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## THE MAXIMAL OPERATOR IN WEIGHTED VARIABLE SPACES ON METRIC MEASURE SPACES

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## 1. INTRODUCTION

We study the boundedness of the maximal operator

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y)$$

in weighted spaces  $L^{p(\cdot)}(X, \varrho)$  with variable exponent  $p(x)$  on a metric measure space  $X$ .

The classical operators of harmonic analysis in variable Lebesgue spaces in the Euclidean setting were extensively studied during last decade, see surveying papers [2], [10], [22] and references therein. A study of these operators in the setting of measure metric spaces began several years ago. In [11], [12], [16], [17] there were proved results on the boundedness, including certain weighted cases, of maximal, singular and potential operators on an arbitrary Carleson curve, which is a typical example of Ahlfors-regular measure metric space with constant dimension. The non-weighted boundedness of the maximal operator on a bounded measure metric space was proved in [6] and [9], see also [3] for an extension to the case where  $p$  satisfies a condition weaker than the standard log-condition. We refer also to [5], where Sobolev-type theorem for potential operators on bounded metric spaces in  $\mathbb{R}^n$  with variable dimension was obtained and to [19], where continuity of Sobolev functions on metric spaces in the limiting case was studied.

In this paper we obtain weighted estimates for the maximal operator on a measure metric space  $(X, d, \mu)$  with doubling condition. In the case where  $X$  is bounded, the weight function belongs to a certain version of a general Muckenhoupt-type condition, which is narrower than the expected Muckenhoupt condition for variable exponent, but coincides with the usual Muckenhoupt class  $A_p$  in the case of constant  $p$ .

We also specially consider the case of radial-type weights  $w(d(x_0, x))$ , namely in the case  $X$  is bounded, we admit the weights of form  $\varrho(x) = \prod_{k=1}^n w_k(d(x, x_k))$ ,  $x_k \in X$ , where  $w_k(r)$  belong to a certain class of almost monotonic functions, and show that in this case the condition of weighted boundedness of the maximal operator may be written in natural terms of relations between certain index numbers of the weight  $w(r)$  and of the function  $\mu B(x, r)$ :

$$-\frac{m(\mu B)}{p(x_k)} < m(w) \leq M(w) < \frac{m(\mu B)}{p'(x_k)},$$

where  $m(\mu B)$  is the infimum with respect to  $x \in X$  of the lower index of  $\mu B(x, \cdot)$ . In the case of unbounded  $X$  we admit also weights of the type  $w_0[1+d(x_0, x)] \prod_{k=1}^n w_k[d(x_k, x)]$ .

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Some of the results are new even in the case of constant  $p$ .

## 2. PRELIMINARIES

**1.1. Definitions related to homogeneous spaces.** Let  $(X, d, \mu)$  be a homogeneous space, that is, measure space with quasimetric  $d$  and a non-negative measure  $\mu$  satisfying the doubling condition  $\mu(B(x, 2r)) \leq C\mu B(x, r)$ , where  $B(x, r) = B_X(x, r) = \{y \in X : d(x, y) < r\}$ ;  $1_\Omega$  will denote the characteristic function of a set  $\Omega \subseteq X$  and  $\ell = \text{diam } X$ .

Following ideas of papers [20]–[21], we make use of the so called index numbers of monotonic function to measure the local (variable) dimensions of the space  $(X, d, \mu)$  at the point  $x \in X$ :

1) the *local lower and upper dimensions*

$$m(\mu B_x) = \sup_{t>1} \frac{\ln \left( \liminf_{r \rightarrow 0} \frac{\mu B(x, rt)}{\mu B(x, r)} \right)}{\ln t}, \quad M(\mu B_x) = \sup_{t>1} \frac{\ln \left( \limsup_{r \rightarrow 0} \frac{\mu B(x, rt)}{\mu B(x, r)} \right)}{\ln t},$$

2) *similar dimensions "influenced" by infinity:*

$$m_\infty(\mu B_x) = \sup_{t>1} \frac{\ln \left( \liminf_{r \rightarrow \infty} \frac{\mu B(x, rt)}{\mu B(x, r)} \right)}{\ln t}, \quad M_\infty(\mu B_x) = \sup_{t>1} \frac{\ln \left( \limsup_{r \rightarrow \infty} \frac{\mu B(x, rt)}{\mu B(x, r)} \right)}{\ln t}$$

the latter appearing only in the case of unbounded  $X$  (In a different form local dimensions were considered in [5], [6], [7]).

In the sequel we will use the bounds

$$m(\mu B) = \inf_{x \in X} m(\mu B_x) \quad \text{and} \quad m_\infty(\mu B) = \inf_{x \in X} m_\infty(\mu B_x), \quad M_\infty(\mu B) = \sup_{x \in X} M_\infty(\mu B_x).$$

We suppose that

$$0 < m(\mu B) < \infty, 0 < m_\infty(\mu B) < \infty \quad \text{and} \quad 0 < M_\infty(\mu B) < \infty. \quad (1)$$

Observe that for an arbitrarily small  $\varepsilon > 0$  we have  $c_1 r^{M(\mu B_x)+\varepsilon} \leq \mu B(x, r) \leq c_1 r^{m(\mu B_x)-\varepsilon}$ ,  $0 < r \leq R < \infty$  and  $c_3 r^{m_\infty(\mu B_x)-\varepsilon} \leq \mu B(x, r) \leq c_4 r^{M_\infty(\mu B_x)+\varepsilon}$ ,  $r_0 \leq r < \infty$ , where  $c_i, i = 1, 2, 3, 4$ , depend on  $\varepsilon > 0$ , but do not depend on  $r$ ; under conditions (1) they also do not depend on  $x$ .

**1.2. Spaces  $L^{p(\cdot)}(X, \varrho)$ .** By  $L^{p(\cdot)}(X, \varrho)$ , where  $\varrho(x) \geq 0$ , we denote the weighted Banach space of measurable functions  $f : X \rightarrow \mathbb{C}$  such that

$$\|f\|_{L^{p(\cdot)}(X, \varrho)} := \|\varrho f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_X \left| \frac{\varrho(x)f(x)}{\lambda} \right|^{p(x)} d\mu(x) \leq 1 \right\} < \infty.$$

We write  $L^{p(\cdot)}(X, 1) = L^{p(\cdot)}(X)$  and  $\|f\|_{L^{p(\cdot)}(X)} = \|f\|_{p(\cdot)}$  in the case  $\varrho(t) \equiv 1$ . The variable exponent  $p(x) : X \rightarrow (1, \infty)$  is supposed to satisfy the conditions

$$p_- = p_-(X) = \inf_{x \in X} p(x) > 1, \quad p^+ = p^+(X) = \sup_{x \in X} p(x) < \infty \quad (2)$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{d(x, y)}}, \quad d(x, y) \leq \frac{1}{2}, \quad x, y \in X. \quad (3)$$

The generalized Lebesgue spaces  $L^{p(\cdot)}(X)$  with variable exponent on metric measure spaces have been considered in [3], [5], [6], [9], see also references there.

**1.3. Index numbers of almost monotonic functions.** Let  $0 < \ell < \infty$  and

$$W([0, \ell]) = \{w \in C([0, \ell]) : w(t) > 0 \text{ for } t > 0, w(t) \text{ is almost increasing}\}$$

and

$$\widetilde{W}([0, \ell]) = \{\varphi : \exists a = a(\varphi) \in \mathbb{R}^1 \text{ such that } x^a \varphi(x) \in W([0, \ell])\}.$$

Similarly we introduce  $W([\ell, \infty)) = \{w \in C([\ell, \infty) : w(t) > 0 \text{ for } t \geq \ell, w(t) \text{ is almost increasing}\}$  and

$$\widetilde{W}([\ell, \infty)) = \{\varphi : \exists a = a(\varphi) \in \mathbb{R}^1 \text{ such that } x^a \varphi(x) \in W([\ell, \infty))\}.$$

For functions  $w \in \widetilde{W}([0, \ell])$  the numbers

$$m(w) = \sup_{t>1} \frac{\ln \left( \liminf_{h \rightarrow 0} \frac{w(ht)}{w(h)} \right)}{\ln t} = \sup_{0 < t < 1} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{w(ht)}{w(h)} \right)}{\ln t} = \lim_{t \rightarrow 0} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{w(ht)}{w(h)} \right)}{\ln t}$$

and

$$M(w) = \sup_{t>1} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{w(ht)}{w(h)} \right)}{\ln t} = \lim_{t \rightarrow \infty} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{w(ht)}{w(h)} \right)}{\ln t}$$

are well defined (see [20]–[21]) and  $0 \leq m(w) \leq M(w) \leq \infty$  in the case  $w \in W$ .

The indices  $m_\infty(w)$  and  $M_\infty(w)$  responsible for the behavior of functions  $w$  at infinity are introduced in a similar way:

$$m_\infty(w) = \sup_{x>1} \frac{\ln \left[ \overline{\lim}_{h \rightarrow \infty} \frac{w(xh)}{w(h)} \right]}{\ln x}, \quad M_\infty(w) = \inf_{x>1} \frac{\ln \left[ \underline{\lim}_{h \rightarrow \infty} \frac{w(xh)}{w(h)} \right]}{\ln x}.$$

### 3. THE MAIN STATEMENTS

Let  $A_{p(\cdot)}(X)$  be the class of weights  $\varrho$  for which the maximal operator is bounded in the space  $L^{p(\cdot)}(X, \varrho)$ . For Theorem A we introduce the  $\widetilde{A}_{p(\cdot)}(X)$  of weights, which satisfy the condition

$$\begin{aligned} & \sup_{x \in X, r > 0} \left( \frac{1}{\mu B(x, r)} \int_{B(x, r)} |\varrho(y)|^{p(y)} d\mu(y) \right) \times \\ & \times \left( \frac{1}{\mu B(x, r)} \int_{B(x, r)} \frac{d\mu(y)}{|\varrho(y)|^{\frac{p(y)}{p-1}}} \right)^{p-1} < \infty. \end{aligned} \quad (4)$$

This class  $\widetilde{A}_{p(\cdot)}(X)$  used in Theorem A is narrower than the naturally expected Muckenhoupt class  $A_{p(\cdot)}$ . This may be seen on power weights, see conditions of Theorem B. However, it coincides with the Muckenhoupt class  $A_p$  in case  $p$  is constant.

In Theorem A, under log-condition on  $p$  and doubling condition on the measure we show that

$$\widetilde{A}_{p(\cdot)}(X) \subset A_{p(\cdot)}(X).$$

In Theorem B we deal with a special class of radial type weights in the Zygmund-Bari-Stechkin class and arrive at the necessity to relate the properties of the weight to those of the measure  $\mu B(x, r)$ . Such a result for the Euclidean case was earlier obtained in [13]. Theorem B is proved by means of Theorem A, but it is not contained in Theorem A, being more general in its range of applicability.

**Theorem A.** *Let  $X$  be a bounded doubling measure metric space, let the exponent  $p(x)$  satisfy conditions (2), (3) and the weight  $\varrho$  fulfill condition (3). Then the operator  $\mathcal{M}$  is bounded in  $L^{p(\cdot)}(X, \varrho)$ .*

When  $X$  is bounded, we consider weights of the form

$$\varrho(x) = \prod_{k=1}^N w_k(d(x, x_k)), \quad x_k \in X, \quad (5)$$

where  $x_k$  are distinct points and  $w_k(r)$  may oscillate between two power functions as  $r \rightarrow 0+$  (radial Zygmund-Bari-Steckin type weights), and when  $X$  is unbounded, we deal with similar weights of the form

$$\varrho(x) = w_0[1 + d(x_0, x)] \prod_{k=1}^N w_k[d(x, x_k)], \quad x_k \in X, k = 0, 1, \dots, N. \quad (6)$$

**Theorem B.** *Let  $X$  be a bounded doubling measure metric space, let  $p(x)$  satisfy conditions (2), (3) on  $X$ . The operator  $\mathcal{M}$  is bounded in  $L^{p(\cdot)}(X, \varrho)$  with weight (5), if  $w_k \in \widetilde{W}([0, \ell])$ ,  $\ell = \text{diam } X$ , and*

$$-\frac{m(\mu B)}{p(x_k)} < m(w) \leq M(w) < \frac{m(\mu B)}{p'(x_k)}, \quad k = 1, 2, \dots, N. \quad (7)$$

In the case where  $X$  is a bounded open set in  $\mathbb{R}^n$ , Theorem B was proved in [13] for weights in Zygmund-Bary-Steckin type class and in [15] for power weights.

**Theorem C.** *Let*

- i)  $X$  be an unbounded metric measure space satisfying the doubling condition;
- ii)  $p$  satisfy conditions (2)–(3) and let there exist a ball  $B(x_0, R)$ ,  $x_0 \in X$  such that  $p(x) \equiv p_\infty = \text{const}$  for  $x \in X \setminus B(x_0, R)$ .

*Then the maximal operator  $\mathcal{M}$  is bounded in the space  $L^{p(\cdot)}(X, w)$  with weight (6), if  $w_k \in \widetilde{W}(\mathbb{R}_+^1)$  and*

$$-\frac{m(\mu B)}{p(x_k)} < m(w_k) \leq M(w_k) < \frac{m(\mu B)}{p'(x_k)}, \quad k = 1, \dots, N, \quad (8)$$

$$-\frac{m_\infty(\mu B)}{p_\infty} < \sum_{k=0}^N m_\infty(w_k) \leq \sum_{k=0}^N M_\infty(w_k) < \frac{m_\infty(\mu B)}{p'_\infty} - \Delta_{p_\infty}, \quad (9)$$

where  $\Delta_{p_\infty} = \frac{M_\infty(\mu B) - m_\infty(\mu B)}{p_\infty}$ .

In particular, for the power type weight  $\varrho(x) = (1 + d(x_0, x))^{\beta_0} \prod_{k=1}^N [d(x, x_k)]^{\beta_k}$ ,  $x_k \in X$ ,  $k = 0, 1, \dots, N$ , conditions (8)–(9) take the form

$$-\frac{m(\mu B)}{p(x_k)} < \beta_k < \frac{m(\mu B)}{p'(x_k)}, \quad k = 1, \dots, N, \quad (10)$$

and

$$-\frac{m_\infty(\mu B)}{p_\infty} < \sum_{k=0}^N \beta_k < \frac{m_\infty(\mu B)}{p'_\infty} - \Delta_{p_\infty}. \quad (11)$$

In the case where  $X$  has a constant dimension  $d > 0$  in the sense that  $C_1 r^d \leq \mu B(x, r) \leq C_2 r^d$ , conditions (10)–(11) take the form

$$-\frac{d}{p(x_k)} < \beta_k < \frac{d}{p'(x_k)}, \quad k = 1, \dots, N, \quad -\frac{d}{p_\infty} < \sum_{k=0}^N \beta_k < \frac{d}{p'_\infty}. \quad (12)$$

In the case of constant  $p \in (1, \infty)$  the boundedness of the maximal operator in  $L^p(X, \varrho)$  on measure metric spaces is known for Muckenhoupt weights  $\varrho \in A_p$ , see [1] and [18]. In the case of variable exponents  $p(\cdot)$  statements on the non-weighted boundedness

of the maximal operator were obtained in [6] and [9]. Weighted boundedness in the case of variable exponents was earlier obtained in the cases where  $X$  is a bounded open set  $\Omega \subset \mathbb{R}^n$  or a Carleson curve on the complex plane in [14], [15] for power type weights and in [12], [13], [17] for radial weights of the Zygmund-Bary-Steckin type class.

#### 4. TOOLS USED IN THE PROOFS

The proof of Theorem A is based on the following pointwise weighted estimate.

**Theorem 1.** *Let  $\mu(X) < \infty$ ,  $p(x)$  satisfy conditions (2) and (3),  $x_0 \in X$  and let  $\varrho(x) = [d(x_0, x)]^\beta$ . If  $0 \leq \beta < \frac{m(\mu B)}{p'(x_0)}$ , then*

$$\left[ \frac{\varrho(x)}{\mu B(x, r)} \int_{B_r(x)} \frac{|f(y)|}{\varrho(y)} dy \right]^{p(x)} \leq C \left( 1 + \frac{1}{\mu B(x, r)} \int_{B(x, r)} |f(y)|^{p(y)} dy \right)$$

for all  $f \in L^{p(\cdot)}(\Omega)$  such that  $\|f\|_{p(\cdot)} \leq c < \infty$ , where  $C = C(c, p, \beta) < \infty$  is a constant not depending on  $x, r$  and  $x_0$ .

When dealing with radial weights we use properties of weights belonging to the Zygmund-Bary-Steckin class  $\Phi_\beta^\alpha$  defined as  $\Phi_\beta^\alpha := \mathcal{Z}^\alpha \cap \mathcal{Z}_\beta$ , where  $\mathcal{Z}^\alpha$  is the class of functions  $w \in \widetilde{W}$  satisfying the condition  $(\mathcal{Z}^\alpha) : \int_0^h \frac{w(t)}{t^{1+\alpha}} dt \leq c \frac{w(h)}{h^\alpha}$  and  $\mathcal{Z}_\beta$  is the class of functions  $w \in \widetilde{W}$  satisfying the condition  $(\mathcal{Z}_\beta) : \int_h^\ell \frac{w(t)}{t^{1+\beta}} dt \leq c \frac{w(h)}{h^\beta}$ , where  $c = c(w) > 0$  does not depend on  $h \in (0, \ell]$ ,  $-\infty < \alpha < \beta < \infty$ .

Similarly, let  $\Psi_\alpha^\beta := \widehat{\mathcal{Z}}^\beta \cap \widehat{\mathcal{Z}}_\alpha$ , where  $\widehat{\mathcal{Z}}^\beta = \{w \in \widetilde{W}([l, \infty)) : \int_r^\infty (\frac{r}{t})^\beta \frac{w(t)}{t} dt \leq cw(r), \text{ as } r \rightarrow \infty\}$  and  $\widehat{\mathcal{Z}}_\alpha = \{w \in \widetilde{W}([l, \infty)) : \int_\ell^r (\frac{r}{t})^\alpha \frac{w(t)}{t} dt \leq cw(r), \text{ } r \rightarrow \infty\}$ ,  $\ell > 0$ . In particular, the following statement, proved in [20],[21] for  $\alpha = 0, \beta = 1$  and [8] for the general case: *Let  $w \in \widetilde{W}([0, \ell])$ ,  $0 < \ell < \infty$ . Then  $w \in \mathcal{Z}^\alpha$  if and only if  $\alpha < m(w) < \infty$ , and  $w \in \mathcal{Z}_\beta$ ,  $\beta > 0$ , if and only if  $-\infty < M(w) < \beta$ , so that*

$$w \in \Phi_\beta^\alpha \iff \alpha < m(w) \leq M(w) < \beta.$$

Besides this, for  $w \in \Phi_\beta^\alpha$  and any  $\varepsilon > 0$  there exist constants  $c_1 = c_1(\varepsilon) > 0$  and  $c_2 = c_2(\varepsilon) > 0$  such that  $c_1 t^{M(w)+\varepsilon} \leq w(t) \leq c_2 t^{m(w)-\varepsilon}$ ,  $0 \leq t \leq \ell$ .

The role of the indices  $m(w_k), M(w_k)$  of weights involved in (5) and (6) and of similar indices related to the measure  $\mu$  may be seen from the following lemmas.

**Lemma 1.** *Let  $X$  be an unbounded doubling metric measure space and let  $\alpha > M_\infty(\mu B)$ . Then for every  $0 < \varepsilon < \alpha - M_\infty(\mu B)$  there exists a constant  $C = C(\varepsilon)$ , not depending on  $x$  and  $r$  such that*

$$\int_{X \setminus B(x, r)} \frac{d\mu(y)}{[d(x, y)]^\alpha} \leq C(x) r^{M_\infty(\mu B) - \alpha + \varepsilon}, \quad 0 < r_0 \leq r < \infty,$$

**Lemma 2.** *Let  $X$  be a doubling metric measure space,  $\varrho$  a weight of form (5) and let  $p(\cdot)$  satisfy conditions (2), (3). Then*

$$\begin{aligned} -\frac{m(\mu B)}{p(x_k)} < m(w_k), \quad k = 1, 2, \dots, N, & \implies \varrho \in L_{loc}^{p(\cdot)}(X). \\ M(w_k) < \frac{m(\mu B)}{p'(x_k)}, \quad k = 1, 2, \dots, N, & \implies \frac{1}{\varrho} \in L_{loc}^{p'(\cdot)}(X). \end{aligned}$$

The following statement also holds.

**Lemma 3.** Let  $X$  be a metric measure space with doubling condition and let  $w \in \widetilde{W}([0, \ell])$ ,  $\ell = \text{diam } X$ . Then the inequality

$$\frac{w(d(x, x_0))}{\mu B(x, r)} \int_{B(x, r)} \frac{d\mu(y)}{w(d(y, x_0))} \leq c$$

holds with  $c > 0$  not depending on  $0 < r < \ell$  and  $x \in X$ , in one of the following cases:

- i)  $d(x, x_0) \geq Nr$  for some  $N \in \mathbb{R}_+^1$ ,
- ii)  $m(w) > 0$  when  $\ell < \infty$ , and  $\min\{m(w), m_\infty(w)\} > 0$  when  $\ell = \infty$ .

In the case  $d(x, x_0) \leq Nr$ , there also holds the estimate

$$\frac{w(r)}{\mu B(x, r)} \int_{B(x, r)} \frac{d\mu(y)}{w(d(y, x_0))} \leq c.$$

**Theorem 2.** Let  $w \in \widetilde{W}([0, \ell])$ ,  $\ell = \text{diam } X$ , let  $p(x)$  satisfy conditions (2)–(3), let  $x_0 \in X$  and the measure  $\mu$  satisfy the doubling condition. If the function  $w$  and the measure  $\mu$  satisfy the conditions

$$\int_0^r \frac{\mu B(x, t)[w(t)]^{p(x_0)}}{t} dt \leq C\mu B(x, r)[w(r)]^{p(x_0)}, \quad (13)$$

$$\int_0^r \frac{\mu B(x, t)}{t[w(t)]^{q_0}} dt \leq C \frac{\mu B(x, r)}{[w(r)]^{q_0}} \quad \text{with} \quad q_0 = \frac{p(x_0)}{p_- - 1}, \quad (14)$$

then the function  $\varrho(x) = w(d(x, x_0))$  belongs to  $\widetilde{A}_{p(\cdot)}(X)$  if either  $\ell < \infty$ , or  $\ell = \infty$  and  $p = \text{const}$ .

**Corollary.** Let  $w \in \widetilde{W}([0, \ell])$ ,  $\ell = \text{diam } X$  and the measure  $\mu$  satisfy the doubling condition, let  $x_0 \in X$  and  $p(x) = p = \text{const}$ . Then under conditions (13)–(14)  $w[d(x, x_0)] \in A_p(X)$ ,  $1 < p < \infty$ .

Note that known examples of weights on doubling metric spaces  $X$  belonging to  $A_p(X)$  even for constant  $p$  were powers  $[\mu B(x_0, d(x_0, x))]^\alpha$  of the measure, see [4], p.42.

The following statement was also proved.

**Theorem 3.** Let  $w \in \widetilde{W}([0, \ell])$ , let  $p(x)$  satisfy conditions (2)–(3) and let  $x_0 \in X$ .

I)  $\ell = \text{diam } X < \infty$ ; if

$$-\frac{m(\mu B)}{p(x_0)} < m(w) \leq M(w) < \frac{m(\mu B)}{q_0},$$

where  $q_0 = \frac{p(x_0)}{p_- - 1}$ , then the function  $\varrho(x) = w(d(x, x_0))$  satisfies condition (3).

II)  $\ell = \text{diam } X = \infty$ ; if  $p = \text{const}$ ,  $1 < p < \infty$  and

$$-\frac{m(\mu B)}{p} < m(w) \leq M(w) < \frac{m(\mu B)}{p'}$$

and

$$-\frac{m_\infty(\mu B)}{p} < m_\infty(w) \leq M_\infty(w) < \frac{m_\infty(\mu B)}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

then  $w(d(x, x_0)) \in A_p(X)$ .

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