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Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, II

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Abstract

In [8], Sobolev-type $p(\cdot) \to q(\cdot)$-theorems were proved for the Riesz potential operator I^α in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ with the variable exponent $p(x)$ and a two-parameter power weight fixed to an arbitrary finite point x_0 and to infinity, under an additional condition relating the weight exponents at x_0 and at infinity. We show in this note that those theorems are valid without this additional condition. Similar theorems for a spherical analogue of the Riesz potential operator in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \rho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ are also improved in the same way.

Key Words and Phrases: weighted Lebesgue spaces, variable exponent, Riesz potentials, spherical potentials, stereographical projection

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1. Introduction

We consider the Riesz potential operator

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad 0 < \alpha < n, \tag{1.1}$$

in the weighted Lebesgue generalized spaces $L^{p(\cdot)}(\mathbb{R}^n,\rho)$ with a variable exponent $p(x)$ defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)} = \inf\left\{\lambda>0 : \int_{\mathbb{R}^n}\rho(x)\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx \le 1\right\}, \tag{1.2}$$

where

$$\rho(x)=\rho_{\gamma_0,\gamma_\infty}(x)=|x|^{\gamma_0}(1+|x|)^{\gamma_\infty-\gamma_0}. \tag{1.3}$$

We refer to [5], [6], [4], [3] for the basics of the spaces $L^{p(\cdot)}$ with variable exponent.

We assume that the exponent $p(x)$ satisfies the standard conditions

$$1<p_-\le p(x)\le p_+<\infty, \quad x\in\mathbb{R}^n, \tag{1.4}$$

$$|p(x)-p(y)|\le\frac{A}{\ln\frac{1}{|x-y|}}, \ \ |x-y|\le\frac{1}{2}, \ \ x,\, y\in\mathbb{R}^n, \tag{1.5}$$

and also the following condition at infinity

$$|p_*(x)-p_*(y)|\le\frac{A_\infty}{\ln\frac{1}{|x-y|}}, \quad |x-y|\le\frac{1}{2}, \ \ x,\, y\in\mathbb{R}^n, \tag{1.6}$$

where $p_*(x) = p\left(\frac{x}{|x|^2}\right)$. Conditions (1.5) and (1.6) taken together are equivalent to the following global condition

$$|p(x)-p_(y)|\le\frac{C}{\ln\left(\frac{2\sqrt{1+|x|^2}\sqrt{1+|y|^2}}{|x-y|}\right)}, \quad x,\, y\in\mathbb{R}^n. \tag{1.7}$$

Let

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n}.$$

The following statement was proved in [8].

Theorem 1.1. *Under assumptions (1.4), (1.5), (1.6) and the condition*

$$p_+<\frac{n}{\alpha} \tag{1.8}$$

the operator I^α is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n,\rho_{\gamma_0,\gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n,\rho_{\mu_0,\mu_\infty})$, where

$$\mu_0=\frac{q(0)}{p(0)}\gamma_0 \quad \textit{and} \quad \mu_\infty=\frac{q(\infty)}{p(\infty)}\gamma_\infty, \tag{1.9}$$

if

$$\alpha p(0)-n<\gamma_0<n[p(0)-1], \qquad \alpha p(\infty)-n<\gamma_\infty<n[p(\infty)-1], \tag{1.10}$$

and the exponents γ_0 and γ_∞ are related to each other by the equality

$$\frac{q(0)}{p(0)}\gamma_0+\frac{q(\infty)}{p(\infty)}\gamma_\infty=\frac{q(\infty)}{p(\infty)}[(n+\alpha)p(\infty)-2n]. \tag{1.11}$$

The goal of this note is to prove that Theorem 1.1 is valid without the additional condition (1.11). We consider also a similar statement for the spherical potential operators

$$(K^\alpha f)(x) = \int\limits_{\mathbb{S}_n} \frac{f(\sigma)}{|x-\sigma|^{n-\alpha}}\, d\sigma, \quad x \in \mathbb{S}_n, \quad 0 < \alpha < n, \tag{1.12}$$

in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \rho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

2. Preliminaries

We need the following theorem for bounded domains proved in [7].

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$ and let $p(x)$ satisfy conditions (1.4), (1.5) and (1.8) in Ω. Then the following estimate*

$$\|I^\alpha f\|_{L^{q(\cdot)}(\Omega, |x-x_0|^\mu)} \le C\, \|f\|_{L^{p(\cdot)}(\Omega, |x-x_0|^\gamma)} \tag{2.1}$$

is valid, if

$$\alpha p(x_0) - n < \gamma < n[p(x_0) - 1] \tag{2.2}$$

and

$$\mu \ge \frac{q(x_0)}{p(x_0)}\, \gamma. \tag{2.3}$$

3. The case of the spatial potential operator

We prove the following theorem

Theorem A. *Under assumptions (1.4), (1.5), (1.6) and (1.8), the operator I^α is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \rho_{\gamma_0,\gamma_\infty})$ into the space $L^{q(\cdot)}(\mathbb{R}^n, \rho_{\mu_0,\mu_\infty})$, where*

$$\mu_0 = \frac{q(0)}{p(0)}\gamma_0 \quad \textit{and} \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty, \tag{3.1}$$

if

$$\alpha p(0) - n < \gamma_0 < n[p(0) - 1], \qquad \alpha p(\infty) - n < \gamma_\infty < n[p(\infty) - 1]. \tag{3.2}$$

Proof. Let $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n, \rho)} \le 1$. To estimate the integral $\int\limits_{\mathbb{R}^n} \rho_{\mu_0,\mu_\infty}(x) |I^\alpha f(x)|^{q(x)} dx$, we split it, as in [8], in the following way

$$\int\limits_{\mathbb{R}^n} \rho_{\mu_0,\mu_\infty}(x) |I^\alpha f(x)|^{q(x)} dx \le c\left(A_{++} + A_{+-} + A_{-+} + A_{--}\right),$$

where

$$A_{++} = \int\limits_{|x|<1} |x|^{\mu_0} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx, \quad A_{+-} = \int\limits_{|x|<1} |x|^{\mu_0} \left| \int\limits_{|y|>1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

and

$$A_{-+} = \int\limits_{|x|>1} |x|^{\mu_\infty} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx, \quad A_{--} = \int\limits_{|x|>1} |x|^{\mu_\infty} \left| \int\limits_{|y|>1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The boundedness of the terms A_{++} and A_{--} was shown in [8] without condition (1.11). So we only have to treat the terms A_{+-} and A_{-+}.

1^0. The term A_{-+}. We split A_{-+} as

$$A_{-+} = A_1 + A_2,$$

where

$$A_1 = \int\limits_{1<|x|<2} |x|^{\mu_\infty} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

and

$$A_2 = \int\limits_{|x|>2} |x|^{\mu_\infty} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The term

$$A_1 \le C \int\limits_{1<|x|<2} |x|^{\mu_0} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx \le C \int\limits_{|x|<2} |x|^{\mu_0} \left| \int\limits_{|y|<2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

is covered by Theorem 2.1. For the term A_2 we have

$$|x-y| \ge |x| - |y| \ge \frac{|x|}{2}.$$

Therefore,

$$A_2 \le C \int\limits_{|x|>2} |x|^{\mu_\infty + (\alpha-n)q(x)} \left(\int\limits_{|y|<1} |f(y)|\,dy \right)^{q(x)} dx.$$

It follows from condition (1.6) (see also (1.7)) that

$$|p(x) - p(\infty)| \le \frac{C}{\ln|x|}, \qquad |x| \ge 2$$

and then the same is valid for $q(x)$, so that

$$A_2 \le C \int\limits_{|x|>2} |x|^{\mu_\infty+(\alpha-n)q(\infty)} \left(\int\limits_{|y|<1} |f(y)|\,dy \right)^{q(x)} dx.$$

Observe that

$$\int\limits_{|y|<1} |f(y)|\,dy \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)}. \tag{3.3}$$

Indeed, denote $g(y) = [\rho(y)]^{-\frac{1}{p(y)}}$; by the Hölder inequality for variable $L^{p(\cdot)}$-spaces we get

$$\int\limits_{|y|<1} |f(y)|\,dy = \int\limits_{|y|<1} g(y)[\rho(y)]^{\frac{1}{p(y)}}|f(y)|\,dy$$

$$\le k\|g\|_{L^{p'(\cdot)}}\|\rho^{\frac{1}{p}}f\|_{L^{p(\cdot)}} = k\|g\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\rho)}. \tag{3.4}$$

To arrive at (3.3), we have to show that $\|g\|_{L^{p'(\cdot)}} < \infty$. Under condition (1.4) one has

$$\|g\|_{L^{p'(\cdot)}} < \infty \quad \Longleftrightarrow \int\limits_{|y|<1} |g(y)|^{p'(y)}dy < \infty. \tag{3.5}$$

As is easily seen, the last integral is finite since $\gamma_0 < n[p(0)-1]$. Therefore, from (3.4) there follows (3.3).

Then $A_2 \le C < \infty$ if we take into account that $\mu_\infty + (\alpha-n)q(\infty) < -n$ under the condition $\gamma_\infty < n[p(\infty)-1]$.

2^0. The term A_{+-} is estimated similarly to A_{-+}: we split A_{+-} as

$$A_{+-} = A_3 + A_4,$$

where

$$A_3 = \int\limits_{|x|<1} |x|^{\mu_0} \left| \int\limits_{1<|y|<2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

and

$$A_4 = \int\limits_{|x|<1} |x|^{\mu_0} \left| \int\limits_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The term A_3 is covered by Theorem 2.1 similarly to the term A_1 in **1**^{0}. For the term A_4, we have $|x-y| \ge |y| - |x| \ge \frac{|y|}{2}$. Then

$$\left| \int\limits_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right| \le C \int\limits_{|y|>2} \frac{|f(y)|\,dy}{|y|^{n-\alpha}} = C \int\limits_{|y|>2} \frac{|f_0(y)|\,dy}{|y|^{n-\alpha+\frac{\gamma_\infty}{p(\infty)}}}$$

where $f_0(y) = |y|^{\frac{\gamma_\infty}{p(\infty)}} f(y)$. It is easily seen that $f_0(y) \in L^{p(\cdot)}(\mathbb{R}^n \backslash B(0,2))$, since $[\rho(y)]^{\frac{1}{p(y)}} f(y) \in L^{p(\cdot)}(\mathbb{R}^n)$ and $[\rho(y)]^{\frac{1}{p(y)}} \sim |y|^{\frac{\gamma_\infty}{p(\infty)}}$ for $|y| \geq 2$ under the log-condition at infinity. Hence by the Hölder inequality and the same log-condition at infinity,

$$\left| \int\limits_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right| \leq C_1 \|f_0\|_{L^{p(\cdot)}(\mathbb{R}^n \backslash B(0,2))} \left\| |y|^{\alpha - n - \frac{\gamma_\infty}{p(\infty)}} \right\|_{L^{p'(\cdot)}(\mathbb{R}^n \backslash B(0,2))}$$

$$\leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n, \rho_{\gamma_0, \gamma_\infty})} \left\| |y|^{\alpha - n - \frac{\gamma_\infty}{p(\infty)}} \right\|_{L^{p'(\cdot)}(\mathbb{R}^n \backslash B(0,2))} \leq C \left\| |y|^{\alpha - n - \frac{\gamma_\infty}{p(\infty)}} \right\|_{L^{p'(\cdot)}(\mathbb{R}^n \backslash B(0,2))},$$

where the last norm is finite under the condition $\alpha p(\infty) - n < \gamma_\infty$ (use the argument given in (3.5)). □

Corollary 3.1. *Let $0 < \alpha < n$, $p(x)$ satisfy conditions (1.4), (1.5), (1.6) and (1.8). Then the operator I^α is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$.*

The statement of the corollary was proved in [1] and [2] under a weaker than (1.6) version of the log-condition at infinity.

4. The case of the spherical potential operator

4.1 The space $L^{p(\cdot)}(\mathbb{S}^n, \rho)$

We consider the weighted space $L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a,\beta_b})$ with a variable exponent on the unit sphere $\mathbb{S}^n = \{\sigma \in \mathbb{R}^{n+1} : \ |\sigma| = 1\}$, defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a,\beta_b})} = \left\{ \lambda > 0 : \int\limits_{\mathbb{S}^n} |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b} \left| \frac{f(\sigma)}{\lambda} \right|^{p(\sigma)} d\sigma \leq 1 \right\},$$

where $\rho_{\beta_a,\beta_b}(\sigma) = |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b}$ and $a \in \mathbb{S}^n$ and $b \in \mathbb{S}^n$ are arbitrary points on $\mathbb{S}^n$, $\ a \neq b$.

We assume that $0 < \alpha < n$ and

$$1 < p_- \leq p(\sigma) \leq p_+ < \frac{n}{\alpha}, \ \ \sigma \in \mathbb{S}^n, \tag{4.1}$$

$$|p(\sigma_1) - p(\sigma_2)| \leq \frac{A}{\ln \frac{3}{|\sigma_1 - \sigma_2|}}, \quad \sigma_1 \in \mathbb{S}^n, \sigma_2 \in \mathbb{S}^n. \tag{4.2}$$

The following theorem is valid.

Theorem B. *Let the function $p : \mathbb{S}^n \longrightarrow [1, \infty)$ satisfy conditions (4.1) and (4.2). The spherical potential operator K^α is bounded from the space $L^{p(\cdot)}(\mathbb{S}^n, \rho_{\beta_a,\beta_b})$ with $\rho_{\beta_a,\beta_b}(\sigma) = |\sigma - a|^{\beta_a} \cdot |\sigma - b|^{\beta_b}$, where $a \in \mathbb{S}^n$ and $b \in \mathbb{S}^n$ are arbitrary points on the unit sphere $\mathbb{S}^n$, $a \neq b$, into the space $L^{q(\cdot)}(\mathbb{S}^n, \rho_{\beta_a,\beta_b})$ with $\rho_{\nu_a,\nu_b}(\sigma) = |\sigma - a|^{\nu_a} \cdot |\sigma - b|^{\nu_b}$, where $\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}$, and*

$$\alpha p(a) - n < \beta_a < np(a) - n, \quad \alpha p(b) - n < \beta_b < np(b) - n, \tag{4.3}$$

$$\nu_a = \frac{q(a)}{p(a)}\beta_a, \quad \nu_b = \frac{q(b)}{p(b)}\beta_b. \tag{4.4}$$

This theorem was proved in [8] under the additional assumption that the weight exponents β_a and β_b are related to each other by the connection

$$\frac{q(a)}{p(a)}\beta_a = \frac{q(b)}{p(b)}\beta_b. \tag{4.5}$$

Now Theorem B without this condition follows from Theorem A by means of the stereographic projection exactly in the same way as in [8], Section 5.

Corollary 4.1. *Under assumptions (4.1) and (4.2), the spherical potential operator K^α is bounded from $L^{p(\cdot)}(\mathbb{S}^n)$ into $L^{q(\cdot)}(\mathbb{S}^n)$, $\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}$.*

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