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Boundedness in Lebesgue Spaces with Variable Exponent of the Cauchy Singular Operator on Carleson Curves

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Abstract. We prove the boundedness of the singular integral operator S_Γ in the spaces $L^{p(\cdot)}(\Gamma,\rho)$ with variable exponent $p(t)$ and power weight ρ on an arbitrary Carleson curve under the assumptions that $p(t)$ satisfy the log-condition on Γ. The curve Γ may be finite or infinite.

We also prove that if the singular operator is bounded in the space $L^{p(\cdot)}(\Gamma)$, then Γ is necessarily a Carleson curve.

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1. Introduction

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell \le \infty\}$ be a simple rectifiable curve with arc-length measure $\nu(t) = s$. In the sequel we denote

$$\gamma(t,r) := \Gamma \cap B(t,r), \qquad t \in \Gamma, \quad r > 0, \tag{1.1}$$

where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$. We also denote for brevity

$$\nu(\gamma(t,r)) = |\gamma(t,r)|.$$

Everywhere below we assume that Γ is a Carleson curve. We remind that a curve is called Carleson curve (regular curve), if there exists a constant $c_0 > 0$ not depending on t and r, such that

$$|\gamma(t,r)| \le c_0 r \tag{1.2}$$

We consider the singular integral operator

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{1.3}$$

on Carleson curves Γ and prove that the operator S is bounded in weighted spaces $L^{p(\cdot)}(\Gamma, w),\ w(t) = \prod_{k=1}^{n} |t-t_k|^{\beta_k},\ t_k \in \Gamma$ with variable exponent $p(t)$ (see definitions in Section 2), under the assumption that $p(t)$ satisfies the standard log-condition. The curve Γ may be finite or infinite. On the latter case we assume also that $p(t)$ satisfies the log-condition at infinity.

2. Definitions

Let p be a measurable function on Γ such that $p : \Gamma \to (1, \infty)$. In what follows we assume that p satisfies the conditions

$$1 < p_- := \operatorname*{ess\,inf}_{t\in\Gamma} p(t) \le \operatorname*{ess\,sup}_{t\in\Gamma} p(t) =: p_+ < \infty, \tag{2.1}$$

$$|p(t) - p(\tau)| \le \frac{A}{ln\ \frac{1}{|t-\tau|}}, \qquad t \in \Gamma, \tau \in \Gamma, \qquad |t-\tau| \le \frac{1}{2}. \tag{2.2}$$

Observe that condition (2.1) may be also written in the form

$$|p(t) - p(\tau)| \le \frac{2\ell A}{ln \frac{2\ell}{|t-\tau|}}, \qquad t, \tau \in \Gamma, \tag{2.3}$$

where ℓ is the length of the curve.

In the case where Γ is an infinite curve, we also assume that p satisfies the following condition at infinity

$$|p(t) - p(\tau)| \le \frac{A_\infty}{ln \frac{1}{\left|\frac{1}{t} - \frac{1}{\tau}\right|}}, \qquad \left|\frac{1}{t} - \frac{1}{\tau}\right| \le \frac{1}{2}, \qquad |t| \ge L,\ |\tau| \ge L \tag{2.4}$$

for some $L > 0$.

By $\mathcal{P} = \mathcal{P}(\Gamma)$ we denote the class of exponents p satisfying condition (2.1) and by $\mathbb{P} = \mathbb{P}(\Gamma)$ the class of those p for which the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma)$.

The generalized Lebesgue space with variable exponent is defined via the modular

$$I_\Gamma^p(f) := \int_\Gamma |f(t)|^{p(t)}\, d\nu(\tau)$$

by the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I_\Gamma^p\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

Observe that

$$\|f\|_{p(\cdot)} = \|f^a\|_{\frac{p(\cdot)}{a}}^{\frac{1}{a}} \tag{2.5}$$

for any $0 < a \le \inf p(t)$.

By $L^{p(\cdot)}(\Gamma, w)$ we denote the weighted Banach space of all measurable functions $f : \Gamma \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma,w)} := \|wf\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int\limits_{\Gamma} \left| \frac{w(t)f(t)}{\lambda} \right|^{p(t)} d\nu(t) \le 1 \right\} < \infty. \tag{2.6}$$

We denote $p'(t) = \frac{p(t)}{p(t)-1}$.

From the Hölder inequality for the $L^{p(\cdot)}$-spaces

$$\left| \int_{\Gamma} u(\tau)v(\tau)\, d\nu(\tau) \right| \le k\|u\|_{L^{p(\cdot)}(\Gamma)} \|v\|_{L^{p'(\cdot)}(\Gamma)}, \qquad \frac{1}{p(\tau)} + \frac{1}{p'(\tau)} \equiv 1,$$

where $k = 1 + \frac{1}{p_-} + \frac{1}{(p')_-} = 1 + \frac{1}{p_-} - \frac{1}{p_+} < 2$, it follows that

$$\left| \int_{\Gamma} u(t)v(t)\, d\nu(t) \right| \le k\|u\|_{L^{p'}\left(\Gamma, \frac{1}{w}\right)} \|v\|_{L^{p}(\Gamma,w)}, \tag{2.7}$$

and for the conjugate space $\left[L^{p(\cdot)}(\Gamma, w)\right]^*$ we have

$$\left[L^{p(\cdot)}(\Gamma, w)\right]^* = L^{p'(\cdot)}(\Gamma, 1/w) \tag{2.8}$$

which is an immediate consequence of the fact that $\left[L^{p(\cdot)}(\Gamma)\right]^* = L^{p'(\cdot)}(\Gamma)$ under conditions (2.1), see [13], [16].

The following value

$$\frac{1}{p_\gamma} = \frac{1}{|\gamma|} \int\limits_{\gamma} \frac{d\nu(t)}{p(t)}, \quad \gamma \subset \Gamma \tag{2.9}$$

will be used, introduced for balls in $\mathbb{R}^n$ by L. Diening [4]. Here $\gamma = \gamma(t, r)$, $t \in \Gamma$, $r > 0$, is any portion of the curve Γ.

By $\chi_\gamma(\tau) = \begin{cases} 1, & \tau \in \gamma \\ 0, & \tau \in \Gamma \backslash \gamma \end{cases}$ we denote the characteristic function of a portion γ of the curve Γ.

3. The main statements

In the sequel we consider the power weights of the form

$$w(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \qquad t_k \in \Gamma \tag{3.1}$$

in the case of finite curve and the weights

$$w(t) = |t - z_0|^{\beta} \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \qquad t_k \in \Gamma, \quad z_0 \notin \Gamma \tag{3.2}$$

in the case of infinite curve.

Theorem A. *Let*

i) Γ *be a simple Carleson curve;*
ii) p *satisfy conditions* (2.1), (2.2) *and also* (2.4) *in the case* Γ *is an infinite curve.*

Then the singular operator S_Γ *is bounded in the space* $L^{p(\cdot)}(\Gamma, w)$ *with weight* (3.1) *or* (3.2), *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, \dots, n, \tag{3.3}$$

and also

$$-\frac{1}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'(\infty)} \tag{3.4}$$

in the case Γ *is infinite.*

Remark 3.1. From (2.4) it follows that there exists $p_\infty = \lim_{\substack{|t|\to\infty \\ t\in\Gamma}} p(t)$ and $|p(t) - p_\infty| \leq \frac{A_\infty}{ln|t|}, \qquad |t| \geq \max\{L, 2\}$.

For constant p Theorem A is due to G. David [3] in the non-weighted case, for the weighted case with constant p see [2]. For earlier results on the subject we refer to [9], Theorem 2.2. The statement of Theorem A for variable $p(\cdot)$ was proved in [11] in the case of finite Lyapunov curves or curves of bounded rotation without cusps.

Theorem B. *Let* Γ *be a finite rectifiable curve. Let* $p : \Gamma \to [1, \infty)$ *be a continuous function. If the singular operator* S_Γ *is bounded in the space* $L^{p(\cdot)}(\Gamma)$, *then the curve* Γ *has the property*

$$\sup_{\substack{t\in\Gamma \\ r>0}} \frac{|\gamma(t, r)|}{r^{1-\varepsilon}} < \infty \tag{3.5}$$

for every $\varepsilon > 0$. *If* $p(t)$ *satisfies the log-condition* (2.2), *then property* (3.5) *holds with* $\varepsilon = 0$, *that is,* Γ *is a Carleson curve.*

Observe that Theorem B for the case of constant p was proved in [15].

Theorem C. *Let assumptions* i)–ii) *of Theorem A be satisfied, and let* $a \in C(\Gamma)$. *In the case where* Γ *is an infinite curve starting and ending at infinity, we assume that* $a \in C(\dot{\Gamma})$, *where* $\dot{\Gamma}$ *is the compactification of* Γ *by a single infinite point, that is,* $a(t(-\infty)) = a(t(+\infty))$. *Then under conditions* (3.3)–(3.4), *the operator*

$$(S_\Gamma aI - aS_\Gamma)f = \frac{1}{\pi i} \int_\Gamma \frac{a(\tau) - a(t)}{\tau - t} f(\tau) d\nu(\tau)$$

is compact in the space $L^{p(\cdot)}(\Gamma, w)$ *with weight* (3.1)–(3.2).

Theorems A, B and C are proved in Sections 6, 7 and 8, respectively.

4. Preliminaries

We base ourselves on the following result for maximal operators on Carleson curves. Let

$$Mf(t) = \sup_{r>0} \frac{1}{\nu\{\gamma(t,r)\}} \int\limits_{\gamma(t,r)} |f(\tau)| d\nu(\tau) \tag{4.1}$$

be the maximal operator on functions defined on a curve Γ in the complex plane. The following statements are valid.

Proposition 4.1. *Let*

i) Γ *be a simple Carleson curve of a finite length;*
ii) p *satisfy conditions* (2.1)–(2.2).

Then the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma, w)$ with weight (3.1), *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, \dots, n. \tag{4.2}$$

Proposition 4.2. *Let*

i) Γ *be an infinite simple Carleson curve;*
ii) p *satisfy conditions* (2.1)–(2.2) *and let there exist a circle $B(0,R)$ such that* $p(t) \equiv p_\infty = \text{const}$ *for* $t \in \Gamma \backslash (\Gamma \cap B(0,R))$.

Then the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma, w)$, with weight (3.2), *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)} \qquad \textit{and} \qquad -\frac{1}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'_\infty}. \tag{4.3}$$

The Euclidean space versions of Propositions 4.1 and 4.2 for variable exponents were proved in [11] and [8], respectively. The proof of Propositions 4.1 and 4.2 for Carleson curves follows similar ideas, but needs some modifications. The proofs of Propositions 4.1 and 4.2 for the case of Carleson curves will be given in another publication.

We will also make use of the following Kolmogorov theorem, see [12], [3], [7].

Theorem 4.3. *Let Γ be a Carleson curve of a finite length. Then for any $s \in (0,1)$*

$$\left(\frac{1}{|\Gamma|} \int\limits_{\Gamma} |S_\Gamma f(t)|^s d\nu(t) \right)^{\frac{1}{s}} \leq c \frac{1}{|\Gamma|} \int\limits_{\Gamma} |f(t)| d\nu(t). \tag{4.4}$$

Theorem 4.3 is a consequence of the fact that the singular operator on Carleson curves has weak (1,1)-type:

$$\nu \left\{ t \in \Gamma : |S_\Gamma f(t)| > \lambda \right\} \leq \frac{c}{\lambda} \int\limits_{\Gamma} |f(t)| d\nu(t)$$

the latter being proved in [3].

Proposition 4.4. *Let $p(t)$ satisfy condition* (2.1) *and the maximal operator M be bounded in $L^{p(\cdot)}(\Gamma)$. Then there exists a constant $C>0$ such that*

$$\|\chi_\gamma\|_{p(\cdot)} \le C|\gamma|^{\frac{1}{p_\gamma}} \qquad \text{for all} \qquad \gamma=\gamma(t,r)\subset\Gamma \tag{4.5}$$

where p_γ is the mean value (2.9).

Proposition 4.4 was proved in [4], Lemma 3.4, for balls in the Euclidean space and remains the same for arcs γ on Carleson curves. For completeness of presentation we expose this proof in the appendix.

5. Auxiliary statements

Let

$$\mathcal{M}^\# f(t) = \sup_{r>0}\frac{1}{|\gamma(t,r)|}\int_{\gamma(t,r)} |f(\tau)-f_{\gamma(t,r)}|\, d\nu(\tau), \qquad t\in\Gamma \tag{5.1}$$

where $f_{\gamma(t,r)} = \frac{1}{|\gamma(t,r)|}\int_{\gamma(t,r)} f(\tau)\ d\nu(\tau)$, be the sharp maximal function on the curve Γ.

Theorem 5.1. *Let Γ be an infinite Carleson curve. Let $p(t)$ satisfy conditions* (2.1)–(2.2) *and $p(t)=p_\infty$ outside some circle $B(t_0,R)$. Let $w(t)=|t-t_0|^\beta$, $t_0\in\mathbf{C}$, where*

$$-\frac{1}{p(t_0)}<\beta<\frac{1}{p'(t_0)} \quad \text{and} \quad -\frac{1}{p_\infty}<\beta<\frac{1}{p'_\infty} \quad \text{if} \quad t_0\in\Gamma$$

and $-\frac{1}{p_\infty}<\beta<\frac{1}{p'_\infty}$ if $t_0\notin\Gamma$. Then for $f\in L^{p(\cdot)}(\Gamma,w)$

$$\|f\|_{L^{p(\cdot)}(\Gamma,w)} \le c\left\|\mathcal{M}^\# f\right\|_{L^{p(\cdot)}(\Gamma,w)}. \tag{5.2}$$

Proof. As is known, $\|f\|_{L^{p(\cdot)}} \sim \sup\limits_{\|g\|_{L^{p'(\cdot)}}\le 1}\left|\int\limits_\Gamma f(t)g(t)\ d\nu(t)\right|$, see [13], Theorem 2.3 or [16], Theorem 3.5. Therefore,

$$\|fw\|_{L^{p(\cdot)}} \le c \sup_{\|g\|_{L^{p'(\cdot)}}\le 1}\left|\int_\Gamma f(t)g(t)w(t)\ d\nu(t)\right|.$$

We make use of the inequality

$$\int\limits_\Gamma |f(t)g(t)|\, d\nu(t) \le \int\limits_\Gamma M^\# f(t) Mg(t)\, d\nu(t) \tag{5.3}$$

where $f\in L^{p(\cdot)}(\Gamma), g\in L^{p'(\cdot)}(\Gamma)$, which is known for the Euclidean space, see [5], Lemma 3.5, and is similarly proved for infinite Carleson curves. We obtain

$$\|fw\|_{L^{p(\cdot)}} \le c \sup_{\|g\|_{L^{p'(\cdot)}}\le 1}\left|\int_\Gamma w(t)M^\# f(t)[w(t)]^{-1}M(gw)\ d\nu(t)\right|$$

and then

$$\|fw\|_{L^{p(\cdot)}} \le c \sup_{\|g\|_{L^{p'(\cdot)}} \le 1} \|wM^{\#}f\|_{L^{p(\cdot)}} \|w^{-1}M(gw)\|_{L^{p'(\cdot)}}$$

by the Hölder inequality. Since $-\frac{1}{p'(t_0)} < -\beta < \frac{1}{p(t_0)}$, we may apply Proposition 4.2 for the space $L^{p'(\cdot)}$ with β replaced by $-\beta$ and conclude that

$$\|fw\|_{L^{p(\cdot)}} \le C \sup_{\|g\|_{L^{p'(\cdot)}} \le 1} \|wM^{\#}f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}} \le C\|wM^{\#}f\|_{L^{p(\cdot)}}$$

which proves (5.2). □

6. Proof of Theorem A

6.1. General remark

Remark 6.1. *It suffices to prove Theorem A for a single weight $|t-t_0|^\beta$ where $t_0 \in \Gamma$ in the case Γ is finite and t_0 may belong or not belong to Γ when Γ is infinite.*

Indeed, in the case of a finite curve let $\Gamma = \bigcup_{k=1}^{n} \Gamma_k$ where Γ_k contains the point t_k in its interior and does not contain $t_j, j \ne k$ in its closure. Then

$$\|f\|_{L^{p(\cdot)}\left(\Gamma, \prod_{k=1}^{n} |t-t_k|^{\beta_k}\right)} \sim \sum_{k=1} \|f\|_{L^{p(\cdot)}\left(\Gamma_k, |t-t_k|^{\beta_k}\right)} \tag{6.1}$$

whenever $1 \le p_- \le p_+ < \infty$. This equivalence follows from the easily checked modular equivalence

$$I_\Gamma^p \left(f(t) \prod_{k=1}^{n} |t-t_k|^{\beta_k} \right) \sim \sum_{k=1} I_{\Gamma_k}^p \left(f(t)|t-t_k|^{\beta_k} \right),$$

since

$$\begin{aligned} c_1 \le \|f\|_{p(\cdot)} \le c_2 \quad &\Longrightarrow \quad c_3 \le I_\Gamma^p(f) \le c_4, \\ C_1 \le I_\Gamma^p(f) \le C_2 \quad &\Longrightarrow \quad C_3 \le \|f\|_{p(\cdot)} \le C_4 \end{aligned} \tag{6.2}$$

with $c_3 = \min\left(c_1^{p_-}, c_1^{p_+}\right), c_4 = \max\left(c_2^{p_-}, c_2^{p_+}\right), C_3 = \min\left(C_1^{\frac{1}{p_-}}, C_1^{\frac{1}{p_+}}\right)$ and $C_4 = \max\left(C_2^{\frac{1}{p_-}}, C_2^{\frac{1}{p_+}}\right)$.

Similarly, in the case of an infinite curve

$$\|f\|_{L^{p(\cdot)}\left(\Gamma, |t-z_0|^\beta \prod_{k=1}^{n} |t-t_k|^{\beta_k}\right)} \sim \|f\|_{L^{p(\cdot)}(\Gamma_\infty, |t-z_0|^\beta)} + \sum_{k=1} \|f\|_{L^{p(\cdot)}\left(\Gamma_k, |t-t_k|^{\beta_k}\right)} \tag{6.3}$$

where Γ_∞ is a portion of the curve outside some large circle, so that Γ_∞ does not contain the points $t_k, k = 1, \ldots, n$.

Then, because of (6.1) and (6.3), the statement of Remark 6.1 is obtained by introduction of the standard partition of unity $1 = \sum\limits_{k=1}^{n} a_k(t)$, where $a_k(t)$ are smooth functions equal to 1 in a neighborhood $\gamma(t_k,\varepsilon)$ of the point t_k and equal to 0 outside its neighborhood $\gamma(t_k, 2\varepsilon)$ (and similarly in a neighborhood of infinity in the case Γ is infinite), so that $a_k(t)|t-t_j|^{\pm\beta_j} \equiv 0$ in a neighborhood of the point t_k, if $k \neq j$.

6.2. Auxiliary results

We start with proving the following statement known for singular integrals in the Euclidean space (T. Alvarez and C. Pérez, [1]).

Proposition 6.2. *Let Γ be a simple Carleson curve. Then the following pointwise estimate is valid*

$$\mathcal{M}^{\#}\left(|S_\Gamma f|^s\right)(t) \leq c[Mf(t)]^s, \qquad 0 < s < 1, \tag{6.4}$$

where the constant $c > 0$ may depend on Γ and s, but does not depend on $t \in \Gamma$ and f.

To prove Proposition 6.2, we need – following ideas in [1] – the following technical lemma.

Lemma 6.3. *Let Γ be a simple Carleson curve, $z_0 \in \Gamma$ and $\gamma_r = \gamma(z_0, r)$ and*

$$H_{r,z_0}(t) = \frac{1}{|\gamma_r|^2} \int\limits_{\gamma_r} \int\limits_{\gamma_r} \left| \frac{1}{z-t} - \frac{1}{\tau - t} \right| d\nu(z) d\nu(\tau). \tag{6.5}$$

Then for any locally integrable function f the pointwise estimate holds

$$\sup_{r>0} \int\limits_{t \in \Gamma : |t-z_0| > 2r} |f(t)| H_{r,z_0}(t) d\nu(t) \leq CMf(z_0) \tag{6.6}$$

where $C > 0$ does not depend on f and z_0.

Proof. We have

$$H_{r,z_0}(t) = \frac{1}{|\gamma_r|^2} \int\limits_{\gamma_r} \int\limits_{\gamma_r} \frac{|\tau - z|}{|z-t| \cdot |\tau - t|} d\nu(z) d\nu(\tau).$$

For $|t - z_0| > 2r$ we have

$$|z-t| \geq |t-z_0| - |z-z_0| \geq |t-z_0| - r \geq \frac{1}{2}|t-z_0|$$

and similarly $|\tau - t| \geq \frac{1}{2}|t-z_0|$ so that

$$H_{r,z_0}(t) \leq \frac{Cr}{|t-z_0|^2}$$

where the constant $C > 0$ depends only on the length of the curve Γ. Then

$$\sup_{r>0} \int\limits_{t\in\Gamma:|t-z_0|>2r} |f(t)|H_{r,z_0}(t)d\nu(t) \le c \sup_{r>0} \sum_{k=0}^{m} \int\limits_{2^k r<|t-z_0|<2^{k+1}r} \frac{r|f(t)|}{|t-z_0|^2} d\nu(t)$$

with $m = m(r)$. Hence

$$\sup_{r>0} \int\limits_{t\in\Gamma:|t-z_0|>2r} |f(t)|H_{r,z_0}(t)d\nu(t) \le 2c \sup_{r>0} \sum_{k=0}^{m} \frac{1}{2^k} \frac{1}{2^{k+1}r} \int\limits_{|t-z_0|<2^{k+1}r} |f(t)|d\nu(t)$$

$$\le 2cMf(z_0) \sum_{k=0}^{m} \frac{1}{2^k} \le c_1 Mf(z_0). \qquad \square$$

We will also need the following technical lemma

Lemma 6.4. *Let f be an integrable function on Γ, $f_\gamma = \frac{1}{|\gamma|} \int\limits_\gamma f(\tau)d\nu(\tau)$. Then*

$$\frac{1}{|\gamma|} \int\limits_\gamma |f(\tau) - f_\gamma| d\nu(\tau) \le \frac{2}{|\gamma|} \int\limits_\gamma |f(\tau) - C| d\nu(\tau) \tag{6.7}$$

for any constant C on the right-hand side.

Proof. The proof is well known:

$$\frac{1}{|\gamma|} \int\limits_\gamma |f(\tau) - f_\gamma| d\nu(\tau) \le \frac{1}{|\gamma|^2} \int\limits_\gamma \int\limits_\gamma |f(\tau) - f(\sigma)| d\nu(\tau)d\nu(\sigma)$$

$$\le \frac{1}{|\gamma|^2} \int\limits_\gamma \int\limits_\gamma (|f(\tau) - C| + |C - f(\sigma)|) d\nu(\tau)d\nu(\sigma)$$

$$= \frac{2}{|\gamma|} \int\limits_\gamma |f(\tau) - C| d\nu(\tau). \qquad \square$$

Proof of Proposition 6.2. To prove estimate (6.4), according to Lemma 6.4 it suffices to show that for any locally integrable function f and any $0 < s < 1$ there exists a positive constant A such that

$$\left(\frac{1}{|\gamma|} \int\limits_\gamma \Big| |S_\Gamma f(\xi)|^s - A^s \Big| d\nu(\xi) \right)^{\frac{1}{s}} \le CMf(z_0), \qquad \gamma = \gamma(z_0, r) \tag{6.8}$$

for almost all $z_0 \in \Gamma$, where $C > 0$ does not depend on f and z_0. We set $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{\gamma(z_0,2r)}$ and $f_2 = f \cdot \chi_{\Gamma\setminus\gamma(z_0,2r)}$. We take

$$A = (S_\Gamma f_2)_\gamma = \frac{1}{|\gamma|} \int\limits_\gamma |S_\Gamma f_2(\xi)| \, d\nu(\xi).$$

Then, taking into account that $||a|^s - |b^s|| \leq |a-b|^s$, for $0<s<1$, we have

$$\left(\frac{1}{|\gamma|}\int\limits_\gamma \left||S_\Gamma f(\xi)|^s - A^s\right| d\nu(\xi)\right)^{\frac{1}{s}} \leq c\left(\frac{1}{|\gamma|}\int\limits_\gamma \left|S_\Gamma f_1(\xi)\right|^s d\nu(\xi)\right)^{\frac{1}{s}} +$$

$$+c\left(\frac{1}{|\gamma|}\int\limits_\gamma \left||S_\Gamma f_2(\xi)| - A\right|^s d\nu(\xi)\right)^{\frac{1}{s}} =: c(I_1+I_2).$$

For I_1 by (4.4) we obtain

$$I_1 \leq \frac{1}{|\gamma|}\int\limits_\gamma |f_1(\xi)|d\nu(t) \leq \frac{1}{|\gamma|}\int\limits_\gamma |f(t)|d\nu(\xi) \leq Mf(z_0). \tag{6.9}$$

For I_2, by Jensen inequality and Fubini theorem after easy estimations we get

$$I_2 \leq \frac{1}{|\gamma|}\int\limits_\gamma \left|(S_\Gamma f_2)(\xi) - \frac{1}{|\gamma|}\int\limits_\gamma (S_\Gamma f_2)(\tau)d\nu(\tau)\right| d\nu(\xi) \leq \int\limits_{\Gamma\backslash\gamma(z_0,2r)} |f(t)|H_{r,z_0}(t)d\nu(t),$$

where $H_{r,z_0}(t)$ is the function defined in (6.5). Therefore, by Lemma 6.3, $I_2 \leq CMf(z_0)$ which completes the proof. □

6.3. Proof of Theorem A itself. Sufficiency part

According to Remark 6.1, we consider the case of a single weight $|t-t_0|^\beta$ where t_0 may be not belonging to Γ in case Γ is infinite.

I). The case of infinite curve and p constant at infinity. First we consider the case where Γ is an infinite curve and we additionally suppose at this step that $p(t) \equiv \text{const} = p_\infty$ outside some large ball $B(0,R)$.

Let $0<s<1$. Observe that

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,w)} = \||S_\Gamma f|^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,w)}.$$

Then by Theorem 5.1 we have

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,w)} \leq C\left\|\mathcal{M}^{\#}(|S_\Gamma f|^s)\right\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,w)}$$

for s sufficiently close to 1. Indeed, Theorem, 5.1 is applicable in this case, because $\frac{p(t)}{s}$ satisfies conditions (2.1)–(2.2) and when s is sufficiently close to 1, then the exponent $\mu(t_0)$ of the weight w satisfies the conditions $-\frac{1}{\frac{p(t_0)}{s}} < \mu(t_0) < \frac{1}{\frac{p'(t_0)}{s}}$, required by Theorem 5.1, since the interval in (3.3) is open. Therefore, by Proposition 6.2 we get

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,w)} \leq c\,\|(Mf)^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,w)} = c\,\|Mf\|_{L^{p(\cdot)}(\Gamma,w)}.$$

It remains to apply Proposition 4.2 to obtain $\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,w)} \leq c\|f\|_{L^{p(\cdot)}(\Gamma,w)}$.

II). The case of finite curve and p constant on some arc. At the next step we consider the case of finite curve under the additional assumption that there exists an arc $\gamma \subset \Gamma$ with $|\gamma| > 0$ on which $p(t) \equiv \text{const}$.

First we observe that the singular integral may be considered in the form

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau \tag{6.10}$$

instead of (1.3), since $d\tau = \tau'(s)d\nu(\tau)$ and $|\tau'(s)| = 1$ on Carleson curves so that $\|f\tau'\|_{L^{p(\cdot)}(\Gamma,w)} = \|f\|_{L^{p(\cdot)}(\Gamma,w)}$.

The case considered now is reduced to the previous case **I)** by the change of variables. Let $z_0 \in \gamma$ be any point of γ (different from t_0 if $t_0 \in \gamma$). Without loss of generality we may assume that $z_0 = 0$. Let

$$\Gamma_* = \{t \in \mathbf{C} : t = \frac{1}{\tau},\ \tau \in \Gamma\} \qquad \text{and} \qquad \widetilde{p}(t) = p\left(\frac{1}{t}\right), \quad t \in \Gamma_*,$$

so that Γ_* is an infinite curve and $\widetilde{p}(z)$ is constant on Γ_* outside some large circle. By the change of variables $\frac{1}{\tau} = w$ and $\frac{1}{t} = z$ we get

$$(S_\Gamma f)(t) = -z(S_{\Gamma_*}\psi)(z), \qquad z \in \Gamma_* \tag{6.11}$$

where $\psi(w) = \frac{1}{w} f\left(\frac{1}{w}\right)$. The following lemma is valid where the equivalence $A \sim B$ means that $c_1 A \leq B \leq c_2 A$ with c_1 and c_2 not depending on A and B.

Lemma 6.5. *The following modular equivalence holds*

$$I_\Gamma^p \left(|t - t_0|^\beta f(t)\right) \sim I_{\Gamma_*}^{\widetilde{p}} \left(\rho(t)\psi(t)\right) \tag{6.12}$$

where $\rho(t) = |t|^\nu\, |t - t_0^*|^\beta$ *with* $t_0^* = \frac{1}{t_0} \in \Gamma_*$ *and*

$$\nu = 1 - \beta - \frac{2}{\widetilde{p}(\infty)} = 1 - \beta - \frac{2}{p(0)}. \tag{6.13}$$

Proof. Indeed,

$$I_\Gamma^p \left(|t - t_0|^\beta f(t)\right) = \int\limits_\Gamma |t - t_0|^{\beta p(t)} |f(t)|^{p(t)} |dt|.$$

After the change of variables $t \to \frac{1}{t}$ we get

$$I_\Gamma^p \left(|t - t_0|^\beta f(t)\right) = \int\limits_{\Gamma_*} \left|\frac{1}{t} - t_0\right|^{\beta\tilde{p}(t)} \left|f\left(\frac{1}{t}\right)\right|^{\tilde{p}(t)} \frac{|dt|}{|t^2|}$$

$$= \int\limits_{\Gamma_*} |t_0|^{\beta\tilde{p}(t)} \frac{|t_0^* - t|^{\beta\tilde{p}(t)}}{|t|^{\beta\tilde{p}(t)}} \left|f\left(\frac{1}{t}\right)\right|^{\tilde{p}(t)} \frac{|dt|}{|t^2|}.$$

Since $|t_0|^{\beta\tilde{p}(t)} \sim \text{const}$, we obtain

$$I_\Gamma^p \left(|t - t_0|^\beta f(t)\right) \sim \int\limits_{\Gamma_*} \frac{|t_0^* - t|^{\beta\tilde{p}(t)}}{|t|^{\beta\tilde{p}(t)+2}} \left|f\left(\frac{1}{t}\right)\right|^{\tilde{p}(t)} |dt|. \tag{6.14}$$

Now we have

$$f\left(\frac{1}{t}\right) = t\psi(t).$$

Therefore, from (6.14) we get

$$I_\Gamma^p\left(|t-t_0|^\beta f(t)\right) \sim \int\limits_{\Gamma_*} \frac{|t_0^*-t|^{\beta\tilde{p}(t)}}{|t|^{\beta\tilde{p}(t)+2}}\,|t\psi(t)|^{\tilde{p}(t)}\,|dt|.$$

Hence

$$I_\Gamma^p\left(|t-t_0|^\beta f(t)\right) \sim \int\limits_{\Gamma_*} \left(\frac{|t_0^*-t|^{\beta}}{|t|^{(\beta-1)+\frac{2}{\tilde{p}(t)}}}\,|t\psi(t)|\right)^{\tilde{p}(t)}|dt|.$$

Observe that the point $z=0$ does not pass through the origin and therefore $|t|^{(\beta-1)+\frac{2}{\tilde{p}(t)}} \sim |t|^{(\beta-1)+\frac{2}{\tilde{p}(\infty)}}$. As a result we arrive at (6.12)–(6.13). □

According to (6.2), from(6.12) we also have

$$\|f\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^\beta))} \sim \|\psi\|_{L^{\tilde{p}(\cdot)}(\Gamma_*,\rho(t))}$$

and

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^\beta)} \sim \|S_{\Gamma_*}\psi\|_{L^{\tilde{p}(\cdot)}(\Gamma_*,\rho(t))} \tag{6.15}$$

where (6.11) was taken into account. Observe also that

$$-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)} \qquad \Longleftrightarrow \qquad -\frac{1}{\tilde{p}(t_0^*)} < \nu < \frac{1}{\tilde{p}'(t_0^*)}. \tag{6.16}$$

Obviously, $\widetilde{p}(t)$ satisfies conditions (2.1)–(2.2). Since $\widetilde{p}(t)$ is constant at infinity, according to part **I)** and Remark 6.1, the operator S_{Γ_*} is bounded in the space $L^{\widetilde{p}(\cdot)}(\Gamma_*,\rho(t))$, the required conditions on the weight $\rho(t)$ being satisfied by (6.16) and by the fact that $\beta+\nu = 1-\frac{2}{\tilde{p}(\infty)}$ is automatically in the interval $\left(-\frac{1}{\tilde{p}(\infty)}, \frac{1}{\tilde{p}'(\infty)}\right)$. Then the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma,|t-t_0|^\beta)$ by (6.15).

III). **The general case of finite curve**. Let $\gamma_1 \subset \Gamma$ and $\gamma_2 \subset \Gamma$ be two disjoint non-empty arcs of Γ, $\overline{\gamma_1}\cap\overline{\gamma_2} = \emptyset$. According to the part **II)** the operator $wS_\Gamma\frac{1}{w}$ with $w(t) = |t-t_0|^\beta, t_0\in\Gamma$, is bounded in the space $L^{p_1(\cdot)}(\Gamma)$, if

1) $p_1(t)$ satisfies conditions (2.1)–(2.2),
2) $p_1(t)$ is constant at γ_1,
3) $-\frac{1}{p_1(t_0)} < \beta < \frac{1}{p_1'(t_0)}$

and similarly in the space $L^{p_2(\cdot)}(\Gamma)$, if

1′) $p_2(t)$ satisfies conditions (2.1)–(2.2),
2′) $p_2(t)$ is constant at γ_2,
3′) $-\frac{1}{p_2(t_0)} < \beta < \frac{1}{p_2'(t_0)}$

Aiming to make use of the Riesz interpolation theorem, we observe that the following statement is valid (see its proof in Appendix 2).

Lemma 6.6. *Given a function $p(t)$ satisfying conditions* (2.1)–(2.2), *there exist arcs $\gamma_2 \subset \Gamma$ and $\gamma_1 \subset \Gamma$ such that $p(t)$ may be represented in the form*

$$\frac{1}{p(t)} = \frac{\theta}{p_1(t)} + \frac{1-\theta}{p_2(t)}, \qquad \theta = \frac{1}{2}, \tag{6.17}$$

where $p_j(t)$, $j = 1, 2$, satisfy the above conditions 1), 2) *and* 1′), 2′), *respectively and*

$$\frac{1}{p_j(t_0)} = \frac{1}{p(t_0)}, \quad j = 1, 2,$$

so that conditions 3) *and* 3′(*are also satisfied whenever they are satisfied for $p(t)$.*

In view of Lemma 6.6, the boundedness of the singular operator in $L^{p(\cdot)}(\Gamma)$ with given p follows from the Riesz-Thorin interpolation theorem for the spaces $L^{p(\cdot)}(\Gamma)$ (proved in [14]).

IV). **The general case of infinite curve**. Obviously, the after the step **III)** the general case of infinite curve, that is, the case where Γ is infinite and p is not necessarily constant outside some circle, is reduced to the case of finite curve by mapping the infinite curve Γ onto a finite curve Γ_* in the same way as it was done in the step **II)**. What is important to note is that thanks to conditions (2.2) and (2.4), the new exponent $\tilde{p}(t), t \in \Gamma_*$ is log-continuous on the curve Γ_*.

Remark 6.7. We emphasize the following. We had to prove the boundedness of the singular operator on an infinite curve under additional assumption that $p(t)$ is constant at infinity, then by the change of variables we could cover the case of a finite curve with $p(t)$ constant at any arc. After that we could use the interpolation theorem to get a result on boundedness on a finite curve without the assumption on $p(t)$ to be constant an an arc. After that it remained to use change of variables to get the general result for an infinite curve. This order is essential. Indeed, a seeming possibility to treat first the general case of a finite curve and then cover the case of infinite curve by the change of variables, is not applicable, because the initial step in the whole proof was based on Theorem 5.1, which was proved for infinite curves.

6.4. Proof of the necessity part of Theorem A

The proof of the necessity is in fact the same as in the case of smooth curves, see [10], p. 153. We dwell on the main points. Let Γ be a finite curve. From the boundedness of S_Γ in $L^{p(\cdot)}(\Gamma, \rho)$ it follows that $S_\Gamma f(t)$ exists almost everywhere for an arbitrary $f \in L^{p(\cdot)}(\Gamma, \rho)$. Thus ρ should be such that $f \in L^1(\Gamma)$ for arbitrary $f \in L^{p(\cdot)}(\Gamma, \rho)$. The function $f = f\rho\rho^{-1}$ belongs to $L^1(\Gamma)$ for arbitrary $f \in L^{p(\cdot)}(\Gamma, \rho)$ if and only if $\rho^{-1} \in L^{q(\cdot)}$. Then the function $\rho^{-1}(t) = |t - t_0|^{-\beta}$, $t_0 \in \Gamma$, belongs to $L^{q(\cdot)}(\Gamma)$ if and only if $\beta < \frac{1}{q(t_0)}$. Indeed, by the log-condition we have

$$|t - t_0|^{-\beta q(t)} \sim |t - t_0|^{-\beta q(t_0)}.$$

On the other hand, since Γ is a Carleson curve, from $|t - t_0|^{-\beta q(t_0)} \in L^1$ we have $\beta < \frac{1}{q(t_0)}$.

The necessity of the condition $-\frac{1}{p(s_0)} < \beta$ follows from the duality argument.

In a similar way, with slight modifications the case of infinite curve and weight fixed to infinity, is treated.

7. Proof of Theorem B

We start with the following remark.

Remark 7.1. If the operator S_Γ is bounded in $L^{p(\cdot)}(\Gamma)$ and γ is a measurable subset of Γ, then the operator $S_\gamma = \chi_\gamma S_\Gamma \chi_\gamma$ is bounded in $L^{p(\cdot)}(\gamma)$ and $\|S_\gamma\|_{L^{p(\cdot)}(\gamma)} \le \|S_\Gamma\|_{L^{p(\cdot)}(\Gamma)}$ (we denote the restriction of $p(\cdot)$ onto γ by the same symbol $p(\cdot)$).

7.1. Auxiliary lemmas

Lemma 7.2. *For every point $t \in \Gamma$ and every $\rho \in \left(0, \frac{1}{6}\operatorname{diam}\Gamma\right)$ there exists a function $\varphi_t := \varphi_{t,\rho}(\tau)$ such that*

$$I_p\left(S_\Gamma \varphi_t\right) \ge m \left(\frac{|\gamma(t,\rho|)}{\rho}\right)^{p_- - 1} I_p\left(\varphi_t\right), \tag{7.1}$$

where $m > 0$ is a constant not depending on t and ρ.

Proof. Let us fix the point $t = t_0$ and consider circles centered at t_0 of the radii $\rho, 2\rho$ and 3ρ and 8 rays with the angle $\frac{\pi}{4}$, one of them being parallel to the axis of abscissas. These rays split the circle $|z - t_0| < \rho$ and the annulus $2\rho < |z - t_0| < 3\rho$ into 16 parts. It suffices to treat only those partes which lie in a semiplane, for example, in the upper semiplane. We denote these parts of the circle $|z - t_0| < \rho$ by $\Gamma_k := \Gamma_{k,t_0,\rho}$ and the parts of the annulus $2\rho < |z - t_0| < 3\rho$ by $\gamma_k := \gamma_{k,t_0,\rho}$, respectively, $k = 1, 2, 3, 4$, counting them, e.g., counter clockwise. These rays may be chosen so that there exists a pair k_0, j_0 such that

$$|\Gamma_{k_0}| \ge \frac{1}{8}|\gamma(t_0,\rho)| \quad \text{and} \quad |\gamma_{j_0}| \ge \frac{1}{8}\rho. \tag{7.2}$$

Without loss of generality we may take $k_0 = 1$.

Let

$$\varphi_{t_0} = \varphi_{t_0,\rho}(t) = \begin{cases} 1, & t \in \Gamma_1 \\ 0, & t \in \Gamma \backslash \Gamma_1 \end{cases} \tag{7.3}$$

We have to estimate the integral

$$I_p(S_\Gamma \varphi_{t_0,\rho}) = \int\limits_\Gamma \left| \int\limits_\Gamma \frac{\varphi_{t_0}(\tau)}{\tau - t}\, d\nu(\tau) \right|^{p(t)} d\nu(t). \tag{7.4}$$

Let $\tau - t = |\tau - t| e^{i\alpha(\tau,t)}$. We have

$$I_p(S_\Gamma \varphi_{t_0}) \ge \int\limits_{\gamma_{j_0}} \left| \int\limits_{\Gamma_1} \frac{\cos\alpha(\tau,t) - i\sin\alpha(\tau,t)}{|\tau - t|}\, d\nu(\tau) \right|^{p(t)} d\nu(t). \tag{7.5}$$

Let first $j_0 = 1$. We put $M_1 = (\rho, 0)$ and $M_2 = \left(2\rho\cos\frac{\pi}{4}, 2\rho\sin\frac{\pi}{4}\right)$. It is easily seen that $\max|\alpha(\tau,t)| \le \beta_1 < \frac{\pi}{2}$, where β_1 is the angle between the vector $\overline{M_2M_2}$ and the axis of abscissas. Similarly it can be seen that

$$\begin{array}{llll} \text{if} & \tau\in\Gamma_1, t\in\gamma_2, & \text{then} & \frac{\pi}{4}\le\alpha(\tau,t)\le\pi-\beta_2, \\ \text{if} & \tau\in\Gamma_1, t\in\gamma_3, & \text{then} & \frac{\pi}{2}\le\alpha(\tau,t)\le\pi-\beta_3, \\ \text{if} & \tau\in\Gamma_1, t\in\gamma_4, & \text{then} & \frac{3\pi}{4}\le\alpha(\tau,t)\le\pi+\beta_4 \end{array}$$

where $\beta_2 = \operatorname{arctg} 2$, $\beta_3 = \operatorname{arctg}\frac{1}{3}$ and $\beta_4 = \operatorname{arctg}\frac{2\sqrt{2}-1}{7}$. Therefore, when $\tau\in\Gamma_1$ and $t\in\gamma_{j_0}, j_0 = 1,2,3,4$, then

$$\text{either} \quad |\cos\alpha(\tau,t)| \ge m_0 > 0, \qquad \text{or} \qquad |\sin\alpha(\tau,t)| \ge m_0 > 0.$$

Moreover, when $\tau\in\Gamma_1$ and $t\in\gamma_2$ or $t\in\gamma_4$, then $\cos\alpha(\tau,t)$ preserves the sign and when $\tau\in\Gamma_1$ and $t\in\gamma_2$ or $t\in\gamma_3$, then $\sin\alpha(\tau,t)$ preserves the sign. Consequently, from (7.5) we get

$$I_p(S_\Gamma\varphi_{t_0}) \ge \int\limits_{\gamma_{j_0}} \max\left(\left|\operatorname{Re}\int\limits_{\Gamma_1}\frac{\varphi_{t_0}(\tau)d\nu(\tau)}{\tau-t}\right|^{p(t)}, \left|\operatorname{Im}\int\limits_{\Gamma_1}\frac{\varphi_{t_0}(\tau)d\nu(\tau)}{\tau-t}\right|^{p(t)}\right).$$

Hence

$$I_p(S_\Gamma\varphi_{t_0}) \ge \int\limits_{\gamma_{j_0}}\left|\int\limits_{\Gamma_1}\frac{m_0}{3\rho}d\nu(\tau)\right|^{p(t)} d\nu(t) \ge \left(\frac{m_0}{3}\right)^{p_+}\int\limits_{\gamma_{j_0}}\left(\frac{|\Gamma_1|}{\rho}\right)^{p(t)} d\nu(t).$$

Then by (7.2)

$$I_p(S_\Gamma\varphi_{t_0}) \ge \left(\frac{m_0}{3\cdot 8}\right)^{p_+}\int\limits_{\gamma_{j_0}}\left(\frac{|\gamma(t,\rho)|}{\rho}\right)^{p(t)} d\nu(t) \ge m_1\left(\frac{|\gamma(t,\rho)|}{\rho}\right)^{p_-}|\gamma_{j_0}|.$$

Since $|\gamma(t,\rho)| \ge I_p(\varphi_t) = |\Gamma_1|$ and $|\gamma_{j_0}| \ge \frac{\rho}{8}$, we obtain

$$I_p(S_\Gamma\varphi_{t_0}) \ge \frac{m_1}{8}\left(\frac{|\gamma(t,\rho)|}{\rho}\right)^{p_-}\frac{\rho}{|\gamma(t,\rho)|}|\Gamma_1| = m\left(\frac{\nu(\gamma(t,\rho))}{\rho}\right)^{p_- -1} I_p(\varphi_{t_0})$$

which proves (7.1) with (7.3). □

We denote for brevity

$$\alpha(f) = \alpha_\gamma(f) = \begin{cases} p_+, & \text{if } \|f\|_{p(\cdot)} \ge 1, \\ p_-, & \text{if } \|f\|_{p(\cdot)} < 1, \end{cases}$$

and

$$\beta(f) = \beta_\gamma(f) = \begin{cases} p_-, & \text{if } \|f\|_{p(\cdot)} \ge 1, \\ p_+, & \text{if } \|f\|_{p(\cdot)} < 1, \end{cases}$$

so that $\alpha(f) + \beta(f) \equiv p_+ + p_-$ and

$$\|f\|_{p(\cdot)}^{\beta(f)} \le I_p(f) \le \|f\|_{p(\cdot)}^{\alpha(f)}. \tag{7.6}$$

Lemma 7.3. *If the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma)$, then for every $t \in \Gamma$ the estimate holds*

$$\frac{|\gamma(t,\rho)|}{\rho} \le c_\Gamma |\gamma(t,\rho)|^{\delta_\Gamma(t)} \tag{7.7}$$

where $\delta_\Gamma(t) = \frac{1}{p_- - 1}\left(\frac{\alpha(S_\Gamma\varphi_t)}{\beta(\varphi_t)} - 1\right)$, $C_\Gamma = \left(\frac{8\|S_\Gamma\|^{p_+}_{L^{p(\cdot)}(\Gamma)}}{m}\right)^{\frac{1}{p_- - 1}}$ *and the function* φ_t *and the constant were defined in* (7.1)*.*

Proof. Let $K = \|S_\Gamma\|_{L^{p(\cdot)}}$ for brevity. By the boundedness $\|S_\Gamma f\|_{L^{p(\cdot)}} \le K\|f\|_{L^{p(\cdot)}}$ and property (7.6) we have

$$I_p(S_\Gamma f) \le K^{\alpha(S_\Gamma f)} \|f\|^{\alpha(S_\Gamma f)}_{L^{p(\cdot)}} \le K^{\alpha(S_\Gamma f)} [I_p(f)]^{\frac{\alpha(S_\Gamma f)}{\beta(f)}}.$$

We choose $f = \varphi_t$ with φ_t from Lemma 7.2 and take (7.1) and (7.3) into account, which yields

$$K^{\alpha(S_\Gamma f)} [I_p(\varphi_t)]^{\frac{\alpha(S_\Gamma f)}{\beta(S_\Gamma \varphi_t)}} \ge I_p(S_\Gamma \varphi_t) \ge m \left(\frac{|\gamma(t,\rho)|}{\rho}\right)^{p_- - 1} I_p(\varphi_t)$$

$$\ge \frac{m}{8}\left(\frac{|\gamma(t,\rho)|}{\rho}\right)^{p_- - 1} |\gamma(t,\rho)|. \tag{7.8}$$

We observe that in the first term in this chain of inequalities we have $I_p(\varphi_t) \le |\gamma(t,\rho)|$ and then (7.8) yields (7.7). □

7.2. Proof of Theorem B itself

Let $\gamma = \gamma(t, 3\rho) = \Gamma \cap \{z : |z - t| < 3\rho\}$. According to Remark 7.1, the operator S_γ is boundeed in $L^{p(\cdot)}(\gamma)$. Then by Lemma 7.3 we obtain

$$\frac{|\gamma(\xi,\rho)|}{\rho} \le c_\gamma |\gamma(\xi,\rho)|^{\delta_\gamma(\xi)} \le c_\Gamma |\gamma(\xi,\rho)|^{\delta_\gamma(\xi)}, \quad \xi \in \gamma, \tag{7.9}$$

where C_Γ is the same as in Lemma 7.3 and

$$\delta_\gamma(\xi) = \frac{1}{p_-(\gamma) - 1}\left(\frac{\alpha(S_\gamma \varphi_\xi)}{\beta(\varphi_\xi)} - 1\right)$$

with $p_-(\gamma) = \min_{\tau\in\gamma} p(\tau)$. Depending on the values $\|S_\gamma\varphi_\xi\|_{L^{p(\cdot)}(\gamma)}$ and $\|\varphi_\xi\|_{L^{p(\cdot)}(\gamma)}$, the exponent $\delta_\gamma(\xi)$ may take only three values $0, \delta_1$and $-\delta_2$, where

$$\delta_1 = \frac{p_+(\gamma) - p_-(\gamma)}{p_+(\gamma)} \frac{1}{p_-(\gamma) - 1}, \qquad \delta_2 = \frac{p_+(\gamma) - p_-(\gamma)}{p_-(\gamma)} \frac{1}{p_-(\gamma) - 1}$$

(in fact, according to (7.9) only two values 0 and $-\delta_2$ are possible, since $\frac{\gamma(\xi,\rho)}{\rho} \ge 1$). Therefore, when ρ is small, $|\delta_\gamma(\xi)|$ also has small values:

$$|\delta_\gamma(\xi)| \le \lambda\omega(p, 6\rho), \qquad \lambda = \frac{1}{(p_-(\Gamma) - 1)p_-(\Gamma)}, \tag{7.10}$$

where $\omega(p,h)$ is the continuity modulus of the function p, since $p(t)$ is continuous on the compact set Γ and consequently is uniformly continuous.

Let $\rho_1 < 1$ be sufficiently small such that $\lambda\omega(p, 6\rho_1) < \varepsilon$.

From (7.9) we have $|\gamma(\xi,\rho)|^{1-\delta_\gamma(\xi)} \le C_\Gamma \rho$ and then

$$|\gamma(\xi,\rho)| < C_\Gamma^{\frac{1}{1-\delta(\xi)}} \rho^{\frac{1}{1-\delta_\gamma(\xi)}} < C_\Gamma^{\frac{1}{1-\varepsilon}} \rho^{\frac{1}{1+\varepsilon}} \qquad \text{for} \qquad \rho < \rho_1 \tag{7.11}$$

(where we took into account that $C_\Gamma > 1$ and $\rho \le \rho_1 < 1$). Thus, (3.5) has been proved.

Let now $p(t)$ satisfy the log-condition (2.2). For the function

$$\psi_\xi(\rho) = |\gamma(\xi,\rho)|^{\delta_\gamma(\xi)}$$

by (7.11) we have

$$|\ln \psi_\xi(\rho)| = |\delta_\gamma(\xi) \ln |\gamma(\xi,\rho)|| \le \lambda\omega(p, 6\rho) \left(\frac{\ln C_\Gamma}{1-\varepsilon} + \frac{|\ln \rho|}{1+\varepsilon} \right).$$

In view of (7.10) and (2.3) we then obtain

$$|\ln \psi_\xi(\rho)| \le \frac{\lambda A}{1-\varepsilon} \, \frac{\ln \frac{C_\Gamma}{\rho}}{\ln \frac{\ell}{3\rho}}, \qquad \rho < \min\left\{\rho_1, \frac{\ell}{6}\right\}. \tag{7.12}$$

It is easy to see that $\frac{\ln \frac{C_\Gamma}{\rho}}{\ln \frac{\ell}{3\rho}}$ is bounded for small ρ, so that $|\ln \psi_\xi(\rho)| \le C < \infty$. Since $\frac{|\gamma(\xi,\rho)|}{\rho} \le C_\Gamma \psi_\xi(\rho)$ by (7.9), we get $\frac{|\gamma(\xi,\rho)|}{\rho} \le C_\Gamma e^C$, which means that Γ is a Carleson curve.

8. Proof of Theorem C

Theorem C is derived from Theorem A, which is standard. Indeed, it is known that any function $a(t)$ continuous on Γ may be approximated in $C(\Gamma)$ by a rational function $r(t)$, whatsoever Jordan curve Γ we have, as is known from the Mergelyan's result, see for instance, [6], p. 169. Therefore, since the singular operator S is bounded in $L_w^{p(\cdot)}(\Gamma)$ by Theorem A, we obtain that the commutator $aS - SaI$ is approximated in the operator norm in $L_w^{p(\cdot)}(\Gamma)$ by the commutator $rS - SrI$ which is finite-dimensional operator, and consequently compact in $L_w^{p(\cdot)}(\Gamma)$. Therefore, $aS - SaI$ is compact.

9. Appendices

9.1. Appendix 1: Proof of Proposition 4.4

Let $f(\tau)=\chi_\gamma(\tau)|\gamma|^{-\frac{1}{p(\tau)}}$, $\gamma=\gamma(t,r)$, so that $\|f\|_{p(\cdot)}=1$. For all $z\in\gamma$ we have

$$CMf(z)\geq\frac{1}{|\gamma|}\int\limits_\gamma f(\tau)\,d\nu(\tau)=\frac{1}{|\gamma|}\int\limits_\gamma|\gamma|^{-\frac{1}{p(\tau)}}\,d\nu(\tau) \quad \text{for any } \gamma=\gamma(t,r). \tag{9.1}$$

Since the function $\Phi(x)=a^{-x}, x\in\mathbb{R}^1_+$, is convex for any $a>0$, by Jensen's inequality

$$\Phi\left(\frac{1}{|\gamma|}\int\limits_\gamma|f(\tau)|d\nu(\tau)\right)\leq\frac{1}{|\gamma|}\int\limits_\gamma\Phi(|f(\tau)|)\,d\nu(\tau) \tag{9.2}$$

we obtain

$$CMf(z)\geq|\gamma|^{-\frac{1}{|\gamma|}\int\limits_\gamma\frac{d\nu(\tau)}{p(\tau)}}=|\gamma|^{-\frac{1}{p_\gamma}},\qquad z\in\gamma.$$

Hence $\|\chi_\gamma(z)|\gamma|^{-\frac{1}{p_\gamma}}\|_{p(\cdot)}\leq C\|Mf\|_{p(\cdot)}$ and by the boundedness of the maximal operator we obtain that $\|\chi_\gamma(z)|\gamma|^{-\frac{1}{p_\gamma}}\|_{p(\cdot)}\leq C$, which yields (4.5).

9.2. Appendix 2: Proof of Lemma 6.6

We have to prove the following. *Let Γ be a Carleson curve and $a(t)$ any function on Γ, satisfying the* log*-condition and such that*

$$0<d\leq a(t)\leq D<1 \qquad on \qquad \Gamma. \tag{9.3}$$

Then there exist non-intersecting non-empty arcs γ_1 and γ_2 on Γ such that

$$a(t)=\frac{b(t)+c(t)}{2} \qquad with \qquad b(t)\equiv 0 \;\; on \;\; \gamma_1 \qquad and \qquad c(t)\equiv 0 \;\; on \;\; \gamma_2 \tag{9.4}$$

and $b(t)$ and $c(t)$ are log*-continuous on Γ, satisfy the same condition* (9.3) *and* $b(t_0)=c(t_0)=a(t_0)$.

We will take γ_1 and γ_2 so that $t_0\notin\overline{\gamma_1\cup\gamma_2}$ and construct the functions $b(t)$ and $c(t)$ as follows

$$b(t)=\begin{cases} A & ,\ t\in\gamma_1\\ \ell(t) & ,\ t\ \in\Gamma\backslash(\gamma_1\cup\gamma_2)\\ 2a(t)-B & ,\ t\in\gamma_2\end{cases} \tag{9.5}$$

where $A,B\in(0,1)$ are some constants. The link $\ell(t)$ between the values of $b(t)$ on γ_1 and on γ_2 may be constructed for instance in the following way: at each of the components of the set $\Gamma\backslash(\gamma_1\cup\gamma_2)$ it is introduced as the linear interpolation between the number A and the values of $2a(t)-B$ at the endpoints of this component, if it does not belong to it, and as the piece-wise linear interpolation between

A the value $a(t_0)$ and the values of $2a(t) - B$ at the endpoints of this component, if it contains t_0. Then

$$c(t) = 2a(t) - b(t) = \begin{cases} 2a(t) - A & , \ t \in \gamma_1 \\ 2a(t) - \ell(t) & , \ t \ \in \Gamma \backslash (\gamma_1 \cup \gamma_2) \\ B & , \ t \in \gamma_2 \end{cases} \tag{9.6}$$

Obviously, $b(t)$ and $c(t)$ are log-continuous on Γ. Checking condition (9.3) for $b(t), c(t)$, we only have to verify this condition for $2a(t) - A$ on γ_1 and for $2a(t) - B$ on γ_2. To this end, we have to choose A and B so that

$$2a(t) - 1 < A < 2a(t) \quad \text{on} \quad \overline{\gamma_1}, \qquad 2a(t) - 1 < B < 2a(t) \quad \text{on} \quad \overline{\gamma_2}$$

Let $a_-(\gamma_i) = \inf\limits_{t \in \gamma_i} a(t)$ and $a_+(\gamma_i) = \sup\limits_{t \in \gamma_i} a(t), \ i = 1, 2$. It suffices to choose A and B in the intervals

$$A \in (\max\{0, 2a_+(\gamma_1) - 1\}, \min\{2a_-(\gamma_1), 1\}),$$
$$B \in (\max\{0, 2a_+(\gamma_2) - 1\}, \min\{2a_-(\gamma_2), 1\})$$

These intervals are non-empty, if $a_+(\gamma_i) - a_-(\gamma_i) > \frac{1}{2}, i = 1, 2$. Obviously, γ_i may be chosen sufficiently small so that the last condition is satisfied.

References

[1] T. Alvarez and C. Pérez. Estimates with A_∞ weights for various singular integral operators. *Boll. Un. Mat. Ital*, A (7) 8(1):123–133, 1994.

[2] A. Böttcher and Yu. Karlovich. *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*. Basel, Boston, Berlin: Birkhäuser Verlag, 1997, 397 pages.

[3] G. David. Opérateurs intégraux singuliers sur certaines courbes du plan complexe. *Ann. Sci. École Norm. Sup. (4)*, 17(1):157–189, 1984.

[4] L. Diening. Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Mathem. Nachrichten*, 268:31–43, 2004.

[5] L. Diening and M. Ružička. Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics. *J. Reine Angew. Math*, 563:197–220, 2003.

[6] D. Gaier. *Vorlesungen über Approximation im Komplexen.* Birkhäuser, Basel, Boston, Stuttgart, 1980, 174 pages.

[7] S. Hofmann. Weighted norm inequalities and vector valued inequalities for certain rough operators. *Indiana Univ. Math. J.*, 42(1):1–14, 1993.

[8] M. Khabazi. Maximal operators in weighted $L^{p(x)}$ spaces. *Proc. A. Razmadze Math. Inst.*, 135:143–144, 2004.

[9] B.V. Khvedelidze. The method of Cauchy type integrals in discontinuous boundary problems of the theory of holomorphic functions of one variable (Russian). In *Collection of papers "Contemporary problems of mathematics" (Itogi Nauki i Tekhniki), t. 7*, pages 5–162. Moscow: Nauka, 1975.

[10] V. Kokilashvili and S. Samko. Singular Integrals in Weighted Lebesgue Spaces with Variable Exponent. *Georgian Math. J.*, 10(1):145–156, 2003.

[11] V. Kokilashvili and S. Samko. Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Revista Matematica Iberoamericana*, 20(2):495–517, 2004.

[12] A.N. Kolmogorov. Sur les fonctions harmoniques conjugées et les séries de Fourier. *Fund. Math.*, 7:24–29, 1925.

[13] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116):592–618, 1991.

[14] J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[15] V.A. Paatashvili and G.A. Khuskivadze. Boundedness of a singular Cauchy operator in L ebesgue spaces in the case of nonsmooth contours. *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR*, 69:93–107, 1982.

[16] S.G. Samko. Differentiation and integration of variable order and the spaces $L^{p(x)}$. Proceed. of Intern. Conference "Operator Theory and Complex and Hypercomplex Analysis", 12–17 December 1994, Mexico City, Mexico, Contemp. Math., Vol. 212, 203–219, 1998.

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