t I + T_1^t \qquad \text{and} \qquad \varphi_t A\psi_t R^t = \varphi_t I + T_2^t, \tag{4.11}$$

where T_1^t, T_2^t are compact operators on the space $\Lambda_0^{s,w}(\Gamma)$. The operators L^t, R^t are called local regularizers of A at the point $t \in \Gamma$.

Below we use the following notation. Let $a \in PSO(\Gamma)$ and $t_k \in F$. Then $diag(a^k(r)) = diag(a_1^k(r), ..., a_{n_k}^k(r))$ is a diagonal matrix with components $a_j^k(r) = a(t_k + re^{\omega_j^k(r)}), r \in (0,\varepsilon)$. We say that an operator $B : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ is a local type operator, if for an arbitrary function $a \in C^\infty(\Gamma)$ the commutator $[B, aI] = BaI - aB$ is a compact operator on $\Lambda_0^{s,w}(\Gamma)$.

Let $\Gamma \subset \mathbb{C}$ be a composed curve introduced in Subsection 4.1 and w be a slowly oscillating weight satisfying conditions (4.8) for every point $t_k \in F$. Let

$$A_\Gamma = aI + bS_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma), 0 < s < 1$$

be a singular integral operator on Γ with piece-wise slowly oscillating coefficients $a, b \in PSO(\Gamma)$. We set

$$\begin{aligned} & \sigma_{t_k}(A_\Gamma)(r,\lambda) \\ = \ & \varphi_{t_k}(r)\left(diag(a^k(r)) + diag(b^k(r))\left(\varepsilon_l S^k_{jl}\left(r, \frac{\lambda + i(\varkappa_k(r) - s)}{1 + i\delta_k(r)}\right)\right)^{n_k}_{j,l=1}\right), \\ r \ \in \ & \mathbb{R}, \lambda \in \mathbb{R}, \end{aligned}$$

where $\varphi_{t_k} \in C^\infty(\mathbb{R})$ is equal to 1 near the point t_k, and has a support in a small neighborhood of the point $t_k \in F$. We say that $\sigma_{t_k}(A_\Gamma)$ is a local symbol of A_Γ at the point $t_k \in F$.

Theorem 4.5. *(i) A_Γ is a locally Fredholm operator at the point $t_k \in F$ if and only if*

$$\lim_{\delta \to +0} \inf_{\substack{0<r<\delta \\ \lambda \in \mathbb{R}}} |\det \sigma_{t_k}(A_\Gamma)(r,\lambda)| > 0. \tag{4.12}$$

(ii) A_Γ is a locally Fredholm operator at the point $t \in \Gamma \backslash F$, if and only if

$$a^2(t) - b^2(t) \neq 0; \tag{4.13}$$

If conditions (4.12) and (4.13) hold, then for every point $t \in \Gamma$ there exist local regularizers of local type.

Proof. (i) Let $t_k \in F$. Then $A_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ is a locally Fredholm operator at the point t_k, if and only if the operator $\Phi_{t_k}\varphi_k A_\Gamma \psi_k \Phi_{t_k}^{-1`} : \tilde{\Lambda}^s_-(\mathbb{R}_+) \to \tilde{\Lambda}^s_-(\mathbb{R}_+)$ if locally Fredholm at the point 0, where the functions φ_k, ψ_k have supports in a small neighborhood at the point t_k, and $\varphi_k(t_k) = \psi_k(t_k) = 1$. Hence, (i) follows from Proposition 3.7. It follows from the construction of local regularizers and Proposition 3.7 that the local regularizers at the point $t_k \in F$ are local type operators.

(ii) The curve Γ is smooth in a small neighborhood of the point $t \in \Gamma \backslash F$, and the space $\Lambda_0^{s,w}(\Gamma)$ coincides with the usual Hölder space $\Lambda^s(\Gamma)$, and the coefficients of A are smooth. Hence (ii) follows from the well-known results for singular integral operators on Lyapunov curves acting on Hölder spaces (see for instance [12], [8]). The local regularizer at the point $t_0 \in \Gamma \backslash F$ has the form

$$R_{t_0} = \varphi_{t_0}(a(t_0)I - b(t_0)S_\Gamma)\psi_{t_0}I,$$

where $\varphi_{t_0}, \psi_{t_0} (\in C^\infty(\Gamma))$ are functions with supports in a small neighborhood of the point t_0. It is clear that R_{t_0} is a regularizer of local type. □

We denote by $sp_t(A_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma))$ the set of complex numbers ζ such that the operator $A - \zeta I$ is not locally Fredholm operator at the point $t \in \Gamma$.

Theorem 3.10 and Theorem 3.2 yield the following result.

Theorem 4.6. *Let $t_k \in F$. Then*

$$sp_{t_k}(A_\Gamma \quad : \quad \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)) \tag{4.14}$$

$$= \bigcup_{h\in\Omega_0(\sigma_{t_k}(A))} \bigcup_{\lambda\in\widetilde{\mathbb{R}}} sp(\sigma_{t_k,h}(A)(\lambda) : \mathbb{C}^{n_k} \to \mathbb{C}^{n_k}), \tag{4.15}$$

where

$$\sigma_{t_k,h}(A_\Gamma)(\lambda) = \lim_{k\to\infty} \sigma_{t_k}(A)(h_m, \lambda), \tag{4.16}$$

and $h = (h_m) \in \Omega_0(\sigma_{t_k}(A))$ is the set of all the sequences such that the limit (4.16) exists.

Note that

$$\sigma_{t_k,h}(A_\Gamma)(\lambda) = diag(a_h^k) + diag(b_h^k)\left(\varepsilon_l S_{jl}^{k,h}\left(\frac{\lambda + i(\varkappa_k^h - s)}{1 + i\delta_k^h}\right)\right)_{j,l=1}^{n_k}, \lambda \in \mathbb{R}, \tag{4.17}$$

where

$$\begin{aligned} a_h^k &= \lim_{m\to\infty} a^k(h_m), b_h^k = \lim_{m\to\infty} b^k(h_m), \qquad \varkappa_k^h = \lim_{m\to\infty} \varkappa_k(h_m), \\ \delta_k^h &= \lim_{m\to\infty} \delta_k(h_m), \theta_j^{k,h} = \lim_{m\to\infty} \theta_j^k(h_m), \qquad j = 1, ..., n_k, \end{aligned} \tag{4.18}$$

and

$$S_{jl}^{k,h}(z) = \begin{cases} \varepsilon_k^l \nu(2\pi + \theta_j^{k,h} - \theta_l^{k,h}, z) & , \quad j < l, \\ \varepsilon_k \nu(0, z) & , \quad j = l, \\ \varepsilon_k \nu(\theta_j^{k,h} - \theta_l^{k,h}, z) & , \quad j > l. \end{cases} . \tag{4.19}$$

Notice that $\sigma_{t_k,h}(A_\Gamma)$ is the symbol of singular integral operator

$$B^{k,h} = a_h^k I + b_h^k S_{\gamma^{k,h}} : \Lambda_0^{s,w^{k,h}}(\gamma^{k,h}) \to \Lambda_0^{s,w^{k,h}}(\gamma^{k,h}), \tag{4.20}$$

where $\gamma^{k,h}$ is a union of logarithmic spirals staring in the node $t = t_k$, that is

$$\gamma^{k,h} = \{t_k\} \bigcup_{j=1}^{n_k} \left\{ t \in \mathbb{C} : t = t_k + re^{i(\delta_k^h \log r + \theta_j^{k,h})}, r \in \mathbb{R}_+ \right\},$$

$w^{k,h} = |t - t_k|^{\varkappa_k^h}$ is a power weight at the node t_k.

It was proved in [2] that the operator $B^{k,h} : L^{p,w^{k,h}}(\gamma^{k,h}) \to L^{p,w^{k,h}}(\gamma^{k,h})$, where $-1/p < w^{k,h} < 1 - 1/p$, is locally Fredholm at the point t_k, if and only if $B^{k,h}$ is invertible. One can prove that the same property holds for $B^{k,h} : \Lambda_0^{s,w^{k,h}}(\gamma^{k,h}) \to \Lambda_0^{s,w^{k,h}}(\gamma^{k,h})$. Hence, the local spectrum of $A_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma))$ is the union of spectrums of singular integral operators $B^{k,h}$ on the logarithmic star $\gamma^{k,h}$ acting on the space $\Lambda_0^{s,w^{k,h}}(\gamma^{k,h})$ with power weight $w^{k,h}$.

Thus, the local massive Fredholm spectrum $sp_{t_k}(A_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma))$ is generated by three forces: 1) oscillation of curves; 2) oscillation of weights; 3) oscillation of coefficients a, b at the node $t_k \in F$.

If t_k is an end-point of a single curve only, then the logarithmic star $\gamma^{k,h}$ is transformed into a single logarithmic spiral. In this case the local essential spectrum at the node t_k is a union of the double logarithmic spirals (see [4], [3], [15], Ch. 4.6).

Theorem 4.7. *Let conditions (4.8) be satisfied for every point $t_k \in F$. Then*

$$A_\Gamma = aI + bS_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$$

is a Fredholm operator, if and only if

(i) for every point $t_k \in F$ condition (4.12) holds;

(ii) for every point $t \in \Gamma \backslash F$ condition (4.13) holds.

If the conditions (i), (ii) are satisfied, then

$$Ind\ A = -\sum_{k=1}^{L} \frac{1}{2\pi}\left[\arg \frac{a(t)+b(t)}{a(t)-b(t)}\right]_{t\in\Gamma_j} - \sum_{k=1}^{L} \frac{1}{2\pi} \lim_{r\to+0} [\arg\det \sigma_{t_k}(A_\Gamma)(r,\lambda)]_{\lambda=-\infty}^{+\infty},$$

where Γ_j are the simple arcs composing the curve Γ with orientation induced by that of Γ.

Proof. From Propositions 3.7 and 4.2 it follows that the singular integral operator $A = aI + bS_\Gamma : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ is a local type operator. Let conditions (4.12), (4.13) be fulfilled. Then for every point $t \in \Gamma$ there exists a local regularizer of local type, that is, for every $t \in \Gamma$ there exists a function $\varphi_t \in C^\infty(\Gamma)$ equal to one in a neighborhood of t, and operator $R^t : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ such that equality (4.11) is fulfilled. Since Γ is a compact set we can construct a partition of unity

$$\sum_{k=0}^{N} \varphi_k(t) = 1, \qquad t \in \Gamma$$

with the following properties:

(a) $\varphi_k \in C^\infty(\Gamma), k = 0, ..., N,\ 0 \leq \varphi_k \leq 1$;

(b) *supp* φ_k contains only one node $t_k, k = 1, ..., N$, and $\varphi_k(t) = 1$ in a neighborhood of the t_k, where $\varphi_0 = 1 - \sum_{k=1}^{N} \varphi_k$ is such that *supp* φ_0 does not contain the nodes;

(c)

$$R^k A \varphi_k I = \varphi_k I + T'_k, \varphi_k A R^k = \varphi_k I + T''_k,$$

where T'_k, T''_k are compact operators.

Let functions $\psi_k \in C^\infty(\Gamma), k = 0, ..., N$, be such that $0 \leq \psi_k \leq 1$, and $\varphi_k \psi_k = \varphi_k$. We set

$$R_{left} = \sum_{k=0}^{N} \psi_k R^k.$$

Then,

$$\begin{aligned} R_{left}A &= \sum_{k=0}^{N}\psi_k R^k A = \sum_{k=0}^{N}\psi_k R^k A\varphi_k I + \sum_{k=0}^{N}\psi_k R^k A(1-\varphi_k)I \\ &= I + \sum_{k=0}^{N}\psi_k T_k' + \sum_{k=0}^{N}\left[\psi_k, R^k A\right] = I + T'. \end{aligned}$$

Note that T' is a compact operator, since R^k and A are operators of local type. Hence, R_{left} is a left regularizer of A. In the same way one can prove that the operator

$$R_{right} = \sum_{k=0}^{N} R^k \psi_k I$$

is a right regularizer of A, that is, $AR_{right} = I + T''$, where T'' is a compact operator. Thus we proved that the singular integral operator A is a Fredholm operator.

Let $A : \Lambda_0^{s,w}(\Gamma) \to \Lambda_0^{s,w}(\Gamma)$ be a Fredholm operator. Then A is a locally Fredholm operator at every point $t \in \Gamma$. Hence condition (4.12) is fulfilled for $t = t_k \in F$ by Theorem 4.6, and condition (4.13) is fulfilled for $t \in \Gamma\backslash F$.

The proof of the index formula is similar to that of the analogous formula for singular integral operators acting on L^p−spaces (see [19], Theorem 4.1.) □

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