# The maximal Operator in Variable Spaces $L^{p(\cdot)}(\Omega, \rho)$ with Oscillating Weights

by

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**Abstract**

We study the boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\rho, \Omega)$ over a bounded open set in $R^n$ with the weight $\rho(x) = \prod_{k=1}^{m} w_k(|x - x_k|)$, $x_k \in \overline{\Omega}$, where $w_k$ has the property that $r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi_n^0$, where $\Phi_n^0$ is a certain Zygmund-type class. The weight functions $w_k$ may oscillate between two power functions with different exponents. It is assumed that the exponent $p(x)$ satisfies the Dini–Lipschitz condition. The final statement on the boundedness is given in terms of the index numbers of the functions $w_k$ (similar in a sense to the Boyd indices for the Young functions defining Orlich spaces).

*Key Words and Phrases*: maximal functions, weighted Lebesgue spaces, variable exponent, potential operators, integral operators with fixed singularity

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# 1 Introduction

Nowadays there is an evident increase of interest to harmonic analysis problems and operator theory in the generalized Lebesgue spaces with variable exponent $p(x)$ and the corresponding Sobolev spaces, we refer, in particular to surveys [10], [13], [25], [7] and to [28], [14] for the basics on the spaces $L^{p(\cdot)}$.

For the boundedness results of maximal operators we refer to L. Diening [5] for bounded domains in
$mathbbR^n$ and to D.Cruz-Uribe, A.Fiorenza and C.J. Neugebauer [4] and A.Nekvinda [21], [20] for unbounded domains, and to V.Kokilashvili and S.Samko [12] for weighted boundedness on bounded domains.

We refer also to L.Diening [6] and D.Cruz-Uribe, A.Fiorenza, J.M.Martell, and C.Perez [3] where there are also given new insights into the problems of boundedness of singular and maximal operators in variable exponent spaces.

In [12] the power weights $|x - x_0|^\gamma$ were considered and one of the main points in the result obtained in [12] was that in condition on $\gamma$ only the values of $p(x)$ at the point $x_0$ are of importance:

$$-\frac{n}{p(x_0)} < \gamma < \frac{n}{q(x_0)}$$

(under the usual log-condition on $p(x)$).

A problem of more general weights remains open. An explicit description of weights for which the maximal operator is bounded in the spaces $L^{p(\cdot)}$ is a challenging problem. What should be the corresponding $A_{p(\cdot)}$-condition? It is natural to suppose that the Muckenhoupt condition written in the natural terms of the inverse Hölder inequality may be the corresponding characterization. Whether this is or not, is an open question.

In this paper we prove weighted boundedness for the maximal operator in the spaces $L^{p(\cdot)}(\Omega, \rho)$ for some class of general, that is, non-power weights, which are ”attached” to a finite number of points $x_k \in \Omega$ (radial type weights of the Zygmund-Bary-Stechkin class). Weights $w$ in this class are almost increasing or almost decreasing and may oscillate between two power functions with different exponents and have non-coinciding upper and lower indices $m_w$ and $M_w$ (of the type of Boyd indices). In comparison with the approach in [12], the main problems arising are related to the situation when the indices $m_w$ and $M_w$ do not coincide, in particular when $m_w$ is negative while $M_w$ is positive.

The paper is organized as follows. In Section 2 we formulate the main result - Theorem A - on the weighted boundedness of the maximal operator In Section 3 we recall the notion of the upper and lower indices of almost increasing non-negative functions and develop some properties of weights in the Zygmund-Bary-Stechkin class, which we need to prove the main result. Section 4 contains some technical lemmas related to the variable exponent $p(x)$. Finally, Section 5 contains the proof of Theorem A.

We recall the main notation. By $\Omega$ we denote an open bounded set in $R^n$,

$n \geq 1$, and $p(x)$ a function on $\overline{\Omega}$ satisfying the conditions

$$1 < p_* \leq p(x) \leq p^* < \infty, \quad x \in \overline{\Omega} \tag{1.1}$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \overline{\Omega}. \tag{1.2}$$

By $L^{p(\cdot)}(\Omega, \rho)$ we denote the weighted Banach space of all measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Omega,\rho)} := \|\rho f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int\limits_{\Omega} \left| \frac{\rho(x) f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} < \infty. \tag{1.3}$$

**Notation**

a.d. =almost decreasing $\Longleftrightarrow$ $f(x) \geq Cf(y)$ for $x \leq y, C > 0$;
a.i. =almost increasing $\Longleftrightarrow$ $f(x) \leq Cf(y)$ for $x \leq y, C > 0$;
$\Omega$ is a open bounded set in $R^n$;
$|\Omega|$ is the Lebesgue measure of $\Omega$;
$\chi_\Omega$ is the characteristic function of a set $\Omega$;
$f \sim g \iff$ there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 f(x) \leq g(x) \leq C_2 f(x)$.
$B_r(x) = \{y \in R^n : \ |y - x| < r\}$;
$|B_r(x)| = \frac{r^n}{n} |S^{n-1}|$ is the volume of $B_r(x)$;
$q(x) = \frac{p(x)}{p(x)-1}, \ 1 < p(x) < \infty, \ \frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$ ;
$p_* = \inf\limits_{x \in \Omega} p(x)$, $p^* = \sup\limits_{x \in \Omega} p(x)$;
$q_* = \inf\limits_{x \in \Omega} q(x) = \frac{p^*}{p^*-1}$, $q^* = \sup\limits_{x \in \Omega} q(x) = \frac{p_*}{p_*-1}$;
$C, c$ may denote different positive constants.

## 2 Statement of the Main Results

Let

$$\mathcal{M}^\rho f(x) = \sup_{r>0} \frac{\rho(x)}{|B_r(x)|} \int\limits_{B_r(x) \cap \Omega} \frac{|f(y)|}{\rho(y)} \, dy, \tag{2.1}$$

where

$$\rho(x) = \prod_{k=1}^{m} w_k(|x - x_k|), x_k \in \overline{\Omega}.$$

We write $\mathcal{M} = \mathcal{M}^0$ when $\rho(t) \equiv 1$.

In [12] there was proved the boundedness of the operator $\mathcal{M}^\rho$ in the case of the power weight $\rho(x) = |x - x_0|^\beta, x_0 \in \overline{\Omega}$ under the following (necessary and sufficient) condition

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p(x_0)}. \tag{2.2}$$

In the main result of this paper, see Theorem A, we deal with a certain class of weights which may oscillate between two power functions (radial Zygmund-Bary-Stechkin type weights).

The Zygmund-Bari-Stechkin class $\Phi_n^0$ of weights and the upper and lower indices of weights (of the type of the Boyd indices) used in the theorem below are defined in Section 3.

**Theorem A** . *Let* $p(x)$ *satisfy conditions (1.1), (1.2). The operator* $\mathcal{M}$ *is bounded in* $L^{p(x)}(\Omega, \rho)$ *with the weight* $\rho(x) = \prod_{k=1}^m w_k(|x - x_k|)$, $x_k \in \Omega$, *where* $w_k(r)$ *are such functions that* $r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi_n^0$, *if*

$$-\frac{n}{p(x_k)} < m_{w_k} \le M_{w_k} < \frac{n}{q(x_k)} \ , \ \ k = 1, 2, ..., m. \tag{2.3}$$

# 3 Preliminaries on Zygmund-Bary-Stechkin classes.

## 3.1 Index numbers $m_w$ and $M_w$ of non-negative a. i. functions

Let

$$W = \{w \in C([0, \ell]) : w(0) = 0, \ w(x) > 0 \ \text{ for } \ x > 0, \ \ w(x) \ \text{ is a.i.}\}. \tag{3.1}$$

The numbers

$$m_w = \sup_{x>1} \frac{\ln\left(\liminf\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x} = \sup_{0<x<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x} = \lim_{x\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x}$$

and

$$M_w = \sup_{x>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x} = \lim_{x\to\infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x}$$

(see [22], [24], [23]), are known as *the lower and upper indices* of the function $w(x)$ (compare these indices with the Matuszewska-Orlicz indices, see [17], p. 20; they are of the type of the Boyd indices, see [15], p. 75; [16], or [2], p. 149 about the Boyd indices). We have $0 \le m_w \le M_w \le \infty$ for $w \in W$.

We call a function $w(x)$ *equilibrated* or *non-oscillating*, if $M_w = m_w$.

**Remark 3.1.** The upper and lower indices may be also well defined for functions $w(x)$ positive for $x > 0$ which do not necessarily belong to $W$, for example, if there is given a function $w(x)$ such that $w_a(x) := x^a w(x)$ is in $W$, then the indices $m_{w_a}$ and $M_{w_a}$ of $w_a(x)$ are well defined and there also exist the indices $m_w$ and $M_w$ of $w(x)$ and

$$m_{w_a} = a + m_w, \quad M_{w_a} = a + M_w$$

in this case.

We find it convenient to introduce the following class of functions, the indices $m_w$ and $M_w$ of which may be negative:

$$\widetilde{W} = \{w : \; t^a w(t) \in W \quad \text{for some} \quad a \in \mathbb{R}^1\}.$$

## 3.2 The Zygmund-Bary-Stechkin class $\Phi^0_\gamma$

Let $\gamma > 0$. The following class $\Phi^0_\gamma$ was introduced and studied in [1] (with integer $\gamma$); there are also known "two-parametrical" classes $\Phi^\beta_\gamma, 0 \leq \beta < \gamma < \infty$, see [19], [18], [27] and [26], p. 253). Observe that in [30], [31] there were considered more general classes $\Phi^{a(x)}_{b(x)}$ with limits which may "oscillate"; the class $\Phi^0_\gamma$ corresponds to the case where $a(x) = x^0 = 1$ and $b(x) = x^\gamma$.

**Definition 3.2.** ([1]) The Zygmund-Bary-Stechkin type class $\Phi^0_\gamma, \; 0 < \gamma < \infty$, is defined as $\Phi^\beta_\gamma := \mathcal{Z}^0 \cap \mathcal{Z}_\gamma$, where $\mathcal{Z}^0$ is the class of functions $w \in W$ satisfying the condition

$$\int_0^h \frac{w(x)}{x} dx \leq c w(h) \tag{$\mathbb{Z}^0$}$$

and $\mathcal{Z}_\gamma$ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(x)}{x^{1+\gamma}} dx \leq c \frac{w(h)}{h^\gamma}, \tag{$\mathbb{Z}_\gamma$}$$

where $c = c(w) > 0$ does not depend on $h \in (0, \ell]$.

In the sequel we refer to the above conditions as ($\mathbb{Z}^0$)- and ($\mathbb{Z}_\gamma$)-conditions.

We note that each of the inequalities ($\mathbb{Z}^0$) and ($\mathbb{Z}_\gamma$) is invertible if they both are satisfied. Namely, the following statement holds.

**Lemma 3.3.** *Let* $w(r) \in \Phi^0_\gamma, \; \gamma > 0$. *Then*

$$\int_0^h \frac{w(r)}{r} dr \sim h^\gamma \int_h^\ell \frac{w(r)}{r^{1+\gamma}} dr \sim w(h) \tag{3.2}$$

*the latter equivalence holding on any subinterval* $[0, \ell - \delta], \; \delta > 0$.

The following statement is valid, see [22],[24] for $\gamma = 1$ and [11] for an arbitrary $\gamma > 0$.

**Theorem 3.4.** *A function $w \in W$ belongs to $\mathcal{Z}^0$ if and only if $m_w > 0$ and it belongs to $\mathcal{Z}_\gamma,\ \gamma > 0$, if and only if $M_w < \gamma$, so that*

$$w \in \Phi_\gamma^0 \Longleftrightarrow 0 < m_w \leq M_w < \gamma. \tag{3.3}$$

*Besides this, for $w \in \Phi_\gamma^0$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that*

$$c_1 t^{M_w+\varepsilon} \leq w(t) \leq c_2 t^{m_w-\varepsilon}, \qquad 0 \leq t \leq \ell. \tag{3.4}$$

*The following properties are also valid*

$$m_w = \sup\{\lambda \in (0,1) :\ t^{-\lambda} w(t) \quad \textit{is a.i.}\}, \tag{3.5}$$

$$M_w = \inf\{\mu \in (0,1) :\ t^{-\mu} w(t) \quad \textit{is a.d.}\}. \tag{3.6}$$

**Corollary 3.5.** *Let $w(t), 0 < t \leq \ell$, be such a function that $t^a w(t) \in \mathcal{Z}^0$ for some $a \in \mathbb{R}^1$. Then for any $\varepsilon > 0$ there exist $c_1 > 0$ such that*

$$w(t) \leq c_1 t^{m_w - \varepsilon}. \tag{3.7}$$

*Similarly, if $t^a w(t) \in \mathcal{Z}_\gamma$, then for any $\varepsilon > 0$ there exist $c_2 > 0$ such that*

$$w(t) \geq c_2 t^{M_w + \varepsilon}. \tag{3.8}$$

*(The indices $m_w$ and $M_w$ may be negative in this case).*

**Remark 3.6.** If $w \in \widetilde{W}$ and $m_\omega > 0$, then $w \in W$.

Indeed, let $a \in \mathbb{R}^1$ be such that $w_a(t) = t^a w(t) \in W$. Then according to (3.5) the function $\frac{w_a(t)}{t^{m_{w_a}-\varepsilon}}$ is a.i. for any $\varepsilon > 0$. But $m_{w_a} = m_w + a$, so that $\frac{w(t)}{t^{m_w-\varepsilon}}$ is a.i. In particular, the function $w$ itself is a.i., which means that it is in $W$.

**Remark 3.7.** Functions $w \in \mathcal{Z}_\gamma, \gamma > 0$, satisfy the doubling condition

$$w(2r) \leq Cw(r), 0 \leq r \leq \ell \tag{3.9}$$

which follows from the fact that the function $\frac{w(r)}{r^\mu}$ is a.d. for every $\mu > M_w$ according to (3.6) (observe that $M_w$ is finite since $M_\omega < \gamma$ by Theorem 3.4).

We shall need the following lemma.

**Lemma 3.8.** *Let $w \in \widetilde{W}$ and $M_\omega < \gamma$. Then $\frac{t^\gamma}{[w(t)]^\lambda} \in \mathcal{Z}^0$ if $\lambda M_\omega < \gamma$, that is,*

$$\int_0^r \frac{t^{\gamma-1}dt}{[w(t)]^\lambda} \leq c\frac{r^\gamma}{[w(r)]^\lambda}, \quad 0 < r \leq \ell. \tag{3.10}$$

Proof. For $w_1(x) = \frac{x^\gamma}{[w(x)]^\lambda}$, from the definition of the lower index we easily obtain

$$m_{w_1} = \gamma - \lambda M_w.$$

Hence $m_{w_1} > 0$. It is easily checked that $w_1 \in \widetilde{W}$, that is, there exists a number $b$ such that $t^b w_1(t)$ is a.i. Then $w_1 \in W$ according to Remark 3.6 and consequently $w_1 \in \mathcal{Z}^0$ by Theorem 3.4. $\square$

### 3.3 On radial $A_p$-weights generated by oscillating functions $w$.

Let $\rho(x) = [w(|x-x_0|)]^\lambda$, where $\lambda \in \mathbb{R}^1,\ x \in \mathbb{R}^n$ and $x_0$ is a fixed point in $\mathbb{R}^n$ and $w(r)$ is a function such that $r^a w(r) \in W$ for some $a \in \mathbb{R}^1$. The following statement provides conditions in terms of the lower and upper indices $m_w$ and $M_w$ of the function $w(r)$, under which $\rho(x) = [w(|x-x_0|)]^\lambda$ is a Muckenhoupt weight of the class $A_p$. According to the definition of the weighted space given in (1.3), we use the following definition of the class $A_p = A_p(\mathbb{R}^n),\ p = const, 1 < p < \infty$,

$$A_p = \left\{ \rho:\ \sup_Q \left( \frac{1}{|Q|} \int_Q \rho(x)dx \right) \left( \frac{1}{|Q|} \int_Q [\rho(x)]^{1-q} dx \right)^{p-1} < \infty \right\} \quad (3.11)$$

where $\sup$ is taken with respect to all cubes, $\frac{1}{p} + \frac{1}{q} = 1$, see e.g. [29] on $A_p$-weights.

**Lemma 3.9.** *Let $w \in \widetilde{W}$, $\lambda \in \mathbb{R}^1$ and $\Omega$ a bounded domain in $\mathbb{R}^n$. Then $[w(|x-x_0|)]^\lambda \in A_p(\Omega)$ if*

$$[w(r)]^{\lambda p} r^n,\ [w(r)]^{-\lambda q} r^n \in \mathcal{Z}^0. \quad (3.12)$$

*Condition (3.12) is equivalent to the following inequalities in terms of the lower and upper indices of the function $w(r)$*

$$-\frac{n}{\lambda p} < m_w \le M_w < \frac{n}{\lambda q} \quad \textit{when} \quad \lambda > 0 \quad (3.13)$$

*and*

$$-\frac{n}{|\lambda| q} < m_w \le M_w < \frac{n}{|\lambda| p} \quad \textit{when} \quad \lambda < 0. \quad (3.14)$$

Proof. For radial weights the $A_p$-condition (3.11) takes the form

$$\int_0^r [\rho(t)]^p t^{n-1} dt \left( \int_0^r [\rho(t)]^{-q} t^{n-1} dt \right)^{p-1} \le C r^{np}, \quad 0 < r \le \ell = diam\,\Omega, \quad (3.15)$$

where $C > 0$ does not depend on $r > 0$, see [8], [9]. We rewrite this for $\rho(t) = [w(t)]^\lambda$ as

$$\int_0^r \frac{w_1(t)}{t} dt \left( \int_0^r \frac{w_2(t)}{t} dt \right)^{p-1} \le C r^{np} \quad (3.16)$$

where $w_1(t) = [w(t)]^{\lambda p} t^n,\ w_2(t) = [w(t)]^{-\lambda q} t^n$. The feasibility of condition (3.16) is obviously connected with validity of $\mathbb{Z}^0$- condition of Subsection 3.2

for functions $w_1(t)$ and $w_2(t)$. By Theorem 3.4, the functions $w_1(t)$ and $w_2(t)$ satisfy this condition if and only if their lower indices $m_{w_1}$ and $m_{w_2}$ are strictly positive. To calculate these indices, suppose that $\lambda > 0$. We have

$$m_{w_1} = n + \lambda p m_w, \qquad m_{w_2} = n - \lambda p M_w$$

and the positivity of these numbers leads to condition (3.13). Similarly the case $\lambda < 0$ is considered.

Under condition (3.13) we then have

$$\int_0^r \frac{w_1(t)}{t}\, dt \left( \int_0^r \frac{w_2(t)}{t}\, dt \right)^{p-1} \leq c w_1(r)[w_2(r)]^{p-1} = c r^{np}.$$

Thus, (3.16) and consequently (3.15) are satisfied. □

# 4 Preliminaries related to the variable exponent space

## 4.1 Some basics

We recall some basic facts for the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and refer e.g. to [14] for details. The Hölder inequality holds in the form

$$\int_\Omega |f(x)g(x)|\, dx \leq k \|f\|_{p(\cdot)} \cdot \|g\|_{q(\cdot)} \tag{4.1}$$

with $k = \frac{1}{p_*} + \frac{1}{q_0}$. The modular $I_p(f) = \int_\Omega |f(x)|^{p(x)}\, dx$ and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater than one and simultaneously less than $1$:

$$\|f\|_{p(\cdot)}^{p^*} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{p_*} \text{ if } \|f\|_{p(\cdot)} \leq 1 \tag{4.2}$$

and

$$\|f\|_{p(\cdot)}^{p_*} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{p^*} \text{ if } \|f\|_{p(\cdot)} \geq 1. \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$c_1 \leq \|f\|_p \leq c_2 \quad \Longrightarrow c_3 \leq I_\Omega^p(f) \leq c_4 \tag{4.4}$$

and

$$C_1 \leq I_\Omega^p(f) \leq C_2 \quad \Longrightarrow C_3 \leq \|f\|_p \leq C_4 \tag{4.5}$$

with $c_3 = \min\left(c_1^{p_*}, c_1^{p^*}\right)$, $c_4 = \max\left(c_2^{p_*}, c_2^{p^*}\right)$, $C_3 = \min\left(C_1^{1/p_*}, C_1^{1/p^*}\right)$ and $C_4 = \max\left(C_2^{1/p_*}, C_2^{1/p^*}\right)$.

The imbedding

$$L^{p(x)} \subseteq L^{r(x)}, \ \ 1 \leq r(x) \leq p(x) \leq p^* < \infty$$

is valid if $|\Omega| < \infty$. In that case

$$\|f\|_{r(\cdot)} \leq m\|f\|_{p(\cdot)}, \quad m = a_2 + (1-a_1)|\Omega|, \tag{4.6}$$

where $a_1 = \inf\limits_{x\in\Omega} \frac{r(x)}{p(x)}$ and $a_1 = \sup\limits_{x\in\Omega} \frac{r(x)}{p(x)}$.

**Lemma 4.1.** *Let $\Omega$ be a bounded set in $\mathbb{R}^n$, the exponent $p$ satisfy conditions (1.1), (1.2) and let $w$ be any function such that there exist exponents $a, b \in \mathbb{R}^1$ and the constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 r^a \leq w(r) \leq c_2 r^{-b}$, $0 \leq r \leq \ell =$ diam $(\Omega)$. Then*

$$\frac{1}{C}[w(|x-x_0|)]^{p(x_0)} \leq [w(|x-x_0|)]^{p(x)} \leq C[w(|x-x_0|)]^{p(x_0)}, \tag{4.7}$$

*where $C > 1$ does not depend on $x, x_0 \in \overline{\Omega}$.*

Proof. Let

$$g(x,x_0) = [w(|x-x_0|)]^{p(x)-p(x_0)}.$$

To show that $\frac{1}{C} \leq g(x,x_0) \leq C$, that is, $|ln\, g(x,x_0)| \leq C_1,\ C_1 = ln\, C$, we observe that $|ln\, g(x,x_0)| = |p(x) - p(x_0)| \cdot |ln\, w(|x-x_0|)|$ Therefore,

$$|ln\, g(x,x_0)| = |p(x)-p(x_0)| \cdot |ln\, w(|x-x_0|)| \leq A\ell \frac{|ln\, w(|x-x_0|)|}{ln\, \frac{2\ell}{|x-x_0|}}$$

which is bounded by the condition on $w$. □

## 4.2 Auxiliary lemma for averages

Let

$$\mathcal{M}_r^{w^\lambda} f(x) = \frac{[w(|x-x_0|)]^\lambda}{|B_r(x)|} \int\limits_{B_r(x)} \frac{|f(y)|}{[w(|y-x_0|)]^\lambda}\, dy, \qquad x_0 \in \Omega, \tag{4.8}$$

denote the weighted means related to the weighted maximal operator (2.1). In (4.8) we assume that $f(y) = 0$ for $y \notin \Omega$. We write $\mathcal{M}_r f(x) := \mathcal{M}_r^{w^\lambda} f(x)\Big|_{\lambda=0}$.

In general, it will be admitted that $\lambda$ may depend on the point $x \in \Omega$. Observe that the function $[w(t)]^{\lambda(x)}$ is also of the type of the function $w(t)$, that is, it also oscillates between two power functions.

**Lemma 4.1** . *Let $w(r) \in \mathcal{Z}_n$ and $\lambda(x) \geq 0$. Then the inequality*

$$\mathcal{M}_r^{w^\lambda}(1) = \frac{[w(|x-x_0|)]^{\lambda(x)}}{|B_r(x)|} \int\limits_{B_r(x)} \frac{dy}{[w(|y-x_0|)]^{\lambda(x)}} \leq c \tag{4.9}$$

*holds with $c > 0$ not depending on $r > 0, x_0 \in \mathbb{R}^n$ and on $x$ in any set $D \subseteq \mathbb{R}^n$ on which $\sup\limits_{x\in D} \lambda(x) < \frac{n}{M_w}$.*

Proof. We distinguish the cases $|x-x_0|\geq 2r$ and $|x-x_0|\leq 2r$.

In the case $|x-x_0|\geq 2r$ we have

$$|y-x_0|\geq |x-x_0|-|y-x|\geq |x-x_0|-r\geq \frac{1}{2}|x-x_0|.$$

Since the function $w\in \mathcal{Z}_n\subset W$ is a.i., we have $w(|y-x_0|)\geq cw\left(\frac{1}{2}|x-x_0|\right)$. Taking also into account the doubling property (3.9), we obtain

$$w(|y-x_0|)\geq cw(|x-x_0|)$$

and then estimate (4.9) becomes evident since $\lambda(x)\geq 0$.

Let $|x-x_0|\leq 2r$. Observe that in this case

$$B(x,r)\subset B(x_0,3r)$$

since $|y-x|<r \quad\Longrightarrow\quad |y-x_0|\leq |y-x|+|x-x_0|<3r$. Hence

$$\mathcal{M}_r^{w^\lambda}(1)\leq \frac{[w(|x-x_0|)]^{\lambda(x)}}{|B_r(x)|}\int\limits_{B_{3r}(x_0)}\frac{dy}{[w(|y-x_0|)]^{\lambda(x)}}$$

$$=\frac{[w(|x-x_0|)]^{\lambda(x)}}{|B_r(x)|}\int\limits_{B_{3r}(0)}\frac{dy}{[w(|y|)]^{\lambda(x)}}=c\frac{[w(|x-x_0|)]^{\lambda(x)}}{r^n}\int\limits_0^{3r}\frac{\rho^{n-1}d\rho}{[w(\rho)]^{\lambda(x)}}.$$

Then by Lemma 3.8 we get

$$\mathcal{M}_r^{w}(1)\leq c\left(\frac{w(|x-x_0|)}{w(3r)}\right)^{\lambda(x)}\leq c\left(\frac{w(2r)}{w(3r)}\right)^{\lambda(x)}\leq c.$$

□

# 5 Proof of Theorem A

## 5.1 Reduction to the case of a single weight

**Remark 5.1.** *It suffices to prove Theorem A for a single weight* $w(|x-x_0|),\ x_0\in\overline{\Omega},\ t^{\frac{n}{p(x_0)}}w(t)\in\Phi_n^0$.

Indeed, let $\Omega=\bigcup\limits_{k=1}^{n}\Omega_k$ where $\Omega_k$ contains the point $x_k$ in its interior and does not contain $x_j, j\neq k$ in its closure. Then

$$\|f\|_{L^{p(\cdot)}\left(\Omega,\prod\limits_{k=1}^{n}w_k(|t-t_k|)\right)}\sim\sum_{k=1}\|f\|_{L^{p(\cdot)}(\Omega_k,w_k(|t-t_k|))} \tag{5.1}$$

whenever $1 \leq p_- \leq p_+ < \infty$. This equivalence follows from the easily checked modular equivalence

$$I_\Omega^p\left(f(x)\prod_{k=1}^n w_k(|x-x_k|)\right) \sim \sum_{k=1} I_\Omega^p\left(f(x)w_k(|x-x_k|)\right),$$

if we take (4.4) and (4.5) into account.

Then, because of (5.1), the statement of Remark 5.1 is obtained via introduction of the standard partition of unity $1 = \sum\limits_{k=1}^{n} a_k(x)$, where $a_k(x)$ are smooth functions equal to $1$ in a neighborhood $B(x_k, \varepsilon)$ of the point $x_k$ and equal to $0$ outside its neighborhood $B(x_k, 2\varepsilon)$, so that $a_k(x)\left[w_k(|x-x_j|)\right]^{\pm 1} \equiv 0$ in a neighborhood of the point $x_k$, if $k \neq j$.

In what follows, $\Omega$ is an open bounded set in $R^n$ and $x_0 \in \overline{\Omega}$.

## 5.2 A pointwise estimate for the weighted means

**Theorem 5.2.** *Let $p(x)$ satisfy conditions (1.1) and (1.2) and let $w \in W$. If*

$$0 \leq m_w \leq M_w < \frac{n}{q(x_0)}, \tag{5.2}$$

*where $\frac{1}{q(x)} = 1 - \frac{1}{p(x)}$, then*

$$\left[\frac{w(|x-x_0|)}{|B_r(x)|}\int\limits_{B_r(x)}\frac{|f(y)|dy}{w(|y-x_0|)}\right]^{p(x)} \leq c\left(1+\frac{1}{|B_r(x)|}\int\limits_{B_r(x)}|f(y)|^{p(y)}\,dy\right) \tag{5.3}$$

*for all $f \in L^{(p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \leq 1$, where $c = c(p, w)$ is a constant not depending on $x$, $r$ and $x_0$.*

Proof. From (5.2) and the continuity of $p(x)$ we conclude that there exists a $d > 0$ such that

$$M_w q(x) < n \ \text{ for all } \ |x-x_0| \leq d. \tag{5.4}$$

Without loss of generality we assume that $d \leq 1$. Let

$$p_r(x) = \min_{|y-x|\leq r} p(y)$$

and $\frac{1}{q_r(x)} = 1 - \frac{1}{p_r(x)}$. From (5.2) it is easily seen that

$$M_w q_r(x) < n \ \text{ if } \ |x-x_0| \leq \frac{d}{2} \ \text{ and } \ 0 < r \leq \frac{d}{4}. \ \square \tag{5.5}$$

□

## $1^0$ The case $|\mathbf{x}-\mathbf{x_0}| \le \frac{\mathbf{d}}{\mathbf{2}}$ and $\mathbf{0} < \mathbf{r} \le \frac{\mathbf{d}}{\mathbf{4}}$ (the main case)

In this case, applying the Hölder inequality with the exponents $p_r(x)$ and $q_r(x)$ to the integral on the right-hand side of the equality

$$\left|\mathcal{M}_r\Big(\frac{f(y)}{w(|y-x_0|)}\Big)\right|^{p(x)} = \frac{c}{r^{np(x)}}\left(\int\limits_{B_r(x)} \frac{|f(y)|}{w(|y-x_0|)}\,dy\right)^{p(x)},$$

we get

$$\left|\mathcal{M}_r\left(\frac{f(y)}{w(|y-x_0|)}\right)\right|^{p(x)} \le$$

$$\le \frac{c}{r^{np(x)}}\left(\int\limits_{B_r(x)} |f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}} \cdot \left(\int\limits_{B_r(x)} \frac{dy}{[w(|y-x_0|)]^{q_r(x)}}\right)^{\frac{p(x)}{q_r(x)}} \tag{5.6}$$

where the last integral converges, since for small $|y-x_0|$ one has

$$[w(|y-x_0|)]^{q_r(x)} \ge c|y-x_0|^{(M_w+\varepsilon)q_r(x)}$$

where one may choose $\varepsilon$ sufficiently small so that according to (5.4), $|y-x_0|^{(M_w+\varepsilon)q_r(x)} \ge |y-x_0|^{n-\delta}$ for some $\delta > 0$.

We may make use of estimate (4.9) in (5.6), since $w \in W$ and $w \in \mathcal{Z}_n$ under the condition $M_w < \frac{n}{q(x_0)} < n$ according to Theorem 3.4. We obtain

$$\left|\mathcal{M}_r\left(\frac{f(y)}{w(|y-x_0|)}\right)\right|^{p(x)} \le c\,\frac{[w(|x-x_0|)]^{-p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left(\int\limits_{B_r(x)} |f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}}.$$

Here

$$\int\limits_{B_r(x)} |f(y)|^{p_r(x)}\,dy \le \int_{B_r(x)} dy + \int\limits_{\substack{B_r(x)\\ \{y:\ |f(y)|\ge 1\}}} |f(y)|^{p(y)}\,dy,$$

since $p_r(x) \le p(y)$ for $y \in B_r(x)$. Since $p(x)$ is bounded, we see that

$$\left|\mathcal{M}_r\left(\frac{f(y))}{w(|y-x_0|)}\right)\right|^{p(x)} \le c_1\,\frac{[w(|x-x_0|)]^{-p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\left[r^n + \frac{1}{2}\int\limits_{B_r(x)} |f(y)|^{p(y)}\,dy\right]^{\frac{p(x)}{p_r(x)}}.$$

Since $r \le \frac{d}{2} \le \frac{1}{2}$ and the second term in the brackets is also less than or equal to $\frac{1}{2}$, we arrive at the estimate

$$|\mathcal{M}_r^w f|^{p(x)} \le \frac{c}{r^{\frac{np(x)}{p_r(x)}}} \left[ r^n + \int\limits_{B_r(x)} |f(y)|^{p(y)}\, dy \right] \le$$

$$\le c\, r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} \left[ 1 + \frac{1}{r^n} \int\limits_{B_r(x)} |f(y)|^{p(y)}\, dy \right].$$

From here (4.2) follows, since

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} \le c.$$

Indeed,

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} = e^{\frac{n}{p_r}\,[p(x)-p_r(x)]\,\ln\frac{1}{r}},$$

where

$$\left|\frac{n}{p_r}\left[p(x)-p_r(x)\right]\ln\frac{1}{r}\right| \le n\big|p(x)-p(\xi_r)\big|\ln\frac{1}{r}$$

with $\xi_r \in B_r(x)$, and then by (2.2),

$$\left|\frac{n}{p_r}\left[p(x)-p_r(x)\right]\ln\frac{1}{r}\right| \le nA\frac{\ln\frac{1}{r}}{\ln\frac{1}{|x-\xi_r|}} \le nA,$$

since $|x-\xi_r| \le r$.

## $2^0$ The case $|\mathbf{x}-\mathbf{x_0}| \ge \frac{\mathbf{d}}{\mathbf{2}}$, $\mathbf{0} < \mathbf{r} \le \frac{\mathbf{d}}{\mathbf{4}}$.

This case is trivial, because

$$|y-x_0| \ge |x-x_0| - |y-x| \ge \frac{d}{2} - \frac{d}{4} = \frac{d}{4}.$$

Taking into account that $\frac{w(t)}{t^2}$ is almost decreasing, we then have $w(|y-x_0|) \ge Cw\big(\frac{d}{4}\big) = const$. Since $w(|x-x_0|) \le Cw(\,\mathrm{diam}\,\Omega)$, it follows that

$$\mathcal{M}_r^w f(x) \le c\mathcal{M}_r f(x),$$

and one may proceed as above for the case $\beta = 0$ (the condition $|x-x_0| \le \frac{d}{2}$ is not needed in this case).

## $3^0$ The case $\mathbf{r \geq \frac{d}{4}}$.

This case is also easy. It suffices to show that $\mathcal{M}_r^w f(x)$ is bounded. We have

$$\mathcal{M}_r^w f(x) \leq \frac{cw(\operatorname{diam}\Omega)}{\left(\frac{d}{4}\right)^n} \left[ \int\limits_{|y-x_0|\leq \frac{d}{8}} \frac{|f(y)|}{w(|y-x_0|)}\, dy + \int\limits_{|y-x_0|\geq \frac{d}{8}} \frac{|f(y)|}{w(|y-x_0|)}\, dy \right].$$

Here the first integral is estimated via the Hölder inequality with the exponents

$$p_{\frac{d}{8}} = \min_{|y-x_0|\leq \frac{d}{8}} p(y) \text{ and } q_{\frac{d}{8}} = p'_{\frac{d}{8}}$$

which is possible since $\alpha q_{\frac{d}{8}} < n$. The estimate of the second integral is trivial since $|y - x_0| \geq \frac{d}{8}$.

**Corollary 5.3.** *Let* $0 \leq m_w \leq M_w < \frac{n}{q(x_0)}$ . *If conditions (1.1), (1.2) are satisfied, then*

$$|\mathcal{M}^w f(x)|^{p(x)} \leq c \left(1 + \mathcal{M}\left[|f(\cdot)|^{p(\cdot)}\right](x)\right) \tag{5.7}$$

*for all* $f \in L^{p(\cdot)}(\Omega)$ *such that* $\|f\|_{p(\cdot)} \leq 1$ .

**Remark 5.4.** In the non-weighted case $\omega(x) \equiv 1$ the estimate (5.7) is known to be valid if $1 \leq p(x) \leq p^* < \infty$ instead of condition (1.1), see [5].

## 5.3 Proof of Theorem A itself

To prove Theorem A, we have to show that

$$\left\|\mathcal{M}^w f\right\|_{p(\cdot)} \leq c \tag{5.8}$$

in some ball $\|f\|_{p(\cdot)} \leq R$, which is equivalent to the inequality

$$I_p(\mathcal{M}^w f) \leq c \text{ for } \left\|f\right\|_{p(\cdot)} \leq R.$$

According to (4.7) we obtain

$$I_p(\mathcal{M}^w f) \leq c \int\limits_{\Omega} w(|x-x_0|)^{p(x_0)} \left| \mathcal{M}\left(\frac{f(y)}{w(|y-x_0|)}\right)(x)\right|^{p(x)} dx.$$

*(5.8) first for* We prove (5.8) first for

$$-\frac{n}{p(x_0)} < m_w \leq M_w < \frac{n}{q_0}, \tag{5.9}$$

where $\frac{1}{q_0} = \frac{p_*-1}{p(x_0)}$. Observe that $\frac{1}{q_0} \leq \frac{1}{q(x_0)}$ so that the interval (5.9) for $m_w, M_w$ is somewhat narrower than the whole interval $\left(-\frac{n}{p(x_0)}, \frac{n}{q_0}\right)$. After that we treat the remaining case.

**$1^0$ The case** $-\frac{n}{p(x_0)} < m_w \leq M_w < \frac{n}{q_0}$.

Following the idea in [5], we represent this as

$$I_p(\mathcal{M}^w f) \leq c \int\limits_{\Omega} \left( [w(|x-x_0|)]^{p_1(x_0)} \left| \mathcal{M}\left(\frac{f(y)}{w(|y-x_0|)}\right)(x)\right|^{p_1(x)} \right)^{p_*} dx, \tag{5.10}$$

where

$$p_1(x) = \frac{p(x)}{p_*}\,.$$

Estimate (5.7) with $w \equiv 1$ says that

$$|\mathcal{M}\psi(x)|^{p_1(x)} \leq c\Big(1 + \mathcal{M}\big[\psi^{p_1(\cdot)}\big](x)\Big) \tag{5.11}$$

(see Remark 5.4) for all $\psi \in L^{p_1(\cdot)}(\Omega)$ with $\|\psi\|_{p_1(\cdot)} \leq 1$ (or equivalently, for all $\psi$ with $\|\psi\|_{p_1} \leq C$ with fixed $C < \infty$).

We intend to choose $\psi(x) = \frac{f(x)}{w(|x-x_0|)}$ with $f \in L^{p(\cdot)}$ in (5.11). Let us show that in this case

$$\|\psi\|_{p_1} = \left\|\frac{f(x)}{w(|x-x_0|)}\right\|_{p_1} \leq C \tag{5.12}$$

for all $f \in L^{p(\cdot)}$ with $\|f\|_p \leq c$. Since $r^{\frac{n}{p(x_0)}} w(r) \in \Phi^0_n$, by Corollary 3.5 we have $w(|x-x_0|) \geq c|x-x_0|^{M_w+\varepsilon},\ \varepsilon > 0$ and then

$$\int\limits_{\Omega} |\psi(x)|^{p_1(x)}\, dx \leq c \int\limits_{\Omega} \frac{|f(x)|^{\frac{p(x)}{p_*}}}{|x-x_0|^{(M_w+\varepsilon)p_1(x_0)}}\, dx. \tag{5.13}$$

To obtain (5.12), it remains to apply the Hölder inequality in (5.13) with the exponents $p_*$ and $q_* = \frac{p_*}{p_*-1}$ and take into account that

$$I_{q_*}\left(\frac{1}{|x-x_0|^{(M_w+\varepsilon)p_1(x_0)}}\right) = \int\limits_{\Omega} \frac{dx}{|x-x_0|^{(M_w+\varepsilon)q_0}}$$

where $(M_w+\varepsilon)q_0 < n$ under the choice of small $\varepsilon < \frac{n}{q_0} - M_w$.

In view of (5.12), we may apply estimate (5.11). Then (5.10) implies

$$I_p(\mathcal{M}^w f) \leq c \int\limits_{\Omega} \left( [w(|x-x_0|)]^{p_1(x_0)} \left[1 + \mathcal{M}\left(\left|\frac{f(y)}{w(|y-x_0|)}\right|^{p_1(y)}\right)\right]\right)^{p_*} dx.$$

By property (4.7), this yields

$$I_p(\mathcal{M}^w f) \le$$

$$\le c \int\limits_\Omega \left\{ [w(|x-x_0|)]^{p(x_0)} + \left[ [w(|x-x_0|)]^{p_1(x_0)} \mathcal{M} \left( \frac{|f(y)|^{p_1(y)}}{[w(|y-x_0|)]^{p_1(x_0)}} \right) \right]^{p_*} \right\} dx.$$

Here the integral of the first term is finite since $w(|x-x_0|) \le |x-x_0|^{m_\omega - \varepsilon}$ according to (3.4) and $m_w p(x_0) > -n$. Hence

$$I_p(\mathcal{M}^w f) \le c + c \int_\Omega \left[ \mathcal{M}^{w^{p_1(x_0)}} (|f(\cdot)|^{p_1(\cdot)})(x) \right]^{p_*} dx \tag{5.14}$$

in notation (2.1).

As is known [29], p. 201, the weighted maximal operator $\mathcal{M}^{w^{p_1}}$ is bounded in $L^{p_*}$ with a constant $p_* > 1$ if the weight $[w(|x-x_0|)]^{p_1(x_0)}$ is in $A_{p_*}$. According to condition (3.13) of Lemma 3.9 this is the case if $-\frac{n}{p(x_0)} < m_w \le M_w < \frac{n}{q_0}$ which is satisfied in the case under consideration.

Therefore, by the boundedness of the weighted operator $\mathcal{M}^{w^{p_1(x_0)}}$ in $L_{p_*}$, from (5.14) we get

$$I_p(\mathcal{M}^w f) \le c + c \int\limits_\Omega |f(y)|^{p_1(y) \cdot p_*} \, dy = c + c \int\limits_\Omega |f(y)|^{p(y)} \, dy < \infty. \tag{5.15}$$

**$2^0$ The remaining case**

To get rid of the right-hand side bound in (5.9), we may split integration over $\Omega$ into two parts, one over a small neighborhood $B_\delta = B_\delta(x_0)$ of the point $x_0$, and another over its exterior $\Omega \backslash B_\delta$, and to choose $\delta$ sufficiently small so that the number $\frac{p_*(B_\delta)-1}{p(x_0)}$ is arbitrarily close to $\frac{p(x_0)-1}{p(x_0)} = \frac{1}{q(x_0)}$. To this end we put

$$\mathcal{M}^w = \chi_{B_\delta} \mathcal{M}^w \chi_{B_\delta} + \chi_{B_\delta} \mathcal{M}^w \chi_{\Omega \backslash B_\delta} + \chi_{\Omega \backslash B_\delta} \mathcal{M}^w \chi_{B_\delta} + \chi_{\Omega \backslash B_\delta} \mathcal{M}^w \chi_{\Omega \backslash B_\delta} \tag{5.16}$$

$$=: \mathcal{M}_1^w + \mathcal{M}_2^w + \mathcal{M}_3^w + \mathcal{M}_4^w.$$

Since the weight is strictly positive beyond any neighborhood of the point $x_0$, we have

$$\mathcal{M}_4^w f(x) \le C \mathcal{M} f(x). \tag{5.17}$$

For $\mathcal{M}_3^\omega$ we have

$$\mathcal{M}_3^w f(x) = \sup_{r>0} \frac{\chi_{\Omega \backslash B_\delta(x_0)}(x)}{|B_r(x)|} \int\limits_{B_r(x) \cap B_\delta(x_0) \cap \Omega} \frac{w(|x-x_0|)}{w(|y-x_0|)} |f(y)| \, dy.$$

Here $|x-x_0| > r > |y-x_0|$. Observe that the function $w_\varepsilon(t) = \frac{w(t)}{t^{M_w+\varepsilon}}$ is a.d. for any $\varepsilon > 0$ according to (3.6). Therefore

$$\frac{w(|x-x_0|)}{w(|y-x_0|)} = \frac{w_\varepsilon(|x-x_0|)}{w_\varepsilon(|y-x_0|)} \cdot \frac{|x-x_0|^{M_w+\varepsilon}}{|y-x_0|M_w+\varepsilon)} \le C \frac{|x-x_0|^{M_w+\varepsilon}}{|y-x_0|M_w+\varepsilon}.$$

Hence

$$\mathcal{M}_3^w f(x) \le C\mathcal{M}^{M_w+\varepsilon} f(x) \tag{5.18}$$

where $\mathcal{M}^{M_w+\varepsilon} f(x)$ is the weighted maximal function with the power weight $|x-x_0|^{M_w+\varepsilon}$. Similarly we conclude that

$$\mathcal{M}_2^w f(x) \le C\mathcal{M}^{m_w-\varepsilon} f(x). \tag{5.19}$$

Thus from (5.16) according to (5.17), (5.18) and (5.19) we have

$$\mathcal{M}^w f(x) \le \chi_{B_\delta}\mathcal{M}^w \chi_{B_\delta} f(x) + \mathcal{M}f(x) + \mathcal{M}^{M_w+\varepsilon} f(x) + \mathcal{M}^{m_w-\varepsilon} f(x). \tag{5.20}$$

Here the operators $\mathcal{M}, \mathcal{M}^{M_w+\varepsilon}$ and $\mathcal{M}^{m_w-\varepsilon}$ are bounded in the space $L^{p(\cdot)}(\Omega)$, because the boundedness condition (2.2 is satisfied for $\beta = M_w + \varepsilon$ and $\beta = m_\omega - \varepsilon$ under a choice of $\varepsilon$ sufficiently small.

It remains to prove the boundedness of the first term on the right-hand side of (5.20). This is nothing else but the boundedness of the same operator $\mathcal{M}^w$ over a small set $\Omega_\delta = B_\delta(x_0) \cap \Omega$. According to the previous case, this boundedness holds if

$$-\frac{n}{p(x_0)} < m_w \le M_w < \frac{n}{q_\delta} \tag{5.21}$$

where $q_\delta = \frac{p_*(\Omega_\delta)-1}{p(x_0)}$ and $p_*(\Omega_\delta) = \min_{x\in\Omega_\delta} p(x)$. Let us show that, given the condition $-\frac{n}{p(x_0)} < m_w \le M_w < \frac{n}{q(x_0)}$, one can always choose $\delta$ sufficiently small such that (5.21) holds. Given $M_w < \frac{n}{q(x_0)}$, we have to choose $\delta$ so that $M_w < \frac{n}{q_\delta} \le \frac{n}{q(x_0)}$. We have

$$\frac{n}{q_\delta} = \frac{n}{q(x_0)} - a(\delta), \qquad \text{where} \qquad a(\delta) = \frac{n}{p(x_0)}\left[p(x_0) - p_*(\Omega_\delta)\right].$$

By the continuity of $p(x)$ we can choose $\delta$ so that $a(\delta) < \frac{n}{q(x_0)} - M_w$. Then $\frac{n}{q_\delta} > M_w$ and condition (5.21) is fulfilled. Then the operator $\mathcal{M}^w$ is bounded in the space $L^{p(\cdot)}(B_\delta)$ which completes the proof.

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