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# Weighted Sobolev theorem in Lebesgue spaces with variable exponent

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## Abstract

For the Riesz potential operator  $I^\alpha$  there are proved weighted estimates

$$\|I^\alpha f\|_{L^{q(\cdot)}(\Omega, w^{\frac{q}{p}})} \leq C \|f\|_{L^{p(\cdot)}(\Omega, w)}, \quad \Omega \subseteq \mathbb{R}^n, \quad \frac{1}{q(x)} \equiv \frac{1}{p(x)} - \frac{\alpha}{n}$$

within the framework of weighted Lebesgue spaces  $L^{p(\cdot)}(\Omega, w)$  with variable exponent. In case  $\Omega$  is a bounded domain, the order  $\alpha = \alpha(x)$  is allowed to be variable as well. The weight functions are radial type functions “fixed” to a finite point and/or to infinity and have a typical feature of Muckenhoupt–Wheeden weights: they may oscillate between two power functions. Conditions on weights are given in terms of their Boyd-type indices. An analogue of such a weighted estimate is also obtained for spherical potential operators on the unit sphere  $\mathbb{S}^n \subset \mathbb{R}^n$ .

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**Keywords:** Sobolev theorem; Hardy inequality; Lebesgue spaces with variable exponents; Riesz potentials; Spherical potentials; Zygmund–Bari–Stechkin conditions

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## 1. Introduction

Last years harmonic analysis in variable exponent spaces attracts enormous interest of researchers due to both mathematical curiosity caused by the difficulties of investigation in variable exponent spaces and also by various applications. We refer in particular to papers [12,22] on the spaces  $L^{p(\cdot)}$  and papers [4,5,10,11,16] on the recent progress in operator theory and harmonic analysis in  $L^{p(\cdot)}$ , the theory of these spaces and the corresponding Sobolev spaces  $W^{m,p(\cdot)}$  with variable exponent being at present rapidly developing, influenced by applications given in [17], see also references therein. The main progress concerns in particular non-weighted theorems on  $p(\cdot) \rightarrow p(\cdot)$ -boundedness of the Hardy–Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \quad (1.1)$$

and  $p(\cdot) \rightarrow q(\cdot)$ -boundedness of the Riesz potential operator

$$I^\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y) dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n. \quad (1.2)$$

There are various challenging open problems in this topic; we refer to the existing surveys [6,8] and [20], one of those problems being related to the weighted theory: to find an analog of the Muckenhoupt condition for the maximal operator and the Muckenhoupt–Wheeden condition [15] for the Riesz potential operator.

In this paper we deal with the weighted estimates for the Riesz potential operator.

A non-weighted Sobolev-type  $p(\cdot) \rightarrow q(\cdot)$ -theorem for variable exponents was first obtained in [21] for bounded domains  $\Omega \subset \mathbb{R}^n$  under the assumption that the maximal operator is bounded in  $L^{p(\cdot)}(\Omega)$ , which became unconditional statement after L. Diening’s result [5] on the boundedness of the maximal operator. For unbounded domains the Sobolev theorem was proved in by C. Capone, D. Cruz-Uribe and A. Fiorenza [2] and D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Perez [3], a weaker version being given in V. Kokilashvili and S. Samko [10].

In the weighted theory, Sobolev embeddings were obtained for power weights. First, for bounded domains there was obtained the weighted  $p(\cdot) \rightarrow p(\cdot)$ -boundedness in V. Kokilashvili and S. Samko [11] and the  $p(\cdot) \rightarrow q(\cdot)$ -inequality with the limiting exponent  $q(\cdot)$  in S. Samko [26]. As is known, the case of unbounded domains is more difficult (when  $p(\cdot)$  is allowed to be variable up to infinity). A generalization of the Stein–Weiss inequality [26], that is, Sobolev embedding with power weight on unbounded domains for variable exponents was considered in [24], where this generalization was obtained with a certain additional restriction on the parameters involved. This restriction was withdrawn in [25]. The progress in [24,25] became possible under the log-condition on the variable exponent at infinity, a little bit stronger than the decay condition.

Up to now, for variable exponents no Stein–Weiss-type inequality with weights more general than power ones was obtained. A characterization of general weights admissible for the Sobolev embedding is still unknown. In the case of variable exponents, one may easily try to write down an analogue of the Muckenhoupt–Wheeden condition in terms of the corresponding norms, but whether this guarantees the weighted boundedness of the Riesz potential, remains an open problem. The generalization to the case of more general weights encountered essential difficulties, caused both by the general reasons—non-invariance of the variable exponent spaces with respect

to translations and dilations—and by the absence of results for the maximal operator with general weights.

In this paper we partially fill in the existing gap. The main novelty of the results obtained in this paper is admission of a certain class of general weights  $w(|x - x_0|)$ ,  $x_0 \in \overline{\Omega}$ , of radial-type (in the case of unbounded domains we admit radial type weights “fixed” also to infinity). We generalize results obtained in [24,25] to the case of general “radial-type” weights by admitting weights which have a typical feature of Muckenhoupt–Wheeden weights: they may oscillate between two power functions. The class of admissible weights may be considered as a kind of Zygmund–Bary–Stechkin class. The idea to use the Zygmund–Bary–Stechkin class of weights is based on the observation that the integral constructions involved in the Muckenhoupt condition for radial weights (in the case of constant  $p$ ) are exactly those which appear in the Zygmund conditions.

A new point is that the conditions for the validity of the  $p(\cdot) \rightarrow q(\cdot)$ -estimate (with the limiting exponent) are given in terms of the so called index numbers  $m(w)$  and  $M(w)$  of the weights  $w(r)$  (similar in a sense to the Boyd indices). These conditions are obtained in the form of the natural numerical intervals

$$\alpha p(x_0) - n < m(w) \leq M(w) < n[p(x_0) - 1] \quad (1.3)$$

“localized” to the points  $x_0$  to which the radial weights  $w(|x - x_0|)$  are fixed. The sufficiency of this condition in terms of the numbers  $m(w)$  and  $M(w)$  is a new result even in the case of constant  $p$ . As is known, even in the case of constant  $p$  the verification of the Muckenhoupt–Wheeden condition for a concrete weight may be an uneasy task. Therefore, independently of finding an analogue of the Muckenhoupt–Wheeden condition for variable exponents, it is always of importance to find easy to check sufficient conditions for weight functions, as for instance in (1.3).

As a corollary to the weighted result for the Riesz potential operator in  $\mathbb{R}^n$  we also obtain a similar theorem for the spherical analogue

$$(K^\alpha f)(x) = \int_{\mathbb{S}_n} \frac{f(\sigma)}{|x - \sigma|^{n-\alpha}} d\sigma, \quad x \in \mathbb{S}^n, \quad 0 < \alpha < n, \quad (1.4)$$

of the Riesz potential in the weighted spaces  $L^{p(\cdot)}(\mathbb{S}^n, \rho)$  on the unit sphere  $\mathbb{S}^n$  in  $\mathbb{R}^{n+1}$ .

The main results of the paper are given in Theorems A–C, see Section 3. Theorem A contains a weighted result for bounded domains and in this case the order  $\alpha = \alpha(x)$  of the Riesz potential may be also variable. Theorem B provides the weighted result for the whole space  $\mathbb{R}^n$  in case of constant  $\alpha$ . Finally, Theorem C contains a similar result for spherical Riesz potentials. Section 2 contains necessary preliminaries. In Section 4 we prove a crucial lemma on estimation of norms of truncated weight functions. The proof of Theorems A–C is given in Sections 5–7, respectively.

## Notation.

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\};$$

$$\mathbb{S}^n \text{ is the unit sphere in } \mathbb{R}^{n+1}, \quad e_{n+1} = (0, 0, 0, \dots, 0, 1);$$

$$p_0 = \inf_{x \in \Omega} p(x), \quad P = \sup_{x \in \Omega} p(x), \quad p'(x) = \frac{p(x)}{p(x)-1};$$

$$\mathbb{P}(\Omega), \text{ see (2.2)–(2.3);}$$

$$W, \text{ see (2.9);}$$

$\tilde{W}$ , see (2.10);

$\Phi_\gamma^\beta$ , see Definition 2.2;

$\Psi_\gamma^\beta$ , see Definition 2.8;

by  $c$  or  $C$  we denote various positive absolute constants.

## 2. Preliminaries

### 2.1. On weighted Lebesgue spaces with variable exponent

We refer to [12,22] for details on the spaces  $L^{p(\cdot)}(\Omega)$ , but give the basic definitions. Let  $\Omega$  be an open set in  $\mathbb{R}^n$ ,  $p : \Omega \rightarrow [1, \infty)$  a measurable function on  $\Omega$  and

$$L^{p(\cdot)}(\Omega, \rho) = \left\{ f : [\rho(x)]^{\frac{1}{p(x)}} f(x) \in L^{p(\cdot)}(\Omega) \right\}, \quad L^{p(\cdot)}(\Omega) := L^{p(\cdot)}(\Omega, 1),$$

$$\|f\|_{L^{p(\cdot)}(\Omega, \rho)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \rho(x) \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \quad (2.1)$$

where  $\rho(x)$  will be of the form  $\rho(x) = w(|x - x_0|)$  with  $x_0 \in \overline{\Omega}$ .

By  $\mathbb{P}(\Omega)$  we denote the set of functions  $p : \overline{\Omega} \rightarrow (1, \infty)$  satisfying the conditions

$$1 < p_0 \leq p(x) \leq P < \infty \quad \text{on } \Omega, \quad (2.2)$$

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}} \quad \text{for all } x, y \in \overline{\Omega}, \text{ with } |x - y| \leq \frac{1}{2}, \quad (2.3)$$

where  $A > 0$  does not depend on  $x$  and  $y$ . In case of a bounded set  $\Omega$  condition (2.3) may be also written in the form

$$|p(x) - p(y)| \leq \frac{NA}{\ln \frac{N}{|x-y|}}, \quad x, y \in \overline{\Omega}, \quad N = 2 \text{ diam } \Omega. \quad (2.4)$$

Under condition (2.2) for the conjugate space  $[L^{p(\cdot)}(\Omega, \rho)]^*$  we have

$$[L^{p(\cdot)}(\Omega, \rho)]^* = L^{p'(\cdot)}(\Omega, [\rho(x)]^{\frac{1}{1-p(x)}}), \quad \frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad (2.5)$$

$$\left| \int_{\Omega} u(x)v(x) dx \right| \leq k \|u\|_{L^{p'(\cdot)}(\Omega, [\rho(x)]^{\frac{1}{1-p(x)}})} \|v\|_{L^{p(\cdot)}(\Omega, \rho)}. \quad (2.6)$$

We will also deal with a similar weighted space  $L^{p(\cdot)}(\mathbb{S}^n, \rho)$  with variable exponent on the unit sphere  $\mathbb{S}^n = \{\sigma \in \mathbb{R}^{n+1} : |\sigma| = 1\}$ , defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{S}^n, \rho)} = \left\{ \lambda > 0 : \int_{\mathbb{S}^n} \rho(\sigma) \left| \frac{f(\sigma)}{\lambda} \right|^{p(\sigma)} d\sigma \leq 1 \right\}.$$

Similarly to the Euclidean case, by  $\mathbb{P}(\mathbb{S}^n)$  we denote the set of exponents  $p(\sigma)$  on  $\mathbb{S}^n$  which satisfy the conditions  $1 < p_- \leq p(\sigma) \leq p_+ < \infty$ ,  $\sigma \in \mathbb{S}^n$ ,

$$|p(\sigma_1) - p(\sigma_2)| \leq \frac{A}{\ln \frac{3}{|\sigma_1 - \sigma_2|}}, \quad \sigma_1 \in \mathbb{S}^n, \sigma_2 \in \mathbb{S}^n.$$

2.2. On Zygmund–Bari–Stechkin classes  $\Phi_\gamma^\beta$  and  $\Psi_\gamma^\beta$

$1^0$ . Classes  $\Phi_\gamma^\beta = \Phi_\gamma^\beta([0, \ell])$ . By  $C_+([0, \ell])$ ,  $0 < \ell < \infty$ , we denote the class of functions  $w(t)$  on  $[0, \ell]$  continuous and positive at every point  $t \in (0, \ell]$  and having a finite or infinite limit  $\lim_{t \rightarrow 0} w(t) =: w(0)$ . A function  $\varphi \in C_+([0, \ell])$  is said to be almost increasing (or almost decreasing) if there exists a constant  $C \geq 1$  such that  $\varphi(x) \leq C\varphi(y)$  for all  $x \leq y$  (or  $x \geq y$ , respectively).

**Definition 2.1.** Let  $-\infty < \beta < \gamma < \infty$ . We define the class  $\mathcal{Z}^\beta = \mathcal{Z}^\beta([0, \ell])$  as the set of functions in  $C_+([0, \ell])$  satisfying the condition

$$\int_0^h \frac{w(x) dx}{x^{1+\beta}} \leq c \frac{w(h)}{h^\beta} \tag{2.7}$$

and the class  $\mathcal{Z}_\gamma = \mathcal{Z}_\gamma([0, \ell])$  as the set of functions  $w \in C_+([0, \ell])$  satisfying the condition

$$\int_h^\ell \frac{w(x) dx}{x^{1+\gamma}} \leq c \frac{w(h)}{h^\gamma}, \tag{2.8}$$

where  $c = c(w) > 0$  does not depend on  $h \in (0, \ell]$ .

Let

$$W_0 = \{\varphi \in C_+([0, \ell]): \varphi(x) \text{ is almost increasing}\} \tag{2.9}$$

and

$$\tilde{W}_0 = \{\varphi \in C_+([0, \ell]): \exists a = a(\varphi) \in \mathbb{R}^1 \text{ such that } x^a \varphi(x) \in W_0\}. \tag{2.10}$$

**Definition 2.2.** We define the Zygmund–Bari–Stechkin class  $\Phi_\gamma^\beta$ , as

$$\Phi_\gamma^\beta = \tilde{W}_0 \cap \mathcal{Z}^\beta \cap \mathcal{Z}_\gamma, \quad -\infty < \beta < \gamma < \infty. \tag{2.11}$$

The class  $\Phi_\gamma^\beta$  is a modification of the class  $\Phi_k^0, k = 0, 1, 2, \dots$ , introduced in [1], where increasing functions  $w$  were considered. We deal with almost monotonic functions as in [18]. We refer in particular to [7] for properties of functions in  $\Phi_\gamma^\beta$ . Observe that

$$\Phi_{\gamma_1}^{\beta_1} \subseteq \Phi_{\gamma_2}^{\beta_2} \subseteq \Phi_{\gamma_2}^0, \quad 0 \leq \beta_2 \leq \beta_1 \leq \gamma_1 \leq \gamma_2. \tag{2.12}$$

**Definition 2.3.** Let  $w \in C_+([0, \ell])$ . The numbers

$$m(w) = \sup_{x>1} \frac{\ln[\overline{\lim}_{h \rightarrow 0} \frac{w(xh)}{w(h)}]}{\ln x}, \quad M(w) = \inf_{x>1} \frac{\ln[\overline{\lim}_{h \rightarrow 0} \frac{w(xh)}{w(h)}]}{\ln x}$$

introduced in such a form in [18], will be referred to as the *lower and upper index numbers* of a function  $w$  (they are close to the Matuszewska–Orlicz indices, see [13, p. 20]).

Observe that

$$m[x^a w(x)] = a + m(w) \quad \text{and} \quad M[x^a w(x)] = a + M(w) \tag{2.13}$$

and

$$m\left(\frac{1}{w}\right) = -M(w), \quad M\left(\frac{1}{w}\right) = -m(w). \tag{2.14}$$

The following statement characterizes the class  $\Phi_\gamma^\beta$  in terms of the indices  $m(w)$  and  $M(w)$ , see its proof in [18, p. 125] for the case  $\beta = 0, \gamma = 1$  and in [7] for the case  $0 \leq \beta < \gamma < \infty$ . The validity of Theorem 2.4 for all  $-\infty < \beta < \gamma < \infty$  follows from the possibility (2.13) to shift the indices.

**Theorem 2.4.** *Let  $-\infty < \beta < \gamma < \infty$ . Then*

I. *A function  $w(x) \in \tilde{W}_0$  is in the Bari–Stechkin class  $\Phi_\gamma^\beta$  if and only if*

$$\beta < m(w) \leq M(w) < \gamma. \tag{2.15}$$

*Besides this, the condition  $m(w) > \beta$  for  $w \in \tilde{W}_0$  is equivalent to inequality (2.7), while the condition  $M(w) < \gamma$  is equivalent to (2.8).*

II. *For  $w \in \Phi_\gamma^\beta$  and every  $\varepsilon > 0$  there exist constants  $c_1 = c_1(w, \varepsilon) > 0$  and  $c_2 = c_2(w, \varepsilon) > 0$  such that*

$$c_1 t^{M(w)+\varepsilon} \leq w(t) \leq c_2 t^{m(w)-\varepsilon}, \quad 0 \leq t \leq \ell. \tag{2.16}$$

III. *If  $w \in \tilde{W}_0 \cap \mathcal{Z}^\beta$ , then  $\frac{w(t)}{t^{\delta_1}}$  is almost increasing for every  $\delta_1 < m(w)$ ; if  $w \in \tilde{W}_0 \cap \mathcal{Z}_\gamma$ , then  $\frac{w(t)}{t^{\delta_2}}$  is almost decreasing for every  $\delta_2 > M(w)$ .*

**Corollary 2.5.** *Let  $0 < \gamma < \infty$ . For every  $w \in \Phi_\gamma^0$  there exists a  $\delta = \delta(w) > 0$  such that  $w \in \Phi_{\gamma-\delta}^0$ .*

**Proof.** Indeed, from part I of Theorem 2.4 it follows that one may take any  $\delta$  in the interval  $0 < \delta < \gamma - M(w)$ .  $\square$

**Lemma 2.6.** *Let a function  $\rho(t) \in C_+([0, \ell])$  have the property: there exist  $a, b \in \mathbb{R}^1$  such that  $t^a \rho(t)$  is almost increasing and  $t^b \rho(t)$  is almost decreasing. Then  $c_1 \rho(\tau) \leq \rho(t) \leq c_2 \rho(\tau)$  for all  $t, \tau \in [0, \ell]$  such that  $\frac{1}{2} \leq \frac{t}{\tau} \leq 2$ , where  $c_1$  and  $c_2$  do not depend on  $t, \tau$ .*

**Proof.** The proof is a matter of direct verification.  $\square$

The following lemma was proved in [9] (see Lemma 4.1 in [9]).

**Lemma 2.7.** *Let  $\Omega$  be an open bounded set,  $x_0 \in \overline{\Omega}$ , let  $w \in \Phi_\gamma^\beta, -\infty < \beta < \gamma < \infty$ , and let  $p(x)$  be a bounded function on  $\Omega$  satisfying the condition  $|p(x) - p(x_0)| \leq \frac{C}{\ln \frac{N}{|x-x_0|}}, x \in \Omega, N = 2 = \text{diam } \Omega$ . Then*

$$c_1 [w(|x - x_0|)]^{p(x_0)} \leq [w(|x - x_0|)]^{p(x)} \leq c_2 [w(|x - x_0|)]^{p(x_0)}, \tag{2.17}$$

where  $c_1 > 0$  and  $c_2 > 0$  do not depend on  $x$ .

$2^0$ . Classes  $\Psi_\gamma^\beta = \Psi_\gamma^\beta([\ell, \infty))$ . Let  $C_+([\ell, \infty))$ ,  $0 < \ell < \infty$ , be the class of functions  $w(t)$  on  $[\ell, \infty)$ , continuous and positive at every point  $t \in [\ell, \infty)$  and having a finite or infinite limit  $\lim_{t \rightarrow \infty} w(t) =: w(\infty)$ . We also denote

$$\tilde{W}_\infty = \{\varphi \in C_+([0, \ell]): \exists a \in \mathbb{R}^1: x^a \varphi(x) \text{ is almost decreasing}\}. \tag{2.18}$$

**Definition 2.8.** Let  $-\infty < \gamma < \beta < \infty$ . We define the class  $\hat{Z}^\beta = \hat{Z}^\beta([0, \ell])$  as the set of functions in  $C_+([\ell, \infty))$  satisfying the condition

$$\int_r^\infty \left(\frac{r}{t}\right)^\beta \frac{w(t) dt}{t} \leq cw(r), \quad r \rightarrow \infty, \tag{2.19}$$

and the class  $\hat{Z}_\gamma = \hat{Z}_\gamma([\ell, \infty))$ ,  $0 < \ell < \infty$ , as the set of functions  $w \in C_+([\ell, \infty))$  satisfying the condition

$$\int_\ell^r \left(\frac{r}{t}\right)^\gamma \frac{w(t) dt}{t} \leq cw(r), \quad r \rightarrow \infty, \tag{2.20}$$

where  $c = c(w) > 0$  does not depend on  $r \in (0, \ell]$ . We define the class  $\Psi_\gamma^\beta$ ,  $-\infty < \beta < \gamma < \infty$ , as

$$\Psi_\gamma^\beta = \tilde{W}_\infty \cap \hat{Z}^\beta \cap \hat{Z}_\gamma. \tag{2.21}$$

The indices  $m(w)$  and  $M(w)$  responsible for the behavior of functions  $w$  at infinity are introduced in the way similar to Definition 2.3:

$$m(w) = \sup_{x>1} \frac{\ln[\overline{\lim}_{h \rightarrow \infty} \frac{w(xh)}{w(h)}]}{\ln x}, \quad M(w) = \inf_{x>1} \frac{\ln[\overline{\lim}_{h \rightarrow \infty} \frac{w(xh)}{w(h)}]}{\ln x}.$$

One can easily reformulate properties of functions of the class  $\Phi_\gamma^\beta$  near the origin, given in Theorem 2.4 and Lemma 2.7 for the case of the corresponding behavior at infinity of functions of the class  $\Psi_\gamma^\beta$ . This reformulation is an easy task since for  $w \in C_+([\ell, \infty))$  one has  $w_*(t) := w(\frac{1}{t}) \in C_+([0, \frac{1}{\ell}])$  and the direct calculation shows that

$$m(w) = -M(w_*), \quad M(w) = -m(w_*). \tag{2.22}$$

Observe in particular that

$$w \in \Psi_\gamma^\beta([1, \infty)) \iff w_* \in \Phi_{-\gamma}^{-\beta}([0, 1]) \tag{2.23}$$

and the analogue of property (2.16) for functions in  $\Psi_\gamma^\beta([1, \infty))$  takes the form

$$c_1 t^{m(w)-\varepsilon} \leq w(t) \leq c_2 t^{M(w)+\varepsilon}, \quad t \geq \ell, w \in \Psi_\gamma^\beta([\ell, \infty)). \tag{2.24}$$

### 2.3. On the maximal operator

The following statement was proved in [9].

**Theorem 2.9.** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$ ,  $\ell = \text{diam } \Omega$  and let  $p \in \mathbb{P}(\Omega)$ . The operator  $\mathcal{M}$  is bounded in the space  $L^{p(x)}(\Omega, w)$  with the weight  $w(|x - x_0|)$ ,  $x_0 \in \overline{\Omega}$ , if

$$w(r) \in \Phi_\gamma^\beta([0, \ell]) \quad \text{with } \beta = -n \text{ and } \gamma = n[p(x_0) - 1]$$

or equivalently

$$w \in \widetilde{W}_0 \quad \text{and} \quad -n < m(w) \leq M(w) < n[p(x_0) - 1]. \tag{2.25}$$

### 3. The main statements

In the case of bounded domain  $\Omega$  we admit that the order  $\alpha$  of the operator  $I^\alpha$  may be also variable, so we deal with the operator

$$I^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(y) dy}{|x - y|^{n-\alpha(x)}}, \quad x \in \Omega. \tag{3.1}$$

We assume that the exponent  $\alpha(x)$  in (3.1) satisfies the assumptions

$$\inf_{x \in \Omega} \alpha(x) > 0 \quad \text{and} \quad \sup_{x \in \Omega} \alpha(x)p(x) < n \tag{3.2}$$

and the logarithmic condition

$$|\alpha(x) - \alpha(y)| \leq \frac{A_1}{\ln \frac{1}{|x-y|}}, \quad x, y \in \Omega, \quad |x - y| \leq \frac{1}{2}. \tag{3.3}$$

Everywhere in the sequel the exponent  $q(x)$  is defined by  $\frac{1}{q(x)} \equiv \frac{1}{p(x)} - \frac{\alpha(x)}{n}$ .

In [23] the following statement for power weights was proved.

**Theorem 3.1.** Let  $\Omega$  be an open bounded set in  $\mathbb{R}^n$  and  $x_0 \in \overline{\Omega}$ , let  $p \in \mathbb{P}(\Omega)$  and  $\alpha$  satisfy conditions (3.2)–(3.3). Then

$$\left\| |x - x_0|^\gamma \int_\Omega \frac{f(y) dy}{|y - x_0|^\gamma |x - y|^{n-\alpha(x)}} \right\|_{L^{q(\cdot)}(\Omega)} \leq C \|f\|_{L^{p(\cdot)}(\Omega)}, \tag{3.4}$$

$$\text{if } \alpha(x_0) - \frac{n}{p(x_0)} < \gamma < \frac{n}{p'(x_0)}.$$

We prove the following generalization of Theorem 3.1.

**Theorem A.** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $x_0 \in \overline{\Omega}$ , let  $p \in \mathbb{P}(\Omega)$  and  $\alpha$  satisfy conditions (3.2)–(3.3). Let also

$$w(r) \in \Phi_\gamma^\beta([0, \ell]) \quad \text{with } \beta = \alpha(x_0)p(x_0) - n, \quad \gamma = n[p(x_0) - 1], \tag{3.5}$$

or equivalently

$$w \in \widetilde{W}_0 \quad \text{and} \quad \alpha(x_0)p(x_0) - n < m(w) \leq M(w) < n[p(x_0) - 1]. \tag{3.6}$$

Then

$$\|I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega, w^{\frac{q}{p}}(|x-x_0|))} \leq C \|f\|_{L^{p(\cdot)}(\Omega, w(|x-x_0|))}. \tag{3.7}$$



The proof of Theorem A will be based on a development—to the case of Zygmund–Bari–Stechkin class—of the technique of weighted estimation of  $L^{p(\cdot)}$ -norms of power functions of distance truncated to exterior of a ball, used in [21] and [23], on properties of such weights developed in [7,18,19] and on Theorem 2.9.

The above mentioned development of the technique from [21,23] is given in the next section. Theorem A itself is proved in Section 5.

For the case of the whole space  $\mathbb{R}^n$  we consider  $\alpha(x) = \alpha = \text{const}$  as in (1.2) and deal with the weight is “fixed” to a finite point  $x_0 = 0$  and to infinity:

$$w(x) = w_0(|x|)w_\infty(|x|), \tag{3.8}$$

where  $w_0(r)$  belongs to some  $\Phi_\gamma^\beta$ -class on  $[0, 1]$  and  $w_\infty(r)$  belongs to some  $\Psi_\gamma^\beta$ -class on  $[1, \infty)$  and both the weights are continued by constant to  $[0, \infty)$ :

$$w_0(r) \equiv w_0(1), \quad 1 \leq r < \infty, \quad \text{and} \quad w_\infty(r) \equiv w_\infty(1), \quad 0 < r \leq 1.$$

As in [25], we also need the following form of the log-condition at infinity:

$$|p_*(x) - p_*(y)| \leq \frac{A_\infty}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^n, \tag{3.9}$$

where

$$p_*(x) = p\left(\frac{x}{|x|^2}\right).$$

**Theorem B.** Let  $0 < \alpha < n$  and let  $p \in \mathbb{P}(\mathbb{R}^n)$  satisfies assumption (3.9) and condition  $\sup_{x \in \mathbb{R}^n} p(x) < \frac{n}{\alpha}$ . The operator  $I^\alpha$  is bounded from  $L^{p(\cdot)}(\mathbb{R}^n, w)$  to  $L^{q(\cdot)}(\mathbb{R}^n, w^{\frac{q}{p}})$ , where  $w(x)$  is the weight of form (3.8) and

$$w^{\frac{q}{p}}(x) = [w_0(|x|)]^{\frac{q(0)}{p(0)}} [w_\infty(|x|)]^{\frac{q(\infty)}{p(\infty)}}, \tag{3.10}$$

if

$$w_0(r) \in \Phi_{\gamma_0}^{\beta_0}([0, 1]), \quad w_\infty(r) \in \Psi_{\gamma_\infty}^{\beta_\infty}([1, \infty)), \tag{3.11}$$

where  $\beta_0 = \alpha p(0) - n$ ,  $\gamma_0 = n[p(0) - 1]$ ,  $\beta_\infty = n[p(\infty) - 1]$ ,  $\gamma_\infty = \alpha p(\infty) - n$ , or equivalently

$$w_0 \in \tilde{W}([0, 1]), \quad \alpha p(0) - n < m(w_0) \leq M(w_0) < n[p(0) - 1], \tag{3.12}$$

and

$$w_\infty \in \tilde{W}([1, \infty)), \quad \alpha p(\infty) - n < m(w_\infty) \leq M(w_\infty) < n[p(\infty) - 1]. \tag{3.13}$$

For the spherical potential operator (1.4) a similar result runs as follows

**Theorem C.** Let  $p \in \mathbb{P}(\mathbb{S}^n)$  and  $\sup_{\sigma \in \mathbb{S}^n} p(\sigma) < \frac{n}{\alpha}$ . Let  $a \in \mathbb{S}^n$  and  $w = w(|\sigma - a|)$ . The spherical potential operator  $K^\alpha$  is bounded from the space  $L^{p(\cdot)}(\mathbb{S}^n, w)$  into the space  $L^{q(\cdot)}(\mathbb{S}^n, w^{\frac{q(a)}{p(a)}})$ , where  $\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}$ , if

$$w \in \Phi_\gamma^\beta([0, 2]), \tag{3.14}$$

with  $\beta = \alpha p(a) - n$  and  $\gamma = n[p(a) - 1]$ , or equivalently

$$w \in \tilde{W} \quad \text{and} \quad \alpha p(a) - n < m(w) \leq M(w) < n[p(a) - 1]. \tag{3.15}$$

4. Estimation of  $\| |x - y|^{-\beta(x)} \chi_r(x - y) \|_{L^{p(\cdot)}(\Omega, \rho)}$

In all the proofs in the sequel, when we pass to a neighborhood  $\{x \in \Omega: |x - x_0| < \delta\}$  of the point  $x_0$ , in the case  $x_0 \in \partial\Omega$  we consider, whenever necessary, the function  $f(x)$  as continued to the whole neighborhood  $\{x \in \mathbb{R}^n: |x - x_0| < \delta\}$  as zero and the exponent  $p(x)$  continued with conservation of the log-property, which is always possible.

4.1. A technical lemma

Let

$$A(x, r) := \int_{\substack{y \in \Omega \\ |y-x| > r}} |y - x|^{-n-a(x)} \rho(|y - x_0|) dy, \quad x_0 \in \Omega. \tag{4.1}$$

**Lemma 4.1.** Let  $d =: \inf_{x \in \Omega} a(x) > 0$  and

$$\rho(t) \in \Phi_d^{-n}([0, \ell]), \quad \ell = \text{diam } \Omega. \tag{4.2}$$

Then the following estimate holds

$$A(x, r) \leq Cr^{-a(x)} \rho(r_x), \quad r_x = \max(r, |x - x_0|), \tag{4.3}$$

where  $C > 0$  does not depend on  $x \in \Omega$  and  $r \in (0, \ell]$ .

**Proof.** We take  $x_0 = 0$  for simplicity and consider separately the cases  $|x| \leq \frac{r}{2}$ ,  $\frac{r}{2} \leq |x| \leq 2r$ ,  $|x| \geq 2r$ .

The case  $|x| \leq \frac{r}{2}$ . We have  $\frac{|y|}{|y-x|} \leq \frac{|y-x|+|x|}{|y-x|} \leq 1 + \frac{x}{r} \leq 2$  and similarly  $\frac{|y|}{|y-x|} \geq 1 - \frac{|x|}{r} \geq \frac{1}{2}$ . Hence  $\frac{1}{2} \leq \frac{|y|}{|y-x|} \leq 2$ . Therefore, by Lemma 2.6 we have  $\rho(|y|) \leq C\rho(|x - y|)$ . Consequently,

$$A(x, r) \leq C \int_{\substack{y \in \Omega \\ |y-x| > r}} |x - y|^{-n-a(x)} \rho(|x - y|) dy \leq C \int_r^\ell t^{-1-a(x)} \rho(t) dt.$$

The inequality  $\int_r^\ell t^{-1-a(x)} \rho(t) dt \leq Cr^{-a(x)} \rho(r)$  with  $C > 0$  not depending on  $x$  and  $r$ , is valid. Indeed, this is nothing else but the statement that  $\rho(t) \in \mathcal{Z}_{a(x)}$  uniformly in  $x \in \Omega$ , which holds because condition (4.2) implies the validity of the uniform inclusion  $\rho(t) \in \mathcal{Z}_{a(x)}$  by property (2.12). Therefore,

$$A(x, r) \leq Cr^{-a(x)} \rho(r). \tag{4.4}$$

The case  $\frac{r}{2} \leq |x| \leq 2r$ . We split the integration in  $A(x, r)$  as follows

$$\begin{aligned} A(x, r) &= \int_{\substack{y \in \Omega \\ r < |y-x| < 2|x|}} |y - x|^{-n-a(x)} \rho(|y|) dy + \int_{\substack{y \in \Omega \\ |y-x| > 2|x|}} |x - y|^{-n-a(x)} \rho(|y|) dy \\ &=: \mathfrak{J}_1 + \mathfrak{J}_2. \end{aligned}$$

For  $\mathfrak{J}_1$  we have

$$\mathfrak{J}_1 \leq r^{-n-a(x)} \int_{\substack{y \in \Omega \\ r < |y-x| < 2|x|}} \rho(|y|) dy.$$

Observe that

$$|y-x| > r \implies |y| \leq |y-x| + |x| \leq |y-x| + 2r \leq 3|y-x|.$$

Consequently,

$$\begin{aligned} \mathfrak{J}_1 &\leq r^{-n-a(x)} \int_{\substack{|y-x| < 2|x| \\ |y| \leq 3|y-x|}} \rho(|y|) dy \leq r^{-n-a(x)} \int_{|y| < 6|x|} \rho(|y|) dy \\ &= Cr^{-n-a(x)} \int_0^{6|x|} t^{n-1} \rho(t) dt. \end{aligned}$$

Since  $t^n \rho(t) \in \Phi_{n+\delta}^0$ , we obtain  $\mathfrak{J}_1 \leq Cr^{-a(x)} \rho(6|x|) \leq Cr^{-a(x)} \rho(|x|)$ . The estimate for  $\mathfrak{J}_2 = A(x, 2|x|)$  is contained in (4.4) with  $r = 2|x|$ .

The case  $|x| \geq 2r$ . We have

$$\begin{aligned} A(x, r) &= \int_{\substack{y \in \Omega \\ r < |y-x| < \frac{1}{2}|x|}} |x-y|^{h(x)} \rho(|y|) dy + \int_{\substack{y \in \Omega \\ |y-x| > \frac{1}{2}|x|}} |x-y|^{-n-a(x)} \rho(|y|) dy \\ &=: \mathfrak{J}_3 + \mathfrak{J}_4. \end{aligned}$$

For the term  $\mathfrak{J}_3$  we have  $\frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x|$ , so that  $\rho(|y|) \leq C\rho(|x|)$  by Lemma 2.6. Therefore,

$$\mathfrak{J}_3 \leq C\rho(|x|) \int_r^{\frac{|x|}{2}} t^{-1-a(x)} dt \leq Cr^{-a(x)} \rho(|x|), \quad |x| \geq 2r. \tag{4.5}$$

The term  $\mathfrak{J}_4$ , coincides with  $A(x, \frac{|x|}{2})$  and its estimate is contained in the preceding case  $\frac{r}{2} \leq |x| \leq 2r$ .

Gathering all the estimates, we arrive at (4.3).  $\square$

#### 4.2. The principal estimate

Let

$$\chi_r(x) = \begin{cases} 1, & \text{if } |x| > r, \\ 0, & \text{if } |x| < r \end{cases}$$

and  $\rho = \rho(|y-x_0|)$ . For the proof of Theorem A we need to estimate the norm

$$n_{\beta, p, \rho}(x, r) = \left\| |x-y|^{-\beta(x)} \chi_r(x-y) \right\|_{L^{p(\cdot)}(\Omega, \rho)} \tag{4.6}$$

as  $r \rightarrow 0$ , the norm being taken with respect to  $y$ . (We will need it with  $\beta(x) = n - \alpha(x)$  and  $\rho(\cdot)$  replaced by  $p'(\cdot)$ .)

**Theorem 4.2.** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$ ,  $x_0 \in \overline{\Omega}$ , and let  $p \in \mathbb{P}(\Omega)$  and  $\beta \in L^\infty(\Omega)$ .  
If

$$\operatorname{ess\,inf}_{x \in \Omega} \beta(x) p(x) > n, \tag{4.7}$$

$$t^n \rho(t) \in \Phi_\gamma^0, \quad \text{with } \gamma = \operatorname{ess\,inf}_{x \in \Omega} \beta(x) p(x), \tag{4.8}$$

then

$$n_{\beta,p,\rho}(x,r) \leq Cr^{\frac{n}{p(x)} - \beta(x)} [\rho(r_x)]^{\frac{1}{p(x)}}, \quad r_x = \max(r, |x - x_0|) \tag{4.9}$$

for all  $x \in \Omega$ ,  $0 < r < \ell = \operatorname{diam} \Omega$ , where  $C > 0$  does not depend on  $x$  and  $r$ .

**Proof.** We follow ideas of the proof of similar estimation for the power weight in [23, Theorem 4.2]; complications arising from the general weight are overcome by means of the properties of Zygmund–Bari–Stechkin weights presented in Section 2.2. For simplicity we take  $x_0 = 0$ . By definition (2.1) of the norm we have

$$\int_{\substack{y \in \Omega \\ |y-x| > r}} \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p(y)} \rho(|y|) dy = 1. \tag{4.10}$$

*1st step.* Values  $n_{\beta,p,\rho}(x,r) \geq 1$  are only of interest. This follows from the fact that the right-hand side of (4.9) is bounded from below:

$$\inf_{\substack{x \in \Omega \\ 0 < r < \operatorname{diam} \Omega}} r^{n-\beta(x)p(x)} \rho(r_x) := c_1 > 0, \quad r_x = \max(r, |x|). \tag{4.11}$$

To verify (4.11), note that from the condition  $\beta(x) p(x) > n$  there follows that

$$r^{n-\beta(x)p(x)} \rho(r_x) \geq \inf \{ r^{n-\beta(x)p(x)} \rho(r), |x|^{n-\beta(x)p(x)} \rho(|x|) \} \geq \min \left\{ C, \inf_{0 < r < 1} r^{n-\gamma} \rho(r) \right\}$$

and to arrive at (4.11), it remains to observe that  $r^{n-\gamma} \rho(r)$  is bounded from below: from the condition  $r^n \rho(r) \in \Phi_\gamma^0$  it follows that  $\frac{r^n \rho(r)}{r^\gamma}$  is almost decreasing (see part III of Theorem 2.4) and consequently bounded from below.

*2nd step.* Small values of  $r$  are only of interest. We assume that  $r$  is small enough,  $0 < r < \varepsilon_0$ . To show that this assumption is possible, we have to check that the right-hand side of (4.9) is bounded from below and  $n_{\beta,p,\rho}(x,r)$  is bounded from above when  $r \geq \varepsilon_0$ . The former was proved at the step 1 even for all  $r > 0$ . To verify the latter for  $r \geq \varepsilon_0$ , we observe that from (4.10) and from the fact that  $n_{\beta,p,\rho} \geq 1$  it follows that  $1 \leq \int_{y \in \Omega, |y-x| > \varepsilon_0} \frac{|y-x|^{-\beta(x)p(y)}}{n_{\beta,p,\rho}} \rho(|y|) dy$  whence

$$n_{\beta,p,\rho}(x,r) \leq \int_{\substack{y \in \Omega \\ |y-x| > \varepsilon_0}} |y-x|^{-\beta(x)p(x)} \rho(|y|) u(x,y) dy,$$

where  $u(x,y) = |y-x|^{-\beta(x)[p(y)-p(x)]}$ . Estimating  $\ln u(x,y)$ , we obtain that  $e^{-NAB} \leq u(x,y) \leq e^{NAB}$ ,  $x, y \in \Omega$ , where  $N$  and  $A$  are the constants from (2.4) and  $B = \sup_{x \in \Omega} \beta(x)$ . Therefore,

$$n_{\beta,p,\rho}(x,r) \leq C \int_{\substack{y \in \Omega \\ |y-x| > \varepsilon_0}} \frac{\rho(|y|) dy}{|y-x|^{\beta(x)p(x)}} \leq C \varepsilon_0^{-BP} \int \rho(|y|) dy = \operatorname{const}$$

1 since  $\rho \in L_1(\Omega)$  which is easily derived from condition (4.8). This proves the boundedness of  
 2  $n_{\beta,p,\rho}(x, r)$  from above.

3 The value of  $\varepsilon_0$  will be chosen later.

4 *3rd step. Rough estimate.* First, we derive a weaker estimate

$$5 \quad n_{\beta,p,\rho}(x, r) \leq Cr^{-\beta(x)} \tag{4.12}$$

6 which will be used later to obtain the final estimate (4.9). To this end, we note that always  
 7  $\lambda^{p(y)} \leq \lambda^{\inf p(y)} + \lambda^{\sup p(y)}$ , so that from (4.10) we have

$$8 \quad 1 \leq \int_{\substack{y \in \Omega \\ |y-x| > r}} \left[ \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p_0} + \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^P \right] \rho(|y|) dy.$$

9 Since  $|y-x| > r$ , we obtain

$$10 \quad 1 \leq \left[ \left( \frac{r^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p_0} + \left( \frac{r^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^P \right] \int_{y \in \Omega} \rho(|y|) dy.$$

11 Hence

$$12 \quad \left( \frac{r^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p_0} + \left( \frac{r^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^P \geq c$$

13 which yields  $\frac{r^{-\beta(x)}}{n_{\beta,p,\rho}} \geq C$  and we arrive at the estimate in (4.12).

14 *4th step.* We split integration in (4.10) as follows

$$15 \quad 1 = \sum_{i=1}^3 \int_{\Omega_i(x, \varepsilon_0)} \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p(y)} \rho(|y|) dy := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{4.13}$$

16 where

$$17 \quad \Omega_1(x, \varepsilon_0) = \left\{ y \in \Omega : r < |y-x| < \varepsilon_0, \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} > 1 \right\},$$

$$18 \quad \Omega_2(x, \varepsilon_0) = \left\{ y \in \Omega : r < |y-x| < \varepsilon_0, \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} < 1 \right\}, \quad \text{and}$$

$$19 \quad \Omega_3(x, \varepsilon_0) = \{ y \in \Omega : |y-x| > \varepsilon_0 \}.$$

20 *5th step. Estimation of  $\mathcal{I}_1$ .* We have

$$21 \quad \mathcal{I}_1 = \int_{\Omega_1(x, \varepsilon_0)} \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p(x)} \rho(|y|) u_r(x, y) dy, \tag{4.14}$$

22 where

$$23 \quad u_r(x, y) = \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\rho}} \right)^{p(y)-p(x)}.$$

24 By direct estimations it may be shown that  $u_r(x, y)$  is bounded from below and from above with  
 25 bounds not depending on  $x, y$  and  $r$  (see details in [23, p. 432]). Therefore,

$$\mathcal{I}_1 \leq \frac{C}{n_{\beta,p,\rho}^{p(x)}} \int_{\substack{y \in \Omega \\ |y-x| > r}} |y-x|^{-\beta(x)p(x)} \rho(|y|) dy. \tag{4.15}$$

We may use here estimate (4.3), which is applicable by (4.8) and (4.7), and get

$$\mathcal{I}_1 \leq \frac{C}{n_{\beta,p,\rho}^{p(x)}} r^{n-\beta(x)p(x)} \rho(r_x). \tag{4.16}$$

6th step. Estimation of  $\mathcal{I}_2$  and the choice of  $\varepsilon_0$ . In the integral  $\mathcal{I}_2$  we have

$$\mathcal{I}_2 \leq C \int_{\Omega_2(x,\varepsilon_0)} \left( \frac{|y-x|^{-\beta(x)}}{n_{\beta,v,p}} \right)^{p_{\varepsilon_0}(x)} \rho(|y|) dy \tag{4.17}$$

where  $p_{\varepsilon_0}(x) = \min_{|y-x| < \varepsilon_0} p(y)$ . Then

$$\mathcal{I}_2 \leq \frac{C}{n_{\beta,p,\rho}^{p_{\varepsilon_0}(x)}} \int_{\substack{y \in \Omega \\ |y-x| > r}} |y-x|^{-\beta(x)p_{\varepsilon_0}(x)} \rho(|y|) dy. \tag{4.18}$$

To be able to apply estimate (4.3), we have to guarantee the validity of the corresponding condition (4.2). To this end, we will have to choose  $\varepsilon_0$  sufficiently small. By conditions (4.8) and (4.7) and Corollary 2.5, there exists a small  $\delta \in (0, \gamma - n)$  such that  $t^n \rho(t) \in \Phi_{\gamma-\delta}^0$ ,  $\gamma = \inf_{x \in \Omega} \beta(x)p(x)$ . Since  $p(x)$  is continuous and  $\beta(x)$  is bounded, we may choose  $\varepsilon_0$  small enough so that  $\beta(x)p_{\varepsilon_0}(x) > \gamma - \delta > n$ . Then condition (4.2) for  $a(x) = a_{\varepsilon_0}(x) = \beta(x)p_{\varepsilon_0}(x) - n$  are satisfied and estimate (4.3) is applicable. It gives

$$\mathcal{I}_2 \leq \frac{C}{n_{\beta,p,\rho}^{p_{\varepsilon_0}(x)}} r^{n-\beta(x)p_{\varepsilon_0}(x)} \rho(r_x), \tag{4.19}$$

where  $C$  does not depend on  $x$  and  $r$ .

7th step. Estimation of  $\mathcal{I}_3$ . We have

$$\mathcal{I}_3 \leq \frac{C}{n_{\beta,p,\rho}^{p_0}} \mathcal{I}_4, \quad \mathcal{I}_4 = \mathcal{I}_4(x) = \int_{\substack{y \in \Omega \\ |y-x| > \varepsilon_0}} |y-x|^{-\beta(x)p(y)} \rho(|y|) dy.$$

The integral  $\mathcal{I}_4(x)$  is obviously a bounded function of  $x$ . Therefore,  $\mathcal{I}_3 \leq \frac{C}{n_{\beta,p,\rho}^{p_0}}$ .

8th step. Gathering the estimates for  $\mathcal{I}_1$ ,  $\mathcal{I}_2$  and  $\mathcal{I}_3$ , we have from (4.13)

$$1 \leq C_0 \left( \frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\rho}^{p(x)}} \rho(r_x) + \frac{r^{-\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,p,\rho}^{p_{\varepsilon_0}(x)}} \rho(r_x) + \frac{1}{n_{\beta,p,\rho}^{p_0}} \right), \tag{4.20}$$

with a certain constant  $C_0$  not depending on  $x$  and  $r$ . We may assume that

$$n_{\beta,p,\rho}(x,r) \geq \left( \frac{1}{2C_0} \right)^{\frac{1}{p_0}} := C_1, \tag{4.21}$$

because for those  $x$  and  $r$  where  $n_{\beta,p,\rho}(x,r) \leq C_1$  there is nothing to prove, the right-hand side of (4.9) being bounded from below according to (4.11). In the situation (4.21) we derive from (4.20) the inequality

$$1 \leq 2C_0 \left( \frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\rho}^{p(x)}} + \frac{r^{-\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,p,\rho}^{p_{\varepsilon_0}(x)}} \right) \rho(r_x). \tag{4.22}$$

Since  $\frac{Cr^{-\beta(x)}}{n_{\beta,p,\rho}} \leq 1$  by (4.12) and  $p_{\varepsilon_0}(x) \leq p(x)$ , we have

$$\frac{r^{-\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,p,\rho}^{p_{\varepsilon_0}(x)}} \leq C \frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\rho}^{p(x)}}.$$

Therefore, from (4.22) we derive the estimate

$$\frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\rho}^{p(x)}} \rho(r_x) \geq C$$

which yields (4.9).  $\square$

### 5. Proof of Theorem A

We first observe that the equivalence of conditions (3.5) and (3.6) follows from Theorem 2.4. We take  $x_0 = 0$  for simplicity.

By the direct application of Hedberg’s approach we can cover only the case when the indices  $m(w)$  and  $M(w)$  of the weight  $w$  belong to an interval narrower than the interval given in (3.6). Namely, this approach will work within the interval

$$-a < m(w) \leq M(w) < n[p(0) - 1], \tag{5.1}$$

where

$$a = [p(0) - 1] \inf_{x \in \Omega} \frac{n - \alpha(x)p(x)}{p(x) - 1}, \quad -a \geq \alpha(0)p(0) - n. \tag{5.2}$$

Then, by duality arguments, we will cover the interval

$$\alpha(0)p(0) - n < m(w) \leq M(w) < b, \tag{5.3}$$

where

$$b = \frac{np(0)}{q(0)} \inf_{x \in \Omega} \frac{q(x)}{p'(x)}, \quad b \leq n[p(0) - 1]. \tag{5.4}$$

The remaining case where  $\alpha(0)p(0) - n < m(w) \leq a$  and  $b \leq M(w) < n[p(0) - 1]$  will be separately treated, based on a possibility to reduce the problem to consideration of the Riesz potential operator  $I^\alpha$  on a small neighborhood of the point  $x_0 = 0$ .

$1^0$ . The case  $-a < m(w) \leq M(w) < n[p(0) - 1]$ . We have

$$I^{\alpha(\cdot)} f(x) = \int_{\substack{x \in \Omega \\ |x-y| < r}} \frac{f(y) dy}{|x-y|^{n-\alpha(x)}} + \int_{\substack{x \in \Omega \\ |x-y| < r}} \frac{f(y) dy}{|x-y|^{n-\alpha(x)}} := \mathbb{A}_r(x) + \mathbb{B}_r(x). \tag{5.5}$$

We make use of the known inequality

$$|\mathbb{A}_r(x)| \leq cr^{\alpha(x)} \mathcal{M}f(x), \tag{5.6}$$

where  $C > 0$  does not depend on  $r$  and  $x$ , which is known in the case of  $\alpha(x) = \text{const}$  and remains valid in case it is variable thanks to the first condition in (3.2). Let  $f(x) \geq 0$  and  $\|f\|_{L^{p(\cdot)}(\Omega, w)} \leq 1$ . By the Hölder inequality (2.6), we obtain

$$|\mathbb{B}_r(x)| \leq kn_{\beta, p', \rho}(x, r) \|f\|_{L^{p(\cdot)}(\Omega, w)} \leq n_{\beta, p', \rho}(x, r), \tag{5.7}$$

where  $\beta(x) = \alpha(x) - n$  and  $\rho(x) = [w(|x|)]^{\frac{1}{1-p(x)}}$ . By Lemma 2.7 we may take  $\rho(x) \equiv \rho_0(|x|) := [w(|x|)]^{\frac{1}{1-p(0)}}$ . To make use of estimate (4.9) for  $n_{\beta, p', \rho_0}$ , we have to check the validity of condition (4.8) which is equivalent to  $m(t^n \rho_0(t)) > 0$  and  $M(t^n \rho_0(t)) < \min[n - \alpha(x)]p'(x)$  by Theorem 2.4. The latter holds since is the same as  $-a < m(w) \leq M(w) < n[p(0) - 1]$ , see (2.14).

We make use of estimate (4.9) and obtain

$$|\mathbb{B}_r(x)| \leq Cr^{-\frac{n}{q(x)}} w^{-\frac{1}{p(x)}}(r_x) \leq Cr^{-\frac{n}{q(x)}} w^{-\frac{1}{p(x)}}(|x|), \tag{5.8}$$

where we took into account that  $r_x \geq |x|$  and the function  $w^{-\frac{1}{p(x)}}(r)$  is almost decreasing in  $r$  (which follows from the condition  $m(w) > 0$  according to part III of Theorem 2.4); and

$$[w(|x|)]^{\frac{1}{1-p(0)}} \sim [w(|x|)]^{\frac{1}{1-p(x)}}$$

by Lemma 2.7.

Therefore, taking into account (5.6) and (5.8) in (5.5), we arrive at

$$I^{\alpha(\cdot)} f(x) \leq C[r^{\alpha(x)} \mathcal{M}f(x) + [w(|x|)]^{-\frac{1}{p(x)}} r^{-\frac{n}{q(x)}}]. \tag{5.9}$$

It remains to choose the value of  $r$  which minimizes the right-hand side (up to a factor which is bounded from below and above):

$$r = [w(|x|)]^{-\frac{1}{n}} [\mathcal{M}f(x)]^{-\frac{p(x)}{n}}.$$

Substituting this into (5.9), we get

$$I^{\alpha(\cdot)} f(x) \leq C[w(|x|)]^{-\frac{\alpha(x)}{n}} [Mf(x)]^{\frac{p(x)}{q(x)}}.$$

Hence

$$\int_{\Omega} [w(|x|)]^{\frac{q(x)}{p(x)}} |I^{\alpha(\cdot)} f(x)|^{q(x)} dx \leq C \int_{\Omega} w(|x|) |\mathcal{M}f(x)|^{p(x)} dx.$$

Finally we make use of Theorem 2.9 and obtain that

$$\int_{\Omega} [w(|x|)]^{\frac{q(x)}{p(x)}} |I^{\alpha(\cdot)} f(x)|^{q(x)} dx \leq C$$

for all  $f \in L^{p(\cdot)}(\Omega, w)$  with  $\|f\|_{L^{p(\cdot)}(\Omega, w)} \leq 1$  which is equivalent to (3.7).

2<sup>0</sup>. The case  $\alpha(0)p(0) - n < m(w) \leq M(w) < b$ . This case is reduced to the previous case by the duality arguments. Observe that the operator conjugate to  $I^{\alpha(\cdot)}$  has the form

$$(I^{\alpha(\cdot)})^* g(x) = \int_{\Omega} \frac{g(y) dy}{|x - y|^{n-\alpha(y)}} \sim \int_{\Omega} \frac{g(y) dy}{|x - y|^{n-\alpha(x)}} = I^{\alpha(\cdot)} g(x) \tag{5.10}$$

thanks to the logarithmic condition for  $\alpha(x)$ .



We pass to the duality statement in Theorem A considering it already proved in the case  $-a < m(w) \leq M(w) < n[p(0) - 1]$ . By (5.10) we obtain from (3.7) that

$$\|I^{\alpha(\cdot)} g\|_{(L^{p(\cdot)}(\Omega, w))^*} \leq C \|g\|_{(L^{q(\cdot)}(\Omega, w^{\frac{q}{p}}))^*}.$$

In view of (2.5) and equivalence (2.17), this takes the form

$$\|I^{\alpha(\cdot)} g\|_{L^{p'(\cdot)}(\Omega, w^{\frac{1}{1-p(0)}})} \leq C \|g\|_{L^{q'(\cdot)}(\Omega, w^{-\frac{q'(0)}{p(0)}})}. \tag{5.11}$$

Now we re-denote

$$[w(|x|)]^{-\frac{q'(0)}{p(0)}} = w_1(|x|), \quad q'(x) = p_1(x).$$

For the exponent  $p_1(x)$  we have

$$p_1(x) = \frac{np(x)}{n[p(x) - 1] + \alpha(x)p(x)} \quad \text{and}$$

$$n - \alpha(x)p_1(x) = \frac{n^2[p(x) - 1]}{n[p(x) - 1] + \alpha(x)p(x)} \geq c > 0.$$

Its Sobolev exponent is

$$q_1(x) = \frac{np_1(x)}{n - p_1(x)\alpha(x)} = p'(x).$$

Under this passage to the new exponent  $p_1(x)$  and the new weight  $w_1(|x|)$ , the whole interval  $\alpha(0)p(0) - n < m(w) \leq M(w) < n[p(0) - 1]$  transforms into the exactly similar interval

$$\alpha(0)p_1(0) - n < m(w_1) \leq M(w_1) \leq n[p_1(0) - 1], \tag{5.12}$$

which can be easily checked via relations (2.14). Besides this, the subinterval  $-a < m(w) \leq M(w) < n[p(0) - 1]$  is transformed into the subinterval  $\alpha(0)p_1(0) - n < m(w_1) \leq M(w_1) < b_1$  where  $b_1 = \frac{np_1(0)}{q_1(0)} \inf_{x \in \Omega} \frac{q_1(x)}{p'_1(x)}$ .

In the new notation, estimate (5.11) has the form

$$\|I^{\alpha(\cdot)} g\|_{L^{q_1(\cdot)}(\Omega, w_1^{\frac{q}{p}})} \leq C \|g\|_{L^{p_1(\cdot)}(\Omega, w_1)} \tag{5.13}$$

which is nothing else but our Theorem A for the subinterval treated in this case  $2^0$ .

$3^0$ . *The remaining case  $m(w) \leq -a$  and  $M(w) \geq b$ .* The possibility to treat the remaining case is based on a simple observation that the left-hand side bound  $-a$  in (5.1) coincides with the natural left bound  $\alpha(0)p(0) - n$  if the infimum in (5.2) is attained at the point  $x = 0$  and similarly the right-hand side bound  $b$  in (5.3) coincides with the natural right bound  $n[p(0) - 1]$  if the infimum in (5.4) is attained at the same point. This leads to the idea to reduce the estimates to those in a small neighborhood of the point  $x = 0$ .

We find it more convenient to pass to the weighted Riesz potential

$$I^{\alpha, w} f(x) = \int_{\Omega} \left[ \frac{w(|x|)}{w(|y|)} \right]^{\frac{1}{p(0)}} \frac{f(y) dy}{|x - y|^{n-\alpha(x)}} \tag{5.14}$$

and we take  $f \geq 0$ . With Lemma 2.7 in mind, we have to prove in the considered case that

$$\|I^{\alpha,w} f\|_{L^q(\cdot)} \leq C \|f\|_{L^p(\cdot)}. \tag{5.15}$$

We split integration over  $\Omega$  in (5.14) into two parts, one over a small neighborhood  $B_\delta = \{y: |y| < \delta\}$  of the point  $x_0 = 0$ , and another over its exterior  $\Omega \setminus B_\delta$ , with the aim to make an appropriated choice of  $\delta$  later. We have

$$\begin{aligned} I^{\alpha,w} &= \chi_{B_\delta} I^{\alpha,w} \chi_{B_\delta} + \chi_{B_\delta} I^{\alpha,w} \chi_{\Omega \setminus B_\delta} + \chi_{\Omega \setminus B_\delta} I^{\alpha,w} \chi_{B_\delta} + \chi_{\Omega \setminus B_\delta} I^{\alpha,w} \chi_{\Omega \setminus B_\delta} \\ &=: I_1^{\alpha,w} + I_2^{\alpha,w} + I_3^{\alpha,w} + I_4^{\alpha,w}. \end{aligned} \tag{5.16}$$

Since the weight  $w(|x|)$  is bounded from below and from above beyond every neighborhood of the point  $x_0 = 0$ , we have

$$I_4^{\alpha,w} f(x) \leq C I^\alpha f(x). \tag{5.17}$$

For  $I_3^{\alpha,w}$  we have

$$I_3^{\alpha,w} f(x) = \chi_{\Omega \setminus B_\delta(x_0)}(x) \int_{B_\delta \cap \Omega} \left[ \frac{w(|x|)}{w(|y|)} \right]^{\frac{1}{p(\cdot)}} \frac{|f(y)| dy}{|x - y|^{n-\alpha(x)}}.$$

Here  $|x| > \delta > |y|$ . Observe that the function  $w_\varepsilon(t) = \frac{w(t)}{t^{M(w)+\varepsilon}}$  is almost decreasing for every  $\varepsilon > 0$  according to part III of Theorem 2.4. Therefore

$$\frac{w(|x|)}{w(|y|)} = \frac{w_\varepsilon(|x|)}{w_\varepsilon(|y|)} \frac{|x|^{M(w)+\varepsilon}}{|y|^{M(w)+\varepsilon}} \leq C \frac{|x|^{M(w)+\varepsilon}}{|y|^{M(w)+\varepsilon}}.$$

Denoting

$$I_\lambda^\alpha f(x) = \int_\Omega \left( \frac{|x|}{|y|} \right)^\lambda \frac{f(y) dy}{|x - y|^{n-\alpha(x)}},$$

we obtain

$$I_3^{\alpha,w} f(x) \leq C I_{\lambda_1}^\alpha f(x), \quad \lambda_1 = \frac{M(w) + \varepsilon}{p(0)}. \tag{5.18}$$

Similarly we conclude that

$$I_2^{\alpha,w} f(x) \leq C I_{\lambda_2}^\alpha f(x), \quad \lambda_2 = \frac{m(w) - \varepsilon}{p(0)}. \tag{5.19}$$

Thus, from (5.16) according to (5.17), (5.18) and (5.19) we have

$$I_w^\alpha f(x) \leq \chi_{B_\delta} I_w^\alpha \chi_{B_\delta} f(x) + I^\alpha f(x) + I_{\lambda_1}^\alpha f(x) + I_{\lambda_1}^\alpha f(x). \tag{5.20}$$

The operator  $I^\alpha$  is known to be bounded from  $L^{p(\cdot)}(\Omega)$  to  $L^{q(\cdot)}(\Omega)$ , see for instance Theorem 3.1, the case  $\gamma = 0$ . The operators  $I_{\lambda_1}^\alpha$  and  $I_{\lambda_2}^\alpha$  are  $p(\cdot) \rightarrow q(\cdot)$ -bounded by the same Theorem 3.1, if

$$\lambda_1, \lambda_2 \in \left( \alpha(0) - \frac{n}{p(0)}, \frac{n}{p'(0)} \right),$$

that is,  $\alpha(0)p(0) - n < m(w) - \varepsilon < n[p(0) - 1]$ ,  $\alpha(0)p(0) - n < M(w) + \varepsilon < n[p(0) - 1]$  which is satisfied by (3.6) with  $\varepsilon$  sufficiently small.

It remains to prove the boundedness of the first term on the right-hand side of (5.20). This boundedness is nothing else but the boundedness of the same operator  $I_w^\alpha$  over a small set  $\Omega_\delta = B_\delta \cap \Omega$ . According to the preceding parts  $1^0$  and  $2^0$ , this boundedness holds if

$$-a_\delta < m(w) \leq M(w) < b_\delta, \tag{5.21}$$

where

$$a_\delta = \frac{np(0)}{p'(0)} \inf_{|x|<\delta} \frac{p'(x)}{q(x)}, \quad b_\delta = \frac{np(0)}{q(0)} \inf_{|x|<\delta} \frac{q(x)}{p'(x)}.$$

For  $m(w)$ ,  $M(w)$  satisfying the condition in (3.5), that is,

$$-a_0 < m(w) \leq M(w) < b_0, \quad \text{with } a_0 = n - \alpha(0)p(0), \quad b_0 = n[p(0) - 1], \tag{5.22}$$

we can choose  $\delta$  sufficiently small so that  $m(w)$  and  $M(w)$  prove to be in the interval (5.21). This follows from the following easily derived estimates

$$0 \leq a_0 - a_\delta \leq A \sup_{|x|<\delta} |p(x) - p(0)| + \sup_{|x|<\delta} |\alpha(x)p(x) - \alpha(0)p(0)|,$$

$$0 \leq b_0 - b_\delta \leq n \sup_{|x|<\delta} |p(x) - p(0)| + B \sup_{|x|<\delta} |\alpha(x)p(x) - \alpha(0)p(0)|,$$

where

$$A = \sup_{x \in \Omega} \frac{n - \alpha(x)p(x)}{p(x) - 1} \quad \text{and} \quad B = \sup_{x \in \Omega} \frac{p(x) - 1}{n - \alpha(x)p(x)},$$

and from the continuity of the functions  $p(x)$  and  $\alpha(x)$ . Condition (5.21) having been satisfied, the theorem in the remaining case is proved.

### 6. Proof of Theorem B

**Proof.** We follow ideas of [25] where Theorem B was proved for power weights. Let

$$A_w^p(f) = \int_{\mathbb{R}^n} w(x) |f(x)|^{p(x)} dx.$$

We have to show that  $A_w^q(I^\alpha \varphi) \leq c < \infty$  for all  $\varphi$  with  $A_w^p(\varphi) \leq 1$ , where  $c > 0$  does not depend on  $\varphi$ , and we denoted

$$w^1(x) = [w(x)]^{\frac{q(x)}{p(x)}} \sim [w_0(|x|)]^{\frac{q(0)}{p(0)}} [w_\infty(|x|)]^{\frac{q(\infty)}{p(\infty)}},$$

the latter equivalence following from Lemma 2.7.

Let  $B_+ = \{x \in \mathbb{R}^n : |x| < 1\}$  and  $B_- = \{x \in \mathbb{R}^n : |x| > 1\}$ . We have

$$A_w^q(I^\alpha \varphi) \leq c(A_{++} + A_{+-} + A_{-+} + A_{--}), \tag{6.1}$$

where

$$A_{++} = \int_{B_+} w_0^1(|x|) \left| \int_{B_+} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

$$A_{+-} = \int_{B_+} w_0^1(|x|) \left| \int_{B_-} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

and

$$A_{-+} = \int_{B_-} w_\infty^1(|x|) \left| \int_{B_+} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

$$A_{--} = \int_{B_-} w_\infty^1(|x|) \left| \int_{B_-} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

with

$$w_0^1(|x|) = [w_0(|x|)]^{\frac{q(0)}{p(0)}}, \quad w_\infty^1(|x|) = [w_\infty(|x|)]^{\frac{q(\infty)}{p(\infty)}}.$$

The term  $A_{++}$ . This term is covered by Theorem A, condition (3.7) of Theorem A being fulfilled by (3.12).

The term  $A_{--}$ . The estimation of  $A_{--}$  is reduced to that of  $A_{++}$  by means of the simultaneous change of variables (inversion):

$$x = \frac{u}{|u|^2}, \quad dx = \frac{du}{|u|^{2n}}, \quad y = \frac{v}{|v|^2}, \quad dy = \frac{dv}{|v|^{2n}}. \tag{6.2}$$

As a result, we obtain

$$A_{--} = \int_{B_+} |x|^{-2n} \left[ w_\infty \left( \frac{1}{|x|} \right) \right]^{\frac{q(\infty)}{p(\infty)}} \left| \int_{B_+} \frac{\varphi(y_*) dy}{|y|^{2n} |x_* - y_*|^{n-\alpha}} \right|^{q_*(x)} dx,$$

where  $x_* = \frac{x}{|x|^2}$  and  $q_*(x) = q(x_*) = q\left(\frac{x}{|x|^2}\right)$ . It is easy to check that  $|x_* - y_*| = \frac{|x-y|}{|x||y|}$ . Since  $p_*(x)$  satisfies the log-condition (3.9), then  $q_*(x)$  does the same, so that

$$|x|^{(n-\alpha)q_*(x)} \simeq c|x|^{(n-\alpha)q_*(0)} = c|x|^{(n-\alpha)q(\infty)}$$

and we get

$$A_{--} \simeq C \int_{B_+} [w_2(x)]^{\frac{q_*(0)}{p_*(0)}} \left| \int_{B_+} \frac{\psi(y) dy}{|x-y|^{n-\alpha}} \right|^{q_*(x)} dx, \tag{6.3}$$

where

$$w_2(|x|) = |x|^{(n+\alpha)p_*(0)-2n} w_\infty \left( \frac{1}{|x|} \right) \quad \text{and} \quad \psi(y) = |y|^{-n-\alpha} \varphi \left( \frac{y}{|y|^2} \right). \tag{6.4}$$

It is easily checked that

$$\int_{B_+} w_2(|x|) |\psi(x)|^{p_*(x)} dx \simeq C \int_{B_-} w_\infty(|x|) |\varphi(x)|^{p(x)} dx < \infty. \tag{6.5}$$

Therefore, the boundedness of the right-hand side of (6.3) is nothing else but the boundedness of the Riesz potential operator over the ball  $B_+$  from the weighted space  $L^{p_*(\cdot)}(B_+, w_2)$  to the space  $L^{q_*(\cdot)}(B_+, [w_2]^{\frac{q(\infty)}{p(\infty)}})$ . According to Theorem A, this boundedness holds if

$$\alpha p_*(0) - n < m(w_2) \leq M(w_2) < n[p_*(0) - 1]. \tag{6.6}$$

In view of (2.22) this coincides with the given condition (3.13). Consequently,  $A_{--} \leq C < \infty$  by Theorem A.

The term  $A_{-+}$ . We split  $A_{-+}$  as  $A_{-+} = A_1 + A_2$ , where

$$A_1 = \int_{1 < |x| < 2} w_\infty^1(|x|) \left| \int_{|y| < 1} \frac{\varphi(y) dy}{|x - y|^{n-\alpha}} \right|^{q(x)} dx,$$

$$A_2 = \int_{|x| > 2} w_\infty^1(|x|) \left| \int_{|y| < 1} \frac{\varphi(y) dy}{|x - y|^{n-\alpha}} \right|^{q(x)} dx.$$

Since the weight functions  $w_0^1(|x|)$ ,  $w_\infty^1(|x|)$  are bounded from below and from above in the layer  $1 \leq |x| \leq 2$ , we get

$$\begin{aligned} A_1 &\leq C \int_{1 < |x| < 2} w_0^1(|x|) \left| \int_{|y| < 1} \frac{\varphi(y) dy}{|x - y|^{n-\alpha}} \right|^{q(x)} dx \\ &\leq C \int_{|x| < 2} [w_0(|x|)]^{\frac{q(0)}{p(0)}} \left| \int_{|y| < 2} \frac{f(y) dy}{|x - y|^{n-\alpha}} \right|^{q(x)} dx \end{aligned}$$

so that  $A_1 \leq C < \infty$  by Theorem A. For the term  $A_2$  we have  $|x - y| \geq |x| - |y| \geq \frac{|x|}{2}$ . Therefore,

$$A_2 \leq C \int_{|x| > 2} |x|^{(\alpha-n)q(x)} w_\infty^1(|x|) \left( \int_{|y| < 1} |\varphi(y)| dy \right)^{q(x)} dx.$$

Since  $|q(x) - q(\infty)| \leq \frac{C}{\ln|x|}$ ,  $|x| \geq 2$ , we have

$$A_2 \leq C \int_{|x| > 2} |x|^{(\alpha-n)q(\infty)} w_\infty^1(|x|) \left( \int_{|y| < 1} |\varphi(y)| dy \right)^{q(x)} dx. \tag{6.7}$$

Denote  $g(y) = [w(y)]^{-\frac{1}{p(y)}}$ ; by the Hölder inequality for variable  $L^{p(\cdot)}$ -spaces we get

$$\int_{|y| < 1} |\varphi(y)| dy \leq k \|g\|_{L^{p'(\cdot)}} \|w^{\frac{1}{p}} \varphi\|_{L^{p(\cdot)}} = k \|g\|_{L^{p'(\cdot)}} \|\varphi\|_{L^{p(\cdot)}(\mathbb{R}^n, w)}. \tag{6.8}$$

We have

$$\int_{|y| < 1} |g(y)|^{p'(y)} dy \leq C \int_{|y| < 1} [w_0(|y|)]^{\frac{1}{1-p(0)}} dy \leq C \int_{|y| < 1} |y|^{\frac{m(w_0)-\varepsilon}{1-p(0)}} dy$$

for arbitrarily small  $\varepsilon > 0$  according to (2.16). The last integral is finite because  $\frac{m(w_0)-\varepsilon}{1-p(0)} > -n$  under the choice of sufficiently small  $\varepsilon$ . Then from (6.7)

$$A_2 \leq C \int_{|x| > 2} |x|^{(\alpha-n)q(\infty)} w_\infty^1(|x|) dx. \tag{6.9}$$

The convergence of the last integral is verified by means of property (2.24):

$$\begin{aligned} |x|^{(\alpha-n)q(\infty)} w_\infty^1(|x|) &= |x|^{(\alpha-n)q(\infty)} [w_\infty(|x|)]^{\frac{q(\infty)}{p(\infty)}} \leq C |x|^{(\alpha-n)q(\infty) + [M(w_\infty) + \varepsilon] \frac{q(\infty)}{p(\infty)}} \\ &= C |x|^{-n-\delta}, \end{aligned}$$

where

$$\delta = \frac{q(\infty)}{p(\infty)} \{n[p(\infty) - 1] - M(w_\infty) - \varepsilon\}$$

is positive for small  $\varepsilon > 0$ . Therefore,  $A_2 \leq C < \infty$ .

The term  $A_{+-}$ . This term is estimated similarly to  $A_{-+}$ : we split  $A_{+-}$  as  $A_{+-} = A_3 + A_4$ , where

$$A_3 = \int_{|x|<1} w_0^1(|x|) \left| \int_{1<|y|<2} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx,$$

$$A_4 = \int_{|x|<1} w_0^1(|x|) \left| \int_{|y|>2} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The term  $A_3$  is covered by Theorem A similarly to the term  $A_1$  above. For the term  $A_4$ , we have  $|x-y| \geq |y|-|x| \geq \frac{|y|}{2}$ . Then

$$\left| \int_{|y|>2} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right| \leq C \int_{|y|>2} \frac{|\varphi(y)| dy}{|y|^{n-\alpha}} = C \int_{|y|>2} \frac{|\varphi_0(y)| dy}{|y|^{n-\alpha} [w_\infty(|y|)]^{\frac{1}{p(y)}}},$$

where

$$\varphi_0(y) = [w_\infty(|y|)]^{\frac{1}{p(\infty)}} \varphi(y) \in L^{p(\cdot)}(\mathbb{R}^n \setminus B(0, 2)),$$

since  $[\rho(y)]^{\frac{1}{p(y)}} \varphi(y) \in L^{p(\cdot)}(\mathbb{R}^n)$  and  $[\rho(y)]^{\frac{1}{p(y)}} \sim |y|^{\frac{\gamma_\infty}{p(\infty)}}$  for  $|y| \geq 2$  under the log-condition at infinity. Then

$$\left| \int_{|y|>2} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right| \leq C_1 \|\varphi_0\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(0,2))} \| |y|^{\alpha-n} [w_\infty(|y|)]^{-\frac{1}{p(y)}} \|_{L^{p'(\cdot)}(\mathbb{R}^n \setminus B(0,2))}.$$

Hence

$$\left| \int_{|y|>2} \frac{\varphi(y) dy}{|x-y|^{n-\alpha}} \right| \leq C \|\varphi\|_{L^{p(\cdot)}(\mathbb{R}^n, w)} N_p \leq C N_p,$$

where the norm

$$N_p = \| |y|^{\alpha-n} [w_\infty(|y|)]^{-\frac{1}{p(y)}} \|_{L^{p'(\cdot)}(\mathbb{R}^n \setminus B(0,2))}$$

is finite under the condition  $\alpha p(\infty) - n < m(w_\infty)$ . Therefore,  $A_4 \leq C < \infty$  which completes the proof.  $\square$

### 7. Proof of Theorem C

The statement of Theorem C is derived from Theorem B by means of the stereographic projection similarly to the case of power weights [25], so we omit details pointing out only the principal points, referring to [25] for more technical details.

Since Theorem B was proved for the product case of two weights, one fixed to a finite point, another fixed to infinity, it will be technically more convenient for us to obtain Theorem C for

the weight of the form  $w = w_1(|\sigma - a|)w_2(|\sigma - b|)$ , where  $a$  and  $b$ ,  $a \neq b$ , are two arbitrary points of the sphere (one may always take  $w_2 \equiv 1$ ). Without loss of generality we may take  $a = e_{n+1} = (0, 0, \dots, 0, 1)$  and  $b = -e_{n+1}$ .

We recall that the stereographic projection (see, for instance, [14, p. 36]) of the sphere  $\mathbb{S}^n$  onto the space  $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$  is the mapping generated by the following change of variables in  $\mathbb{R}^{n+1}$ :  $\xi = s(x) = \{s_1(x), s_2(x), \dots, s_{n+1}(x)\}$  where

$$s_k(x) = \frac{2x_k}{1 + |x|^2}, \quad k = 1, 2, \dots, n, \quad \text{and} \quad s_{n+1}(x) = \frac{|x|^2 - 1}{|x|^2 + 1}.$$

Via the known formulas

$$|x| = \frac{|\xi + e_{n+1}|}{|\xi - e_{n+1}|}, \quad \sqrt{1 + |x|^2} = \frac{2}{|\xi - e_{n+1}|}, \quad |x - y| = \frac{2|\sigma - \xi|}{|\sigma - e_{n+1}| \cdot |\xi - e_{n+1}|},$$

$$dy = \frac{2^n d\sigma}{|\sigma - e_{n+1}|^{2n}},$$

where  $\xi = s(x)$ ,  $\sigma = s(y)$ ,  $x, y \in \mathbb{R}^{n+1}$ , we obtain the relation

$$\int_{\mathbb{R}^n} \frac{\varphi(y) dy}{|x - y|^{n-\alpha}} = 2^\alpha \int_{S_n} \frac{\psi(\sigma) d\sigma}{|\xi - \sigma|^{n-\alpha}}, \tag{7.1}$$

where  $\psi(\sigma) = \frac{\varphi[s^{-1}(\sigma)]}{|\sigma - e_{n+1}|^{n+\alpha}}$ . The following modular equivalence holds

$$\int_{\mathbb{S}^n} w_1(|\sigma - e_{n+1}|) \cdot w_2(|\sigma + e_{n+1}|) \cdot |\psi(\sigma)|^{p(\sigma)} d\sigma \sim \int_{\mathbb{R}^n} w_0(|x|)w_\infty(|x|)|\varphi(x)|^{\tilde{p}(x)} dx \tag{7.2}$$

where  $\tilde{p}(x) = p[s(x)]$ ,  $w_0(r) = w_2(r)$ , if  $0 < r \leq 1$ ,  $w_0(r) = w_2(1)$ , if  $r \geq 1$ , and  $w_\infty(r) = w_1(\frac{1}{r})r^{(n+\alpha)p(\infty)-2n}$ , if  $r \geq 1$ ,  $w_\infty(r) = w_1(1)$ , if  $0 < r \leq 1$ .

Relation (7.2) is verified directly via the inverse formulas

$$|\xi - e_{n+1}| = \frac{2}{\sqrt{1 + |x|^2}}, \quad |\xi + e_{n+1}| = \frac{2|x|}{\sqrt{1 + |x|^2}},$$

$$|\xi - \sigma| = \frac{2|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad d\sigma = \frac{2^n dy}{(1 + |y|^2)^n}$$

with the property  $w_k(cr) \sim w_k(r)$ ,  $k = 1, 2$ , of our weights taken into account.

In view of relation (7.1) and equivalence (7.2), the statement of Theorem C follows from Theorem B. Indeed, conditions (3.15) of Theorem C for the exponent  $p(\sigma)$  and the point  $a = e_{n+1}$  and similar conditions for the point  $b = -e_{n+1}$  after recalculation coincide with the corresponding conditions (3.12) and (3.13) of Theorem B for the exponent  $\tilde{p}(x)$  for the points 0 and  $\infty$ , if we take into account that  $m(w_0) = m(w_2)$ ,  $M(w_0) = M(w_2)$  and  $m(w_\infty) = (n + \alpha)p(\infty) - 2n - M(w_1)$ ,  $M(w_\infty) = (n + \alpha)p(\infty) - 2n - m(w_1)$ .

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