

Singular operators in variable spaces $L^{p(\cdot)}(\Omega, \rho)$ with oscillating weights

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To Professor Frank-Olme Speck on the occasion of his 60th birthday

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The weight functions w_k may oscillate between two power functions with different exponents. It is assumed that the exponent $p(\cdot)$ satisfies the Dini–Lipschitz condition. The final statement on the boundedness is given in terms of the index numbers of the functions w_k (similar in a sense to the Boyd indices for the Young functions defining Orlicz spaces).

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1 Introduction

Nowadays there is an evident increase of interest to harmonic analysis problems and operator theory in the generalized Lebesgue spaces with variable exponent $p(x)$ and the corresponding Sobolev spaces, we refer, in particular to surveys [10], [16], [19], [35], and to [25] and [38], for the basics on the spaces $L^{p(\cdot)}$.

For the boundedness results of maximal operators we refer to [7] for bounded domains in \mathbb{R}^n , to [6] and [31] for unbounded domains, and to [23] for weighted boundedness on bounded domains.

We refer also to [5] and [8] where there are also given new insights into the problems of boundedness of singular and maximal operators in variable exponent spaces.

In [23] the maximal operator with power weights $\rho(x) = \prod_{k=1}^m |x - x_k|^{\alpha_k}$ with $-\frac{n}{p(x_k)} < \alpha_k < \frac{n}{q(x_k)}$ was considered (under the usual log-condition on $p(x)$).

Recently, in [21] we proved the weighted boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\Omega, \rho)$ for a certain class of non-power weights,

$$\rho(x) = \prod_{k=1}^m w_k(|x - x_k|), \quad x_k \in \overline{\Omega}, \quad (1.1)$$

which are still “fixed” to a finite number of points $x_k \in \Omega$ (radial type weights of the Zygmund–Bary–Stechkin class).

The problem of more general weights remains open. An explicit description of weights for which the maximal operator is bounded in the spaces $L^{p(\cdot)}$ is a challenging problem. The most progress in that direction was done in [8].

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Boundedness of singular operators with standard kernels in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ was proved in [11] and [12].

We refer also to [13] and [14] where the results from [11] and [12] were extended to Calderón–Zygmund singular operators related to the half space \mathbb{R}_+^{n+1} .

Statements on weighted boundedness of Calderón–Zygmund type singular operators with power weights were given in [22]. We refer also for similar results for the Cauchy singular integral on Carleson curves obtained in [20] and [24].

In this paper we prove a theorem on boundedness of Calderón–Zygmund type singular operators over bounded domains in \mathbb{R}^n with weight (1.1) basing ourselves on the weighted result for the maximal function obtained in [21]. A similar statement for the Cauchy singular operator on Carleson curves is also given.

The main results are given in Theorems 3.6, 3.7 and 4.3.

Notation Throughout the paper we denote by:

- $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$;
- $|B(x, r)|$ the volume of $B(x, r)$;
- C, c different positive constants;
- Ω an open set in \mathbb{R}^n ;
- $|\Omega|$ the Lebesgue measure of Ω ;
- χ_Ω the characteristic function of a set Ω ;
- $p'(x) = \frac{p(x)}{p(x)-1}, 1 < p(x) < \infty$;
- $p_- = \inf_{x \in \Omega} p(x), p_+ = \sup_{x \in \Omega} p(x)$;
- $\widetilde{W} = \widetilde{W}([0, \ell])$ as defined in (2.11);
- Φ_0^γ , as defined in Definition 2.4.

2 Preliminaries

2.1 On maximal function in weighted Lebesgue generalized spaces

Let Ω be an open set in $\mathbb{R}^n, n \geq 1$, and let $p(x)$ be a function on Ω with values in $[1, \infty)$. By

$$\mathcal{M}_\Omega f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} |f(y)| dy, \quad x \in \mathbb{R}^n, \tag{2.1}$$

we denote the maximal operator. We write $\mathcal{M} = \mathcal{M}_{\mathbb{R}^n}$ in the case where $\Omega = \mathbb{R}^n$.

We use the following notation for classes of the exponents $p(x)$ related to the boundedness of the maximal operator:

$$\mathbb{P}(\Omega) = \{p(x) : 1 < p_- \leq p_+ < \infty\}, \quad p_+ = \inf_{x \in \Omega} p(x), \quad p_- = \sup_{x \in \Omega} p(x).$$

$\mathcal{P}(\Omega) \subset \mathbb{P}(\Omega)$ is the class of functions in $\mathbb{P}(\Omega)$ for which the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$.

$\mathcal{P}_{log}(\Omega)$ will stand for the class of exponents $p \in \mathbb{P}(\Omega)$ which satisfy the log-condition

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega. \tag{2.2}$$

As is known ([7]), $\mathcal{P}_{log}(\Omega) \subset \mathcal{P}(\Omega)$ in case of a bounded domain Ω .

By $L^{p(\cdot)}(\Omega, \rho)$ we denote the weighted Banach space of all measurable functions $f : \Omega \rightarrow \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Omega, \rho)} := \|\rho f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{\rho(x)f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} < \infty. \tag{2.3}$$

A weight function ρ is said to be in the class $\mathcal{A}_{p(\cdot)}(\Omega)$ if the maximal operator \mathcal{M}_Ω is bounded in the weighted spaces $L^{p(\cdot)}(\Omega, \rho)$.

Obviously,

$$p \in \mathbb{P} \cap \{p : 1 \in \mathcal{A}_{p(\cdot)}\} \iff p \in \mathcal{P}.$$

Lemma 2.1 *Let $\rho \in \mathcal{A}_p(\Omega)$. Then $\rho^s \in \mathcal{A}_{\frac{p(\cdot)}{s}}(\Omega)$ for any $s \in (0, 1]$.*

Proof. We follow the known arguments, see [9, p. 43]. We have

$$\mathcal{M}_\Omega(|f|^s)(x) \leq [\mathcal{M}_\Omega f(x)]^s, \quad 0 < s < 1, \tag{2.4}$$

which is obtained by the Hölder inequality, and

$$\|f\|_{p(\cdot)} = \|f^s\|_{\frac{p(\cdot)}{s}}^{\frac{1}{s}} \quad \text{for any } s \in (0, 1]. \tag{2.5}$$

Then by (2.4) and (2.5) we obtain

$$\|\rho^s \mathcal{M}_\Omega f\|_{\frac{p(\cdot)}{s}} = \|\rho(\mathcal{M}_\Omega f)^{\frac{1}{s}}\|_{p(\cdot)}^s \leq \|\rho(\mathcal{M}_\Omega f^{\frac{1}{s}})\|_{p(\cdot)}^s \leq C \|\rho f^{\frac{1}{s}}\|_{p(\cdot)}^s = C \|\rho^s f\|_{p(\cdot)}. \tag{2.6}$$

□

In [22] there was proved that when $p \in \mathcal{P}_{\log}(\Omega)$ and Ω is bounded, the power weights

$$\rho(x) = \prod_{k=1}^m |x - x_k|^{\alpha_k}, \quad x_k \in \Omega,$$

are in $\mathcal{A}_{p(\cdot)}$ if and only if

$$-\frac{n}{p(x_k)} < \alpha_k < \frac{n}{p'(x_k)}. \tag{2.7}$$

In [21] this was generalized to the case of oscillating weights of the form

$$\rho(x) = \prod_{k=1}^m w_k(|x - x_k|), \quad x_k \in \overline{\Omega}, \tag{2.8}$$

where the weight functions $w_k(r)$ have the property that $r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi_n^0$, where Φ_n^0 is a certain Zygmund–Bari–Stechkin class. For weights of form (2.8), in [21] there was obtained a sufficient condition for such weights to belong to $\mathcal{A}_{p(\cdot)}(\Omega)$. It was given in terms of the upper and lower indices m_{w_k} and M_{w_k} (of the type of the Boyd indices) of the weight functions $w_k(r)$; see definitions of the indices m_{w_k} and M_{w_k} in Subsection 2.3. Weights w in this class are almost increasing or almost decreasing and may oscillate between two power functions with different exponents and have non-coinciding upper and lower indices m_w and M_w . Namely, in [21] the following theorem was proved.

Theorem 2.2 *Let Ω be a bounded open set in \mathbb{R}^n and $p \in \mathcal{P}_{\log}(\Omega)$. The operator \mathcal{M} is bounded in the space $L^{p(\cdot)}(\Omega, \rho)$ with weight (2.8), if*

$$r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi_n^0, \tag{2.9}$$

and

$$-\frac{n}{p(x_k)} < m_{w_k} \leq M_{w_k} < \frac{n}{p'(x_k)}, \quad k = 1, 2, \dots, m. \tag{2.10}$$

Remark 2.3 Condition (2.9) may be replaced by $w_k \in \widetilde{W}$, where the class \widetilde{W} is defined in (2.11), because condition (2.9) for $w_k \in \widetilde{W}$ coincides with condition (2.10) according to statement (2.12) of Theorem 2.5. Note also that condition (2.9) is equivalent to the condition $r^{\frac{n}{p'(x_k)}} \frac{1}{w_k(r)} \in \Phi_n^0$ in view of property (2.17).

2.2 Examples of weights

Following [32], we observe that besides the trivial examples of non-power weights

$$w_k(r) = r^{\lambda_k} \left(\ln \frac{C}{r}\right)^\alpha, \quad w_k(r) = r^{\lambda_k} \left(\ln \ln \frac{C}{r}\right)^\alpha, \quad \text{etc.},$$

where $-\frac{n}{p(x_k)} < \lambda_k < \frac{n}{p'(x_k)}$, the weight functions

$$w_k(r) = r^{\lambda_k + \frac{c}{\ln^{\alpha_k} \frac{1}{r}}},$$

where λ_k is the same and $\alpha_k \geq 1$, are also examples of weights admissible for Theorem 2.2. More generally, one may take $w_k(r) = r^{\gamma_k(r)}$ where $\gamma_k(r)$ satisfies the the Dini condition

$$|\gamma_k(r+h) - \gamma_k(r)| = o\left(\frac{1}{|\ln|h||}\right)$$

and $\gamma_k(0) \in \left(-\frac{n}{p(x_k)}, \frac{n}{p'(x_k)}\right)$.

The last example may be also generalized in the following way: if the weight function $w_k(r)$ fulfills the condition

$$\lim_{h \rightarrow 0} \frac{w_k(rh)}{w_k(h)} = r^{\gamma_k}, \quad \gamma_k = \text{constant},$$

then it is admissible for Theorem 2.2, if $\gamma_k \in \left(-\frac{n}{p(x_k)}, \frac{n}{p'(x_k)}\right)$.

However, all the above examples have coinciding indices $m_{w_k} = M_{w_k}$ (see their definition in the next subsection). Examples of oscillating weights with non-coinciding indices m_{w_k}, M_{w_k} are more complicated. We refer for such examples to [34].

2.3 Index numbers m_w and M_w and Bary–Stechkin–Zygmund class Φ_0^γ

Let $W = \{w \in C([0, \ell]) : w(0) = 0, w(x) > 0 \text{ for } x > 0, w(x) \text{ is almost increasing}\}$, where $0 < \ell < \infty$. (In the sequel, $\ell = \text{diam } \Omega$ in case Ω is bounded).

Following [32]–[34], we introduce the notation

$$m_w = \sup_{x>1} \frac{\ln \left(\liminf_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x} = \sup_{0 < x < 1} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x} = \lim_{x \rightarrow 0} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x}$$

and

$$M_w = \sup_{x>1} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x}.$$

The numbers m_w and M_w are known as *the lower and upper indices* of the function $w(x)$ (compare these indices with the Matuszewska–Orlicz indices, see [28, p. 20]; they are of the type of the Boyd indices, see [26, p. 75], [27], or [3, p. 149] about the Boyd indices). We have $0 \leq m_w \leq M_w \leq \infty$ for $w \in W$.

The upper and lower indices may be also well-defined for functions $w(x)$ positive for $x > 0$ which do not necessarily belong to W , for example, if $w_a(x) := x^a w(x)$ is in W , but $w(x)$ is not, then the indices m_{w_a} and M_{w_a} of $w_a(x)$ are well-defined and there also exist the indices m_w and M_w of $w(x)$ and $m_{w_a} = a + m_w, M_{w_a} = a + M_w$ in this case.

By this reason we introduce also the following class of functions, which may have negative indices m_w and M_w :

$$\widetilde{W} = \{w : t^a w(t) \in W \text{ for some } a \in \mathbb{R}^1\}. \tag{2.11}$$

Let $\gamma > 0$. The following class Φ_γ^0 was introduced and studied in [2] (with integer γ); there are also known “two-parametrical” classes $\Phi_\gamma^\beta, 0 \leq \beta < \gamma < \infty$, see [29], [30], [36, p. 253] and [37]. Observe that in [39]

and [40] there were considered more general classes $\Phi_{b(x)}^{a(x)}$ with limits which may “oscillate”; the class Φ_γ^0 corresponds to the case where $a(x) = x^0 = 1$ and $b(x) = x^\gamma$.

Definition 2.4 ([2]) The Zygmund–Bary–Stechkin type class Φ_γ^0 , $0 < \gamma < \infty$, is defined as $\Phi_\gamma^\beta := \mathcal{Z}^0 \cap \mathcal{Z}_\gamma$, where \mathcal{Z}^0 is the class of functions $w \in W$ satisfying the condition

$$\int_0^h \frac{w(x)}{x} dx \leq cw(h) \tag{Z^0}$$

and \mathcal{Z}_γ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(x)}{x^{1+\gamma}} dx \leq c \frac{w(h)}{h^\gamma}, \tag{Z_\gamma}$$

where $c = c(w) > 0$ does not depend on $h \in (0, \ell]$.

In the sequel we refer to the above conditions as (Z^0) - and (Z_γ) -conditions.

The following statement is valid, see [32] and [34] for $\gamma = 1$ and [18] for an arbitrary $\gamma > 0$.

Theorem 2.5 A function $w \in W$ belongs to \mathcal{Z}^0 if and only if $m_w > 0$ and it belongs to \mathcal{Z}_γ , $\gamma > 0$, if and only if $M_w < \gamma$, so that

$$w \in \Phi_\gamma^0 \iff 0 < m_w \leq M_w < \gamma. \tag{2.12}$$

Besides this, for $w \in \Phi_\gamma^0$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$c_1 t^{M_w + \varepsilon} \leq w(t) \leq c_2 t^{m_w - \varepsilon}, \quad 0 \leq t \leq \ell. \tag{2.13}$$

The following properties are also valid

$$m_w = \sup \{ \lambda \in (0, 1) : t^{-\lambda} w(t) \text{ is almost increasing} \}, \tag{2.14}$$

$$M_w = \inf \{ \mu \in (0, 1) : t^{-\mu} w(t) \text{ is almost decreasing} \}. \tag{2.15}$$

Remark 2.6 Observe that

$$m_{\frac{1}{w}} = -M_w \quad \text{and} \quad M_{\frac{1}{w}} = -m_w. \tag{2.16}$$

Therefore, as a corollary of (2.12) we have

$$w(t) \in \Phi_\gamma^0 \iff \frac{t^\gamma}{w(t)} \in \Phi_\gamma^0. \tag{2.17}$$

Remark 2.7 Functions $w \in \mathcal{Z}_\gamma$, $\gamma > 0$, satisfy the doubling condition

$$w(2r) \leq Cw(r), \quad 0 \leq r \leq \ell, \tag{2.18}$$

which follows from the fact that the function $\frac{w(r)}{r^\mu}$ is almost decreasing for every $\mu > M_w$ according to (2.15) (observe that M_w is finite since $M_w < \gamma$ by Theorem 2.5).

3 Weighted boundedness of Calderón–Zygmund type singular operators

We consider Calderón–Zygmund type operators

$$Tf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} k(x, y) f(y) dy. \tag{3.1}$$

We suppose that the kernel $k(x, y)$ is *standard* in the well-known sense ([4], [15, p. 99] and [17]), that is, satisfies the assumptions:

$$\text{i)} \quad |k(x, y)| \leq A |x - y|^{-n}; \quad (3.2)$$

$$\text{ii)} \quad |k(x, y) - k(z, y)| \leq A \frac{|x - z|^\delta}{|x - y|^{\delta+n}}, \quad |k(y, x) - k(y, z)| \leq A \frac{|x - y|^\delta}{|x - y|^{\delta+n}}, \quad (3.3)$$

for all $|x - z| \leq \frac{1}{2}|x - y|$ with some $A > 0$ and $\delta > 0$. It is known that any such operator, if bounded in $L^2(\mathbb{R}^n)$, is also bounded in any space $L^p(\mathbb{R}^n)$, $1 < p < \infty$, $p = \text{constant}$, see [4].

3.1 Preliminaries

[12, Theorem 4.8] on the boundedness of singular operators with standard kernels in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ runs as follows.

Theorem 3.1 *Let $k(x, y)$ be a standard kernel and let the operator T be of weak $(1, 1)$ -type. If $p \in \mathcal{P}(\mathbb{R}^n)$, then the operator T is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.*

Theorem 3.1 was formulated in [11] and [12] under the assumption that $p \in \mathcal{P}(\mathbb{R}^n)$ and that there exists an $s \in (0, 1)$ such that $(\frac{p}{s})' \in \mathcal{P}$. From the later result $p \in \mathcal{P}(\mathbb{R}^n) \iff p' \in \mathcal{P}(\mathbb{R}^n)$ (see [8, Theorem 8.1]) and the simple fact that $p \in \mathcal{P}(\mathbb{R}^n) \implies \frac{p}{s} \in \mathcal{P}(\mathbb{R}^n)$ for $s \in (0, 1)$, see (2.6), it follows that it suffices to assume only that $p \in \mathcal{P}(\mathbb{R}^n)$.

With reference to the known results for unbounded domains in [6] we have also the following corollary.

Corollary 3.2 *Let $k(x, y)$ be a standard kernel. The Calderón–Zygmund-type singular operator*

$$T_\Omega f(x) = \lim_{\varepsilon \rightarrow 0} \int_{y \in \Omega: |y-x| > \varepsilon} k(x, y) f(y) dy \quad (3.4)$$

of weak $(1, 1)$ -type in Ω , is bounded in the space $L^{p(\cdot)}(\Omega)$, if p is in \mathcal{P}_{\log} , when Ω is bounded, and satisfies also the condition

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(e + |x|)}, \quad x \in \Omega, \quad (3.5)$$

when Ω is unbounded.

We give a weighted version of Theorem 3.1 in Subsection 3.2, see Theorem 3.6 and then make use of that weighted version in the case of Bary–Stechkin–Zygmund type weights in Subsection 3.3.

Let

$$\mathcal{M}_\Omega^\# f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} |f(y) - f_{B(x, r)}| dy, \quad x \in \mathbb{R}^n, \quad (3.6)$$

be the sharp maximal function, where $f_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} f(z) dz$.

We write $\mathcal{M}^\# = \mathcal{M}_{\mathbb{R}^n}^\#$ in the case where $\Omega = \mathbb{R}^n$.

Similarly to [12], in the proof of Theorem 3.6 in Subsection 3.2 will follow the known approach based on the following statement.

Theorem 3.3 ([1]) *Let $k(x, y)$ be a standard kernel and let the operator T be of weak $(1, 1)$ -type. Then for arbitrary s , $0 < s < 1$, there exists a constant $c_s > 0$ such that*

$$[\mathcal{M}^\#(|Tf|^s)(x)]^{\frac{1}{s}} \leq c_s \mathcal{M}f(x) \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n), \quad x \in \mathbb{R}^n. \quad (3.7)$$

The following statement holds (see [11, Lemma 3.5]).

Theorem 3.4 *Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ there holds*

$$\left| \int_{\mathbb{R}^n} f(x)g(x) dx \right| \leq c \int_{\mathbb{R}^n} \mathcal{M}^\# f(x) \mathcal{M}g(x) dx$$

with a constant $c > 0$ not depending on f .

Theorem 3.5 *Let $p(x) \in \mathbb{P}(\mathbb{R}^n)$, and let $\frac{1}{w} \in \mathcal{A}_{p'(\cdot)}(\mathbb{R}^n)$. Then*

$$\|wf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|w\mathcal{M}^\# f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with a constant $c > 0$ not depending on f .

Proof. We have

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \sup_{\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1} \left| \int_{\mathbb{R}^n} f(x)g(x)w(x) dx \right|.$$

Then by Theorem 3.4

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \sup_{\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1} \left| \int_{\mathbb{R}^n} \mathcal{M}^\# f(x)w(x)[w(x)]^{-1} \mathcal{M}(gw) dx \right|.$$

Making use of the Hölder inequality for $L^{p(\cdot)}$, we derive

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \sup_{\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1} \|w\mathcal{M}^\# f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|w^{-1} \mathcal{M}(gw)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

The operator $w^{-1} \mathcal{M}w$ is bounded in the space $L^{p'(\cdot)}(\mathbb{R}^n)$ by assumption. Therefore,

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c_1 \sup_{\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1} \|w\mathcal{M}^\# f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq c_1 \|w\mathcal{M}^\# f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad \square$$

3.2 A general statement

Theorem 3.6 *Let $p \in \mathbb{P}(\mathbb{R}^n)$ and let the weight function ρ satisfy the assumptions*

- i) $|\{x \in \mathbb{R}^n : \rho(x) = 0\}| = 0$;
- ii) $\rho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ and $\frac{1}{\rho} \in \mathcal{A}_{p'(\cdot)}(\mathbb{R}^n)$;
- iii) *there exists an $s \in (0, 1)$ such that $\frac{1}{\rho^s} \in \mathcal{A}_{(\frac{p(\cdot)}{s})'}(\mathbb{R}^n)$.*

Then a singular operator T with a standard kernel $k(x, y)$ and of weak $(1, 1)$ -type is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, \rho)$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $0 < s < 1$. By (2.5) we have

$$\|\rho Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|\rho^s |Tf|^s\|_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n)}^{\frac{1}{s}}. \tag{3.8}$$

Applying Theorem 3.5 with $w(x) = [\rho(x)]^s$ and $p(\cdot)$ replaced by $\frac{p(\cdot)}{s}$, we obtain

$$\|\rho Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|\rho^s \mathcal{M}^\#(|Tf|^s)\|_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n)}^{\frac{1}{s}},$$

the application of Theorem 3.5 being possible by assumption iii) of the theorem. Then by (2.5), Theorem 3.3 and the assumption $\rho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, it follows

$$\|\rho Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \left\| \rho [\mathcal{M}^\#(|Tf|^s)]^{\frac{1}{s}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|\rho(Mf)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|\rho f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{3.9}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. To complete the proof of the theorem, it remains to observe that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n, \rho)$, see [22, Theorem 4.1]. (In [22] this denseness was proved under assumption i) and the condition $[\rho(x)]^{p(x)} \in L_{loc}^1(\mathbb{R}^n)$, but the latter follows from the assumption $\rho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ in ii). \square

3.3 The case of Bary–Stechkin–Zygmund type weights

From Theorem 3.6 we derive the following statement.

Theorem 3.7 *Let Ω be a bounded open set in \mathbb{R}^n and let $p \in \mathcal{P}_{\log}(\Omega)$. A singular operator T_Ω with a standard kernel $k(x, y)$ and bounded from $L^1(\Omega)$ to $L^{1,\infty}(\Omega)$ is bounded in the space $L^{p(\cdot)}(\Omega, \rho)$ with weight*

$$\rho(x) = \prod_{k=1}^m w_k(|x - x_k|), \quad x_k \in \overline{\Omega}, \tag{3.10}$$

where

$$w_k(r), \frac{1}{w_k(r)} \in \widetilde{W}([0, \ell]), \quad \ell = \text{diam } \Omega, \tag{3.11}$$

if

$$-\frac{n}{p(x_k)} < m_{w_k} \leq M_{w_k} < \frac{n}{p'(x_k)}, \quad k = 1, 2, \dots, m. \tag{3.12}$$

Proof. To apply Theorem 3.6, we make an extension

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

of a function $f \in L^{p(\cdot)}\Omega$, extend $p(x)$ outside Ω as $p^*(x)$ with preservation of the log-condition in \mathbb{R}^n and either constant at infinity, or satisfying condition (3.5) so that $p^* \in \mathcal{P}(\mathbb{R}^n)$. This is always possible, as is known. We also extend the weight $\rho(x)$ to be constant outside some big ball:

$$\tilde{\rho}(x) = \prod_{k=1}^m \tilde{w}_k(|x - x_k|), \quad \text{where} \quad \tilde{w}_k(r) = \begin{cases} w_k(r), & 0 \leq r \leq \ell, \\ w_k(\ell), & r \geq \ell. \end{cases} \tag{3.13}$$

We have

$$\|\rho T_\Omega f\|_{L^{p(\cdot)}(\Omega)} \leq \|\tilde{\rho} T \tilde{f}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}.$$

To apply Theorem 3.6 to the right-hand side, we have to check that assumptions i)–iii) of that theorem are satisfied in the case of weight function (3.13) with conditions (3.11) and (3.12). Assumption i) is obviously satisfied, since the set $\{x \in \mathbb{R}^n : \rho(x) = 0 \text{ or } \rho(x) = \infty\}$ is just the finite set of points x_1, x_2, \dots, x_m .

Let us check condition ii). We have

$$I_{p^*}(\tilde{\rho} \mathcal{M} \tilde{f}) \leq \int_\Omega |\rho(x) \mathcal{M}_\Omega f(x)|^{p(x)} dx + C \int_{\mathbb{R}^n \setminus \Omega} |\mathcal{M} f(x)|^{p^*(x)} dx.$$

The first term here is covered by Theorem 2.2, while the second term does not involve weight and is bounded by the well-known Diening–Cruz–Uribe–Fiorenza–Neugebauer non-weighted result, since p^* satisfies the required conditions.

Therefore, $\tilde{\rho} \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$.

To check that, $\frac{1}{\tilde{\rho}} \in \mathcal{A}_{p'(\cdot)}(\mathbb{R}^n)$, it suffices to verify conditions (2.9) and (2.10) for $\frac{1}{\tilde{\rho}}$ with respect to p' . According to Remark 2.3 and by assumption (3.13), we only have to verify condition (2.10):

$$-\frac{n}{p'(x_k)} < m_{\frac{1}{w_k}} \leq M_{\frac{1}{w_k}} < \frac{n}{p(x_k)}, \quad k = 1, 2, \dots, m,$$

which coincides with the same condition (3.12) in view of (2.16). □

It remains to verify condition iii) of Theorem 3.6. Again, it suffices to check condition (2.10) for $\frac{1}{\tilde{\rho}^s}$ with respect to the exponent $(\frac{p(\cdot)}{s})'$. After easy calculation with the formulas $m_{s\omega} = sm_\omega$ and (2.16) taken into account, it turns to be

$$-\frac{n}{p(x_k)} < m_{w_k} \leq M_{w_k} < \frac{n}{p'(x_k)} \cdot \frac{\frac{p(x_k)}{s} - 1}{p(x_k) - 1},$$

which is automatically satisfied for any $0 < s \leq 1$ in view of (2.10), since $\frac{\frac{p(x_k)}{s} - 1}{p(x_k) - 1} \geq 1$.

4 Weighted boundedness of the Cauchy singular operator on Carleson curves

4.1 Preliminaries

Let $\Gamma = \{t \in \mathbb{C} : t = t(s), 0 \leq s \leq \ell \leq \infty\}$ be a simple rectifiable curve with arc-length measure $\nu(t) = s$. We denote

$$\gamma(t, r) := \Gamma \cap B(t, r), \quad t \in \Gamma, \quad r > 0, \tag{4.1}$$

where $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$ and for brevity write

$$\nu(\gamma(t, r)) = |\gamma(t, r)|.$$

We assume that Γ is a Carleson curve, that is, there exists a constant $c_0 > 0$ not depending on t and r , such that

$$|\gamma(t, r)| \leq c_0 r. \tag{4.2}$$

As usual, $p : \Gamma \rightarrow (1, \infty)$ is a measurable function on Γ with

$$1 < p_- := \operatorname{ess\,inf}_{t \in \Gamma} p(t) \leq \operatorname{ess\,sup}_{t \in \Gamma} p(t) =: p_+ < \infty, \tag{4.3}$$

$$|p(t) - p(\tau)| \leq \frac{A}{\ln \frac{1}{|t - \tau|}}, \quad t \in \Gamma, \quad \tau \in \Gamma, \quad |t - \tau| \leq \frac{1}{2}. \tag{4.4}$$

In the case where Γ is an infinite curve, Theorem 4.1 below uses also the condition

$$|p(t) - p(\tau)| \leq \frac{A_\infty}{\ln \frac{1}{|\frac{1}{t} - \frac{1}{\tau}|}}, \quad \left| \frac{1}{t} - \frac{1}{\tau} \right| \leq \frac{1}{2}, \quad |t| \geq L, \quad |\tau| \geq L, \tag{4.5}$$

for some $L > 0$.

Similarly to the euclidean case we define $L^{p(\cdot)}(\Gamma, w)$ as the Banach space of measurable functions $f : \Gamma \rightarrow \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma, \rho)} := \|\rho f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Gamma \left| \frac{\rho(t)f(t)}{\lambda} \right|^{p(t)} d\nu(t) \leq 1 \right\} < \infty. \tag{4.6}$$

We consider the weighted boundedness of the singular operator

$$S_\Gamma f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{4.7}$$

along a Carleson curve Γ .

In [20] and [24] the following Theorem 4.1 was proved taking

$$\rho(t) = \prod_{k=1}^n |t - t_k|^{\beta_k}, \quad t_k \in \Gamma, \tag{4.8}$$

in the case of finite curve, and the weights

$$\rho(t) = |t - z_0|^\beta \prod_{k=1}^n |t - t_k|^{\beta_k}, \quad t_k \in \Gamma, \quad z_0 \notin \Gamma, \tag{4.9}$$

in the case of infinite curve.

Theorem 4.1 *Let*

- i) Γ be a simple Carleson curve;
 ii) p satisfy conditions (4.3) and (4.4), and also (4.5) in the case Γ is an infinite curve.
 Then the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, \rho)$ with weight (4.8) or (4.9), if and only if

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \quad k = 1, \dots, n, \quad (4.10)$$

and also

$$-\frac{1}{p_\infty} < \beta + \sum_{k=1}^n \beta_k < \frac{1}{p'(\infty)} \quad (4.11)$$

in the case when Γ is infinite.

4.2 The case of Bary–Stechkin–Zygmund type weights

Let now

$$\rho(t) = \prod_{k=1}^m w_k(|t - t_k|), \quad t_k \in \Gamma. \quad (4.12)$$

As in Theorem 3.7, we suppose that

$$w_k(r), \frac{1}{w_k(r)} \in \widetilde{W}([0, \ell]), \quad \ell = \nu(\Gamma). \quad (4.13)$$

First of all we note that for the maximal operator

$$M_\Gamma f(t) = \sup_{r>0} \frac{1}{\nu\{\gamma(t, r)\}} \int_{\gamma(t, r)} |f(\tau)| d\nu(\tau) \quad (4.14)$$

on a Carleson curve Γ the following analogue of Theorem 2.2 is valid.

Theorem 4.2 *Let Γ be a simple finite Carleson curve and let p satisfy conditions (4.3) and (4.4). The operator \mathcal{M}_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, \rho)$ with weight (4.12)–(4.16), if*

$$-\frac{1}{p(t_k)} < m_{w_k} \leq M_{w_k} < \frac{n}{p'(t_k)}, \quad k = 1, 2, \dots, m. \quad (4.15)$$

The statement of this theorem for power weights $w(|t - t_k|) = |t - t_k|^{\alpha_k}$ when $m_{w_k} = M_{w_k} = \alpha_k$ was given in [24]. The proof of Theorem 4.2 for non-power weights is similar to the euclidean case of Theorem 2.2, so we do not dwell on it.

Theorem 4.3 *Let*

- i) Γ be a simple finite Carleson curve;
 ii) p satisfy conditions (4.3) and (4.4).
 Then the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, \rho)$ with weight (4.12), where

$$w_k(r), \frac{1}{w_k(r)} \in \widetilde{W}([0, \ell]), \quad \ell = \nu(\Gamma), \quad (4.16)$$

if

$$-\frac{1}{p(t_k)} < m_{w_k} \leq M_{w_k} < \frac{1}{p'(t_k)}, \quad k = 1, 2, \dots, m. \quad (4.17)$$

With Theorem 4.2 taken into account, to prove Theorem 4.3 we repeat the arguments we have used in the proof of Theorems 3.3 and 3.7 and therefore we omit its proof.

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