

The Maximal Operator in Weighted Variable Spaces $L^{p(\cdot)}$

by

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Abstract

We study the boundedness of the maximal operator in the weighted spaces $L^{p(\cdot)}(\rho)$ over a bounded open set Ω in the Euclidean space \mathbb{R}^n or a Carleson curve Γ in a complex plane. The weight function may belong to a certain version of a general Muckenhoupt-type condition, which is narrower than the expected Muckenhoupt condition for variable exponent, but coincides with the usual Muckenhoupt class A_p in the case of constant p . In the case of Carleson curves there is also considered another class of weights of radial type of the form $\rho(t) = \prod_{k=1}^m w_k(|t - t_k|)$, $t_k \in \Gamma$, where w_k has the property that $r^{\frac{1}{p(t_k)}} w_k(r) \in \Phi_1^0$, where Φ_1^0 is a certain Zygmund-Bari-Stechkin-type class. It is assumed that the exponent $p(t)$ satisfies the Dini–Lipschitz condition. For such radial type weights the final statement on the boundedness is given in terms of the index numbers of the functions w_k (similar in a sense to the Boyd indices for the Young functions defining Orlich spaces).

Key Words and Phrases: maximal functions, weighted Lebesgue spaces, variable exponent, Carleson curve, Zygmund conditions, Bari-Stechkin class.

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1 Introduction

Within the frameworks of variable exponent spaces $L^{p(\cdot)}(\Omega)$, the boundedness of maximal operators was proved in L. Diening [7] for bounded domains in \mathbb{R}^n and in D.Cruz-Urbe, A.Fiorenza and C.J. Neugebauer [6] and A.Nekvinda [22], [21] for unbounded domains. The weighted boundedness with power weights was proved in V.Kokilashvili and S.Samko [11] in the case of bounded domains. We refer also to L.Diening [8] and D.Cruz-Urbe, A.Fiorenza, J.M.Martell, and C.Perez [5] for problems of boundedness of maximal operators in variable exponent spaces.

In [11] the power weights $|x - x_0|^\gamma$ were considered and one of the main points in the result obtained in [11] was that in condition on γ only the values of $p(x)$ at the point x_0 are of importance: $-\frac{n}{p(x_0)} < \gamma < \frac{n}{q(x_0)}$ (under the usual log-condition on $p(x)$).

However, an explicit description in terms of Muckenhoupt-type condition of general weights for which the maximal operator is bounded in the spaces $L^{p(\cdot)}$ still remains an open problem.

A certain subclass of general weights was considered in [10], where for the case of bounded domains Ω in the Euclidean space, the boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\Omega, \rho)$ was proved. This subclass may be characterized as a class of radial type weights which satisfy the Zygmund-Bari-Stechkin condition. Radial weights w in this class are almost increasing or almost decreasing and may oscillate between two power functions with different exponents and have non-coinciding upper and lower indices m_w and M_w (of the type of Boyd indices). In comparison with the approach in [11], the main problems arising are related to the situation when the indices m_w and M_w do not coincide, in particular when m_w is negative while M_w is positive.

In this paper, because of applications to weighted boundedness of singular integral operator along Carleson curves, we prove similar results for the maximal operator along Carleson curves. This extension from the Euclidean space to the case of Carleson curves required an essential modification of certain means used in [10]. To obtain this result, we first prove a certain general theorem with a certain version of the Muckenhoupt-type condition.

The weighted results obtained for the maximal operator pave the way to the study of Fredholmness of singular integral equations on Carleson curves in case of more general weights. In fact the main Theorems A, A' and B may be rewritten in terms on function spaces defined on metric spaces. However, because of application to the theory of singular integral equations, we prefer to present the results in the context of Carleson curves.

The paper is organized as follows. In Section 2 we formulate the main results - Theorems A, A' and B - on the weighted boundedness of the maximal operator. In Section 4 we recall the notion of the upper and lower indices of almost increasing non-negative functions and develop some properties of weights in the Zygmund-Bari-Stechkin class, which we need to prove the main result. In Sections 5 and 6 we give the proof of Theorems A, A' and B.

Notation

a.d. =almost decreasing $\iff f(x) \geq C f(y)$ for $x \leq y, C > 0$,

a.i. =almost increasing $\iff f(x) \leq C f(y)$ for $x \leq y, C > 0$; where f is a non-negative function on \mathbb{R}_+^1 ;

Γ is an arbitrary bounded Carleson curve on the complex plane, closed or open;

γ denotes an arbitrary portion of Γ ;

$\gamma_r(t) = \{\tau \in \Gamma : |\tau - t| < r\}$;

$d\nu(t) = ds$ denotes the arc measure on Γ ;

Ω is a bounded open set in \mathbb{R}^n ;

$B(x, r)$ is a ball in \mathbb{R}^n centered at x of radius r ;

$|\gamma|$ is the arc length of γ ; $|\Omega|$ is the Euclidean measure of Ω ;

χ_γ is the characteristic function of γ ;

$f \sim g \iff$ there exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 f(t) \leq g(t) \leq C_2 f(t)$.

$q(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$, $1 < p(\cdot) < \infty$, $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \equiv 1$;

$p_* = p_*(X) = \inf_{t \in X} p(t)$, $p^* = p^*(X) = \sup_{t \in X} p(t)$, where $X = \Gamma$ or $X = \Omega$;

$q_* = \inf_{x \in \Omega} q(t) = \frac{p^*}{p^*-1}$, $q^* = \sup_{t \in \Gamma} q(t) = \frac{p_*}{p_*-1}$;

C, c may denote different positive constants.

In what follows, X will always denote either a bounded open set Ω in \mathbb{R}^n , or a bounded Carleson curve Γ . The variable exponent $p(\cdot)$ defined on X is supposed to satisfy the conditions

$$1 < p_* \leq p(t) \leq p^* < \infty, \quad t \in X \quad (1.1)$$

and

$$|p(t) - p(\tau)| \leq \frac{A}{\ln \frac{1}{|t-\tau|}}, \quad |t - \tau| \leq \frac{1}{2}, \quad t, \tau \in X. \quad (1.2)$$

By $L^{p(\cdot)}(X, \rho)$, where $\rho(t) \geq 0$, we denote the weighted Banach space of measurable functions $f : X \rightarrow \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(X, \rho)} := \|\rho f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_X \left| \frac{\rho(t)f(t)}{\lambda} \right|^{p(t)} d\mu(t) \leq 1 \right\} < \infty. \quad (1.3)$$

where $d\mu(t)$ stands for the arc-length measure $d\nu(t)$ in case $X = \Gamma$ and $d\mu(t) = dt$ in case $X = \Omega$.

2 Statement of the Main Results

We use the notation \mathcal{M}^ρ both for

$$\mathcal{M}^\rho f(x) = \sup_{r>0} \frac{\rho(x)}{|B(x, r)|} \int_{B(x, r)} \frac{|f(y)|}{\rho(y)} dy \quad (2.1)$$

and

$$\mathcal{M}^\rho f(t) = \sup_{r>0} \frac{\rho(t)}{|\gamma_r(t)|} \int_{\gamma_r(t)} \frac{|f(\tau)|}{\rho(\tau)} d\nu(\tau) \quad (2.2)$$

We write $\mathcal{M} = \mathcal{M}^\rho|_{\rho \equiv 1}$.

The boundedness of the operator \mathcal{M}^ρ was proved in the case of the power weight $\rho(x) = |x - x_0|^\beta$, $x_0 \in \Omega$ in [11] and $\rho(t) = |t - t_0|^\beta$, $t_0 \in \Gamma$ in [12] under the following (necessary and sufficient) condition $-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}$ or

$$-\frac{1}{p(t_0)} < \beta < \frac{1}{q(t_0)}, \quad (2.3)$$

respectively. We prove two main results given in Theorems A and B. In Theorem A stated below we consider some general Muckenhoupt type weights, the proof being the same both for Carleson curves and domains in \mathbb{R}^n . In Theorem B, in the case of Carleson curves we deal with a special class of radial type weights in the Zygmund-Bari-Steckin class. Such a result for the Euclidean case was earlier obtained in [10]. The proof for the case of Carleson curves required an essential modification of the technique used.

The class of weights in Theorem A is narrower than the naturally expected Muckenhoupt class $A_{p(\cdot)}$ should be. However, it coincides with the Muckenhoupt class A_p in case p is constant. Theorem B is proved by means of Theorem A, but it is not contained in Theorem A, being more general in its range of applicability.

We introduce the following "ersatz" of the Muckenhoupt condition

$$\sup_{x \in \Omega, r > 0} \left(\frac{1}{|B(x, r)|} \int_{B(x, r)} |\rho(y)|^{p(y)} dy \right) \left(\frac{1}{|B(x, r)|} \int_{B(x, r)} \frac{dy}{|\rho(y)|^{\frac{p(y)}{p_*-1}}} \right)^{p_*-1} < \infty, \quad (2.4)$$

which coincides with the Muckenhoupt condition in the case $p(x) \equiv p_*$ is constant, as well as its version

$$\sup_{t \in \Gamma, r > 0} \left(\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} |\rho(\tau)|^{p(\tau)} d\nu(\tau) \right) \left(\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{|\rho(\tau)|^{\frac{p(\tau)}{p_*-1}}} \right)^{p_*-1} < \infty \quad (2.5)$$

for Carleson curves in the complex plane.

Observe that the class of weights satisfying condition (2.4)-(2.5) is evidently narrower than what we expect from the "real" Muckenhoupt class $A_{p(\cdot)}$. Thus, in the case of power weights $|x - x_0|^\beta$, condition (2.4) yields $-\frac{1}{p(x_0)} < \beta < \frac{1}{q_0}$ with $q_0 = \frac{p(x_0)}{p_*-1}$ which is narrower than the interval $-\frac{1}{p(x_0)} < \beta < \frac{1}{q(x_0)}$, where the boundedness of the maximal operator holds [11]. Obviously conditions (2.4)-(2.5) are sharp on those power functions which are "fixed" to a point at which the minimum of $p(\cdot)$ is reached.

Theorem A. *Let the exponent $p(t)$ satisfy conditions (1.1), (1.2) and the weight ρ fulfill condition (2.4). Then the operator \mathcal{M} is bounded in $L^{p(\cdot)}(\Omega, \rho)$.*

Theorem A'. *Let the exponent $p(t)$ satisfy conditions (1.1), (1.2) and the weight ρ fulfill condition (2.5). Then the operator \mathcal{M} is bounded in $L^{p(\cdot)}(\Gamma, \rho)$.*

In the next theorem, we deal with weights of the form

$$\rho(t) = \prod_{k=1}^m w_k(|t - t_k|), t_k \in \Gamma, \quad (2.6)$$

where $w_k(x)$ may oscillate as $x \rightarrow 0+$ between two power functions (radial Zygmund-Bari-Steckin type weights).

The Zygmund-Bari-Steckin class Φ_1^0 of weights and the upper and lower indices of weights (of the type of the Boyd indices) used in the theorem below are defined in Section 4. Note that various non-trivial examples of functions in Zygmund-Bari-Steckin-type classes with coinciding indices may be found in [23], Section II; [24], Section 2.1, and with non-coinciding indices in [26].

Theorem B. *Let Γ be a bounded Carleson curve and $p(t)$ satisfy conditions (1.1), (1.2) on Γ . The operator \mathcal{M} is bounded in $L^{p(\cdot)}(\Gamma, \rho)$ with the weight (2.6), where $w_k(r)$ are such functions that $r^{\frac{1}{p(t_k)}} w_k(r) \in \Phi_1^0$, if*

$$-\frac{1}{p(t_k)} < m_{w_k} \leq M_{w_k} < \frac{1}{q(t_k)}, \quad k = 1, 2, \dots, m. \quad (2.7)$$

A similar statement for bounded domains in \mathbb{R}^n was proved in [10].

3 Some basics for variable exponent spaces

The weighted space $L^{p(\cdot)}(\Gamma, \rho)$ was introduced in (1.3). We write $L^{p(\cdot)}(\Gamma, 1) = L^{p(\cdot)}(\Gamma)$ in the case $\rho(t) \equiv 1$.

We recall some basic facts for the variable exponent spaces $L^{p(\cdot)}(\Gamma)$ and refer e.g. to [14] for details.

The Hölder inequality holds in the form

$$\int_{\Gamma} |f(t)g(t)| d\nu(t) \leq k \|f\|_{p(\cdot)} \cdot \|g\|_{q(\cdot)} \quad (3.1)$$

with $k = \frac{1}{p^*} + \frac{1}{q^*}$. The modular $I_p(f) = \int_{\Gamma} |f(t)|^{p(t)} d\nu(t)$ and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater than one and simultaneously less than 1: $\|f\|_{p(\cdot)}^{p^*} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{p^*}$ if $\|f\|_{p(\cdot)} \leq 1$ and $\|f\|_{p(\cdot)}^{p^*} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{p^*}$ if $\|f\|_{p(\cdot)} \geq 1$. Hence

$$c_1 \leq \|f\|_p \leq c_2 \implies c_3 \leq I_{\Gamma}^p(f) \leq c_4 \quad (3.2)$$

and

$$C_1 \leq I_{\Gamma}^p(f) \leq C_2 \implies C_3 \leq \|f\|_p \leq C_4 \quad (3.3)$$

with $c_3 = \min(c_1^{p^*}, c_1^{p^*})$, $c_4 = \max(c_2^{p^*}, c_2^{p^*})$, $C_3 = \min(C_1^{1/p^*}, C_1^{1/p^*})$ and $C_4 = \max(C_2^{1/p^*}, C_2^{1/p^*})$.

Lemma 3.1. *Let Γ be a bounded Carleson curve, the exponent p satisfy condition (1.2) and let w be any function such that there exist exponents $a, b \in \mathbb{R}^1$ and the constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 r^a \leq w(r) \leq c_2 r^{-b}$, $0 \leq r \leq \ell = \text{diam}(\Gamma)$. Then*

$$\frac{1}{C} [w(|t - t_0|)]^{p(t_0)} \leq [w(|t - t_0|)]^{p(t)} \leq C [w(|t - t_0|)]^{p(t_0)}, \quad (3.4)$$

where $C > 1$ does not depend on $t, t_0 \in \bar{\Gamma}$.

Proof. Let

$$g(t, t_0) = [w(|t - t_0|)]^{p(t) - p(t_0)}.$$

To show that $\frac{1}{C} \leq g(t, t_0) \leq C$, that is, $|\ln g(t, t_0)| \leq C_1$, $C_1 = \ln C$, we

$$|\ln g(t, t_0)| = |p(t) - p(t_0)| \cdot |\ln w(|t - t_0|)| \leq A\ell \frac{|\ln w(|t - t_0|)|}{\ln \frac{2\ell}{|t - t_0|}}$$

which is bounded by the condition on w . □

4 Preliminaries on Zygmund-Bari-Steckin classes.

4.1 Index numbers m_w and M_w of non-negative a. i. functions

Let

$$W = \{w \in C([0, \ell]) : w(0) = 0, w(x) > 0 \text{ for } x > 0, w(x) \text{ is a.i.}\}. \quad (4.1)$$

The numbers

$$m_w = \sup_{x>1} \frac{\ln \left(\liminf_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x} = \sup_{0 < x < 1} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x} = \lim_{x \rightarrow 0} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x}$$

and

$$M_w = \sup_{x>1} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x} = \lim_{x \rightarrow \infty} \frac{\ln \left(\limsup_{h \rightarrow 0} \frac{w(hx)}{w(h)} \right)}{\ln x}$$

(see [23], [26], [25]), will be referred to as *the lower and upper indices* of the function $w(x)$ (compare these indices with the Matuszewska-Orlicz indices, see [18], p. 20; they are of the type of the Boyd indices, see [15], p. 75; [16], or [3], p. 149 about the Boyd indices). We have $0 \leq m_w \leq M_w \leq \infty$ for $w \in W$.

The indices m_w and M_w may be also well defined for functions $w(x)$ positive for $x > 0$ which do not necessarily belong to W , for example, if there exists an $a \in \mathbb{R}^1$ such that $w_a(x) := x^a w(x)$ is in W . Obviously,

$$m_{w_a} = a + m_w, \quad M_{w_a} = a + M_w.$$

So we also introduce the class

$$\widetilde{W} = \{w : x^a w(x) \in W \text{ for some } a \in \mathbb{R}^1\}.$$

4.2 The Zygmund-Bari-Steckin class Φ_γ^0

Let $\delta > 0$. The following class Φ_δ^0 was introduced and studied in [2] (with integer δ); there are also known "two-parametrical" classes Φ_δ^β , $0 \leq \beta < \delta < \infty$, see [20], [19], [28] and [27], p. 253).

Definition 4.1. ([2]) The Zygmund-Bari-Steckin type class Φ_δ^0 , $0 < \delta < \infty$, is defined as $\Phi_\delta^0 := \mathcal{Z}^0 \cap \mathcal{Z}_\delta$, where \mathcal{Z}^0 is the class of functions $w \in W$ satisfying the condition

$$\int_0^h \frac{w(x)}{x} dx \leq cw(h) \quad (\mathcal{Z}^0)$$

and \mathcal{Z}_δ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(x)}{x^{1+\delta}} dx \leq c \frac{w(h)}{h^\delta}, \quad (\mathcal{Z}_\delta)$$

where $c = c(w) > 0$ does not depend on $h \in (0, \ell]$.

In the sequel we refer to the above conditions as (\mathcal{Z}^0) - and (\mathcal{Z}_δ) -conditions.

The following statement is valid, see [23],[26] for $\delta = 1$ and [9] for an arbitrary $\delta > 0$.

Theorem 4.2. *Let $w \in W$. Then $w \in \mathcal{Z}^0$ if and only if $m_w > 0$, and $w \in \mathcal{Z}_\delta$, $\delta > 0$, if and only if $M_w < \delta$, so that*

$$w \in \Phi_\delta^0 \iff 0 < m_w \leq M_w < \delta. \quad (4.2)$$

Besides this, for $w \in \Phi_\delta^0$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$c_1 x^{M_w + \varepsilon} \leq w(x) \leq c_2 x^{m_w - \varepsilon}, \quad 0 \leq x \leq \ell. \quad (4.3)$$

The following properties are also valid

$$m_w = \sup\{\mu \in \mathbb{R}^1 : x^{-\mu} w(x) \text{ is a.i.}\}, \quad (4.4)$$

$$M_w = \inf\{\nu \in \mathbb{R}^1 : x^{-\nu} w(x) \text{ is a.d.}\}. \quad (4.5)$$

Statements (4.3)-(4.5) remain valid for the case when M_w or m_w may be non-positive. Namely, the following corollary from Theorem 4.2 is valid.

Corollary 4.3. *Let $w(x)$, $0 < x \leq \ell$, be such a function that $t^a w(t) \in \mathcal{Z}^0$ for some $a \in \mathbb{R}^1$. Then formula (4.4) remains valid and for any $\varepsilon > 0$ there exists $c_1 > 0$ such that*

$$w(x) \leq c_1 x^{m_w - \varepsilon}. \quad (4.6)$$

Similarly, if $x^a w(x) \in \mathcal{Z}_\delta$, then (4.5) is valid and for any $\varepsilon > 0$ there exists $c_2 > 0$ such that

$$w(x) \geq c_2 x^{M_w + \varepsilon}. \quad (4.7)$$

Remark 4.4. If $w \in \widetilde{W}$ and $m_w > 0$, then $w \in W$.

Indeed, let $a \in \mathbb{R}^1$ be such that $w_a(x) = x^a w(x) \in W$. Then according to (4.4) the function $\frac{w_a(x)}{x^{m_{w_a}-\varepsilon}}$ is a.i. for every $\varepsilon > 0$. But $m_{w_a} = m_w + a$, so that $\frac{w(x)}{x^{m_w-\varepsilon}}$ is a.i. for every $\varepsilon > 0$. Since $m_w > 0$, then the function w itself is a.i., which means that it is in W .

Remark 4.5. Every function $w \in \widetilde{W}$ with $M_w < \infty$ satisfies the doubling condition

$$w(2r) \leq Cw(r), \quad 0 \leq r \leq \ell \quad (4.8)$$

which follows from the fact that the function $\frac{w(r)}{r^\mu}$ with $\mu > M_w$ is a.d. according to (4.5).

4.3 On examples of functions in Φ_γ^0 .

Power and power-logarithmic functions $w(x) = x^\mu$, $x^\mu \left(\ln \frac{1}{x}\right)^\alpha$, $x^\mu \left(\ln \ln \frac{1}{x}\right)^\alpha$ etc, are in Φ_γ^0 in the case $0 < \mu < \gamma$ and have coinciding indices $m(w) = M(w) = \mu$.

Apart from this trivial examples, observe that the condition

$$\lim_{h \rightarrow 0} \frac{w(th)}{w(h)} = t^\mu, \quad \mu = \text{const}, \quad (4.9)$$

is sufficient for $w(x)$ to have coinciding indices. The function $w(x) = x^{\mu + \frac{c}{\ln^\alpha x}}$, $\alpha \geq 1$, and more generally $w(x) = x^{\mu(x)}$ where $\mu(x)$ satisfies the Dini condition $|\mu(x+h) - \mu(x)| = o\left(\frac{1}{|\ln |h||}\right)$, fulfills condition (4.9) and $x^{\mu(x)} \in \Phi_\gamma^0$ if $0 < \mu(0) < \gamma$.

Examples of non-equilibrated characteristics are much less trivial. An example of such a function w with different indices $m(w)$ and $M(w)$ was given in [1]; in the context of submultiplicative convex functions another example of functions with non-coinciding Matuszewska-Orlicz indices was given in [17], the latter example been also exposed in [18], p.93. In [26] there was explicitly constructed a family of functions with different indices belonging to the class Φ_γ^0 .

4.4 Auxiliary lemmas

Lemma 4.6. Let $w \in \widetilde{W}([0, \ell])$, $0 < \ell < \infty$, and $-\infty < m_w \leq M_w < \infty$, let $\lambda \geq 0$ and $\lambda M_w < \delta$, where $\delta > 0$. Then $\frac{t^\delta}{[w(t)]^\lambda} \in \mathcal{Z}^0$, that is,

$$\int_0^r \frac{x^{\delta-1} dx}{[w(x)]^\lambda} \leq C \frac{r^\delta}{[w(r)]^\lambda}, \quad 0 < r \leq \ell, \quad (4.10)$$

where the constant $C > 0$ does not depend on $r \in [0, \ell]$; it does not depend also on $0 \leq \lambda \leq d$ where $d > 0$ is arbitrary if $M_w \leq 0$ and $d < \frac{\delta}{M_w}$, if $M_w > 0$.

Proof. The function $w_1(x) = \frac{x^\delta}{[w(x)]^\lambda}$ is almost increasing, because the function $\frac{w(x)}{x^{\delta/\lambda}}$ is almost decreasing when $\frac{\delta}{\lambda} > M_w$, according to formula (4.5), the validity of which follows from Corollary 4.3. Therefore, $w_1 \in W$. By the definition of the lower index, we easily obtain that $m_{w_1} = \delta - \lambda M_w$. Hence $m_{w_1} > 0$ and consequently $w_1 \in \mathcal{Z}^0$ by Theorem 4.2.

In order to show that the constant C in (4.10) does not depend on the appropriate choice of λ , we proceed as follows

$$\int_0^r \frac{x^{\delta-1} dx}{[w(x)]^\lambda} = \frac{1}{[w(r)]^\lambda} \int_0^r \left[\frac{w(r)}{w(x)} \cdot \frac{x^{M_w+\varepsilon}}{r^{M_w+\varepsilon}} \right]^\lambda \frac{r^{(M_w+\varepsilon)\lambda} dx}{x^{(M_w+\varepsilon)\lambda+1-\delta}}.$$

The function $\frac{w(x)}{x^{M_w+\varepsilon}}$ is almost decreasing for every $\varepsilon > 0$ by Corollary 4.3. Therefore, the expression in the brackets is bounded from above. Since $\lambda \geq 0$, we get

$$\int_0^r \frac{x^{\delta-1} dx}{[w(x)]^\lambda} \leq C \frac{r^{(M_w+\varepsilon)\lambda}}{[w(r)]^\lambda} \int_0^r \frac{dx}{x^{(M_w+\varepsilon)\lambda+1-\delta}} = \frac{C}{\delta - \lambda(M_w + \varepsilon)} \cdot \frac{r^\delta}{[w(r)]^\lambda} \leq \frac{C_1 r^\delta}{[w(r)]^\lambda}$$

under the choice of ε sufficiently small: $0 < \varepsilon < \frac{\delta}{d} - M_w$. \square

Lemma 4.7. *Let Γ be a bounded Carleson curve, $\lambda(t) \geq 0$ on Γ and $\Lambda := \sup_{t \in \Gamma} \lambda(t)$. Let also $w \in \widetilde{W}([0, \ell])$, $\ell = |\Gamma|$ and $-\infty < m_w \leq M_w < \frac{1}{\Lambda}$. Then*

$$\int_{\gamma_r(t)} \frac{d\nu(\tau)}{[w(|t - \tau|)]^{\lambda(t)}} \leq C \frac{r}{[w(r)]^{\lambda(t)}} \quad (4.11)$$

where $C > 0$ does not depend on $t \in \Gamma$ and $r \in [0, \ell]$.

Proof. Let $\mu < m_w$ and $w_\mu(x) = \frac{w(x)}{x^\mu}$ so that $w_\mu(x)$ is an a.i. function according to (4.4) and Corollary 4.3. We proceed as follows:

$$J := \int_{\gamma_r(t)} \frac{d\nu(\tau)}{[w(|t - \tau|)]^{\lambda(t)}} = \sum_{k=0}^{\infty} \int_{L_k(t,r)} \frac{|t - \tau|^{-\mu\lambda(t)} d\nu(\tau)}{[w_\mu(|t - \tau|)]^{\lambda(t)}} \quad (4.12)$$

where $L_k(t, r) = \{\tau \in \Gamma : 2^{-k-1}r < |t - \tau| < 2^{-k}r\}$. Since the function w_μ is almost increasing, and $|t - \tau|^{-\mu\lambda(t)} \leq C (2^{-k}r)^{-\mu\lambda(t)}$ for $\tau \in L_k(t, r)$, we obtain

$$J \leq C \sum_{k=0}^{\infty} \frac{(2^{-k}r)^{-\mu\lambda(t)} |L_k(t, r)|}{[w_\mu(2^{-k-1}r)]^{\lambda(t)}} \leq C \sum_{k=0}^{\infty} \frac{2^{-k}r}{[w(2^{-k-1}r)]^{\lambda(t)}}. \quad (4.13)$$

The inequality

$$\frac{2^{-k}}{[w(2^{-k-1}r)]^{\lambda(t)}} \leq C \int_{2^{-k-1}}^{2^{-k}} \frac{dx}{[w(xr)]^{\lambda(t)}} \quad (4.14)$$

is valid, which follows from the direct estimation similar to the above arguments:

$$\int_{2^{-k-1}}^{2^{-k}} \frac{dx}{[w(xr)]^{\lambda(t)}} \geq \frac{C}{[w_\mu(2^{-k}r)]^{\lambda(t)}} \int_{2^{-k-1}}^{2^{-k}} (xr)^{-\mu\lambda(t)} dx \geq \frac{C 2^{-k}}{[w(2^{-k}r)]^{\lambda(t)}}.$$

By (4.14) from (4.13) we get

$$J \leq Cr \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \frac{dx}{[w(xr)]^{\lambda(t)}} = Cr \int_0^1 \frac{dx}{[w(xr)]^{\lambda(t)}} = c \int_0^r \frac{dx}{[w(x)]^{\lambda(t)}} \leq C \frac{r}{[w(r)]^{\lambda(t)}},$$

where in the last inequality we used Lemma 4.6 with $\delta = 1$. \square

5 Proof of Theorems A and A'.

5.1 Proof of Theorem A.

To Prove Theorem A, we have to show that

$$I_p(\mathcal{M}^\rho f) \leq c \text{ for } \|f\|_{p(\cdot)} \leq R.$$

Following the idea in [7], we represent $I_p(\mathcal{M}^\rho f)$ as

$$I_p(\mathcal{M}^\rho f) = \int_{\Omega} \left([\rho(x)]^{p_1(x)} \left| \mathcal{M} \left(\frac{f(y)}{\rho(y)} \right) (x) \right|^{p_1(x)} \right)^{p_*} d\nu(t), \quad (5.1)$$

where

$$p_1(x) = \frac{p(x)}{p_*}.$$

We make use of the known estimate

$$|\mathcal{M}\psi(x)|^{p_1(x)} \leq c \left(1 + \mathcal{M}[\psi^{p_1(\cdot)}](x) \right) \quad (5.2)$$

(see [7], valid for for all $\psi \in L^{p_1(\cdot)}(\Omega)$ with $\|\psi\|_{p_1(\cdot)} \leq C$.)

We intend to choose $\psi(y) = \frac{f(y)}{\rho(y)}$ with $f \in L^{p(\cdot)}(\Omega)$ in (5.2). This is possible because

$$\int_{\Omega} \left| \frac{f(y)}{\rho(y)} \right|^{\frac{p(y)}{p_*}} dy \leq C, \quad (5.3)$$

for all $f \in L^{p(\cdot)}$ with $\|f\|_p \leq c$. Estimate (5.3) is obtained by means of the usual Hölder inequality with the constant exponents p_* and $q_* = \frac{p_*}{p_*-1}$, taking into account that $\int_{\Omega} \frac{dy}{[\rho(y)]^{\frac{p(y)}{p_*-1}}} < \infty$, the latter following from condition (2.4).

In view of (5.3), we may apply estimate (5.2). Then (5.1) implies

$$I_p(\mathcal{M}^\rho f) \leq c \int_{\Omega} [\rho(x)]^{p(x)} \left[1 + \mathcal{M} \left(\left| \frac{f(y)}{\rho(y)} \right|^{p_1(y)} \right) \right]^{p_*} dx.$$

Since $\int_{\Omega} [\rho(x)]^{p(x)} dx < \infty$ by (2.4), we obtain

$$I_p(\mathcal{M}^\rho f) \leq c + c \int_{\Omega} [\mathcal{M}^{p_1}(|f(\cdot)|^{p_1(\cdot)})(x)]^{p_*} dx \quad (5.4)$$

under notation (2.1) with $\rho_1(x) = [\rho(x)]^{p_1(x)}$.

As is known [29], p. 201, the weighted maximal operator \mathcal{M}^{ρ_1} is bounded in L^{p_*} with a constant $p_* > 1$, if the weight $[\rho(x)]^{p_1(x)}$ is in A_{p_*} , which is nothing else but condition (2.4).

Therefore, by the boundedness of the weighted operator \mathcal{M}^{ρ_1} in L_{p_*} , from (5.4) we get

$$I_p(\mathcal{M}^\rho f) \leq c + c \int_{\Omega} |f(y)|^{p_1(y) \cdot p_*} dy = c + c \int_{\Omega} |f(y)|^{p(y)} dy < \infty. \quad (5.5)$$

Hence, by (3.2)-(3.3) we conclude that $\|\mathcal{M}f\|_{L^{p(\cdot)}(\Omega, \rho)} \leq C$ for all $f \in L^{p(\cdot)}(\Omega, \rho)$ with $\|f\|_{L^{p(\cdot)}(\Omega, \rho)} \leq 1$. Since \mathcal{M} is sublinear, this yields its boundedness in the space $L^{p(\cdot)}(\Omega, \rho)$.

5.2 Proof of Theorem A'.

The proof of Theorem A' is essentially the same. We only mention that an analogue of the pointwise estimate (5.2) for Carleson curves is also known, see Theorem 3.3 in [13] and Subsection 4.2 in [12], and the boundedness of the maximal operator along Carleson curves with Muckenhoupt weights satisfying the A_p -condition ($p \equiv \text{const}$)

$$\sup_{t \in \Gamma} \sup_{r > 0} \left(\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} [\rho(\tau)]^p d\nu(\tau) \right) \left(\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} [\rho(\tau)]^{-q} d\nu(\tau) \right)^{p-1} < \infty, \quad (5.6)$$

is also known, see [4], p.149.

6 Proof of Theorem B

It suffices to prove Theorem B for a single weight $w(|t - t_0|)$, $t_0 \in \Gamma$, $t^{\frac{1}{p(t_0)}} w(t) \in \Phi_1^0$, which may be shown by standard arguments, we refer for instance to [10], Subsection 5.1, where the Euclidean case was considered.

6.1 On condition (2.5) for weights in \widetilde{W} .

Lemma 6.1. *Let Γ be a bounded Carleson curve and let $w \in \widetilde{W}([0, \ell])$, $\ell = |\Gamma|$ and $-\infty < m_w \leq M_w < 1$. Then the inequality*

$$\mathcal{M}_r^w(1) := \frac{w(|t - t_0|)}{|\gamma_r(t)|} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{w(|\tau - t_0|)} \leq c \quad (6.1)$$

holds with $c > 0$ not depending on $0 < r < \ell$ and $t \in \Gamma$, if either $m_w > 0$ or $|t - t_0| \geq 2r$. In the case $|t - t_0| \leq 2r$, the estimate

$$\frac{w(r)}{|\gamma_r(t)|} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{w(|\tau - t_0|)} \leq c. \quad (6.2)$$

also holds.

Proof. 1⁰. The case $|t - t_0| \geq 2r$. We have

$$|\tau - t_0| \geq |t - t_0| - |\tau - t| \geq |t - t_0| - r \geq \frac{1}{2}|t - t_0|. \quad (6.3)$$

As in the proof of Lemma 4.7, let $\mu < m_w$ and $w_\mu(x) = \frac{w(x)}{x^\mu}$. Since w_μ is an a.i. function, we have $w_\mu(|\tau - t_0|) \geq cw_\mu(\frac{1}{2}|t - t_0|)$. Taking also into account the doubling property (4.8), we obtain

$$w_\mu(|\tau - t_0|) \geq cw_\mu(|t - t_0|).$$

Then we have

$$\mathcal{M}_r^w(1) \leq C \frac{w(|t - t_0|)}{rw_\mu(|t - t_0|)} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{|\tau - t_0|^\mu} = C \frac{|t - t_0|^\mu}{r} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{|\tau - t_0|^\mu}$$

If $\mu \geq 0$, we use (6.3) again and obtain (6.1). If $\mu < 0$, then

$$\begin{aligned} \frac{1}{|\tau - t_0|^\mu} &= |\tau - t_0|^{|\mu|} \leq C (|\tau - t|^{|\mu|} + |t - t_0|^{|\mu|}) \\ &\leq C (r^{|\mu|} + |t - t_0|^{|\mu|}) \leq C_1 |t - t_0|^{|\mu|}, \end{aligned}$$

whence (6.1) again follows.

2⁰. The case $|t - t_0| \leq 2r$. Observe that in this case $\gamma(t, r) \subset \gamma(t_0, 3r)$, since $|\tau - t| < r \implies |\tau - t_0| \leq |\tau - t| + |t - t_0| < 3r$. Hence

$$\mathcal{M}_r^w(1) \leq \frac{w(|t - t_0|)}{|\gamma_r(t)|} \int_{\gamma_{3r}(t_0)} \frac{d\nu(\tau)}{w(|\tau - t_0|)}$$

and then by Lemma 4.7 (with $\lambda(t) \equiv 0$) and Remark 4.5 we get

$$\mathcal{M}_r^w(1) \leq \frac{w(|t - t_0|)}{w(r)}. \quad (6.4)$$

This gives (6.2). In the case $m_\omega > 0$ the function $w(x)$ is almost increasing and then (6.4) yields (6.1). \square

Corollary 6.2. Let $w \in \widetilde{W}([0, \ell])$, $\ell = |\Gamma|$, and $p(t)$ be a bounded non-negative function on Γ satisfying condition (1.2). Then

$$\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} [w(|\tau - t_0|)]^{p(\tau)} d\nu(\tau) \leq C [w(\xi)]^{p(t_0)}, \quad \text{if } m_\omega > -\frac{1}{p(t_0)} \quad (6.5)$$

and

$$\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{[w(|\tau - t_0|)]^{p(\tau)}} \leq \frac{C}{[w(\xi)]^{p(t_0)}}, \quad \text{if } M_\omega < \frac{1}{p(t_0)} \quad (6.6)$$

where $\xi = \max(r, |t - t_0|)$.

Proof. By Lemma 3.1, the exponent $p(\tau)$ on the left-hand side of (6.5) and (6.6) may be replaced by $p(t_0)$ from the very beginning. It easily seen that $M_{[w(\cdot)]^{p(t_0)}} = p(t_0)M_w$ and $M_{[w(\cdot)]^{-p(t_0)}} = p(t_0)m_w$. Then (6.5)-(6.6) follow directly from (6.1)-(6.2). \square

Theorem 6.3. Let $w(x) \in \widetilde{W}(0, \ell)$, $\ell = |\Gamma|$ and $t_0 \in \Gamma$. A function $\rho(t) = w(|t - t_0|)$ satisfies condition (2.5) if

$$-\frac{1}{p(t_0)} < m_w \leq M_w < \frac{1}{q_0}, \quad (6.7)$$

where $\frac{1}{q_0} = \frac{p_*-1}{p(t_0)} \leq \frac{1}{q(t_0)}$.

Proof. By Corollary 6.2, we have

$$\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} [w(|\tau - t_0|)]^{p(\tau)} d\nu(\tau) \leq C[w(\xi)]^{p(t_0)}$$

and

$$\left(\frac{1}{|\gamma_r(t)|} \int_{\gamma_r(t)} \frac{d\nu(\tau)}{[w(|\tau - t_0|)]^{\frac{p(\tau)}{p_*-1}}} \right)^{p_*-1} \leq \frac{C}{[w(\xi)]^{p(t_0)}}$$

just under condition (6.7), which yields the validity of (2.5). \square

6.2 Proof itself of Theorem B.

1⁰ The case (6.7). This case is covered by Theorem A', because in the case (6.7) the weight $w(|t - t_0|)$ satisfies condition (2.5) by Theorem 6.3.

2⁰ The remaining case. To get rid of the right-hand side bound in (6.7), we may split integration over Γ into two parts, one over a small neighborhood $\gamma_\delta = \gamma_\delta(t_0)$ of the point t_0 , and another over its exterior $\Gamma \setminus \gamma_\delta$, and to choose δ sufficiently small so that the number $\frac{p_*(\gamma_\delta)-1}{p(t_0)}$ is arbitrarily close to $\frac{p(t_0)-1}{p(t_0)} = \frac{1}{q(t_0)}$. To this end we put

$$\begin{aligned} \mathcal{M}^w &= \chi_{\gamma_\delta} \mathcal{M}^w \chi_{\gamma_\delta} + \chi_{\gamma_\delta} \mathcal{M}^w \chi_{\Gamma \setminus \gamma_\delta} + \chi_{\Gamma \setminus \gamma_\delta} \mathcal{M}^w \chi_{\gamma_\delta} + \chi_{\Gamma \setminus \gamma_\delta} \mathcal{M}^w \chi_{\Gamma \setminus \gamma_\delta} \\ &=: \mathcal{M}_1^w + \mathcal{M}_2^w + \mathcal{M}_3^w + \mathcal{M}_4^w. \end{aligned} \quad (6.8)$$

Since the weight is strictly positive beyond any neighborhood of the point t_0 , we have

$$\mathcal{M}_4^w f(t) \leq C \mathcal{M} f(t). \quad (6.9)$$

For \mathcal{M}_3^w we have

$$\mathcal{M}_3^w f(t) = \sup_{r>0} \frac{\chi_{\Gamma \setminus \gamma_\delta(t_0)}(t)}{|\gamma_r(t)|} \int_{\gamma_r(t) \cap \gamma_\delta(t_0) \cap \Gamma} \frac{w(|t - t_0|)}{w(|\tau - t_0|)} |f(\tau)| d\nu(\tau).$$

Here $|t - t_0| > r > |\tau - t_0|$. Observe that the function $w_\varepsilon(t) = \frac{w(t)}{t^{M_w + \varepsilon}}$ is a.d. for any $\varepsilon > 0$ according to (4.5). Therefore

$$\frac{w(|t - t_0|)}{w(|\tau - t_0|)} = \frac{w_\varepsilon(|t - t_0|)}{w_\varepsilon(|\tau - t_0|)} \cdot \frac{|t - t_0|^{M_w + \varepsilon}}{|\tau - t_0|^{M_w + \varepsilon}} \leq C \frac{|t - t_0|^{M_w + \varepsilon}}{|\tau - t_0|^{M_w + \varepsilon}}.$$

Hence

$$\mathcal{M}_3^w f(t) \leq C \mathcal{M}^{M_w + \varepsilon} f(t) \quad (6.10)$$

where $\mathcal{M}^{M_w + \varepsilon} f(t)$ is the weighted maximal function with the power weight $|t - t_0|^{M_w + \varepsilon}$. Similarly we conclude that

$$\mathcal{M}_2^w f(t) \leq C \mathcal{M}^{m_w - \varepsilon} f(t). \quad (6.11)$$

Thus from (6.8) according to (6.9), (6.10) and (6.11) we have

$$\mathcal{M}^w f(t) \leq \chi_{\gamma_\delta} \mathcal{M}^w \chi_{\gamma_\delta} f(t) + \mathcal{M} f(t) + \mathcal{M}^{M_w + \varepsilon} f(t) + \mathcal{M}^{m_w - \varepsilon} f(t). \quad (6.12)$$

Here the operators \mathcal{M} , $\mathcal{M}^{M_w + \varepsilon}$ and $\mathcal{M}^{m_w - \varepsilon}$ are bounded in the space $L^{p(\cdot)}(\Gamma)$, because the boundedness condition (2.3) is satisfied for $\beta = M_w + \varepsilon$ and $\beta = m_w - \varepsilon$ under a choice of ε sufficiently small.

It remains to prove the boundedness of the first term on the right-hand side of (6.12). This is nothing else but the boundedness of the same operator \mathcal{M}^w over a small set $\Gamma_\delta = \gamma_\delta(t_0) \cap \Gamma$. According to the previous case, this boundedness holds if

$$-\frac{1}{p(t_0)} < m_w \leq M_w < \frac{1}{q_\delta} \quad (6.13)$$

where $q_\delta = \frac{p_*(\Gamma_\delta) - 1}{p(t_0)}$ and $p_*(\Gamma_\delta) = \min_{t \in \Gamma_\delta} p(t)$. Let us show that, given the condition $-\frac{1}{p(t_0)} < m_w \leq M_w < \frac{1}{q(t_0)}$, one can always choose δ sufficiently small such that (6.13) holds. Given $M_w < \frac{n}{q(t_0)}$, we have to choose δ so that $M_w < \frac{1}{q_\delta} \leq \frac{n}{q(t_0)}$. We have

$$\frac{1}{q_\delta} = \frac{1}{q(t_0)} - a(\delta), \quad \text{where} \quad a(\delta) = \frac{1}{p(t_0)} [p(t_0) - p_*(\Gamma_\delta)].$$

By the continuity of $p(t)$ we can choose δ so that $a(\delta) < \frac{1}{q(t_0)} - M_w$. Then $\frac{1}{q_\delta} > M_w$ and condition (6.13) is fulfilled. Then the operator \mathcal{M}^w is bounded in the space $L^{p(\cdot)}(\gamma_\delta)$ which completes the proof.

References

- [1] V. D. Aslanov and Yu. I. Karlovich. One-sided invertibility of functional operators in reflexive Orlicz spaces. *Akad. Nauk Azerbaidzhan. SSR Dokl.*, 45(11-12):3–7, 1989.
- [2] N.K. Bari and S.B. Stechkin. Best approximations and differential properties of two conjugate functions (in Russian). *Proceedings of Moscow Math. Soc.*, 5:483–522, 1956.
- [3] C. Bennett and R. Sharpley. *Interpolation of Operators.*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

- [4] A. Böttcher and Yu. Karlovich. *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*. Basel, Boston, Berlin: Birkhäuser Verlag, 1997. 397 pages.
- [5] D. Cruz-Uribe, A. Fiorenza, J.M. Martell, and C. Perez. The boundedness of classical operators on variable L^p spaces. *Preprint, Universidad Autonoma de Madrid, Departamento de Matematicas*, [http : //www.uam.es/personal_pdi/ciencias/martell/Investigacion/research.html](http://www.uam.es/personal_pdi/ciencias/martell/Investigacion/research.html), 2004. 26 pages.
- [6] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer. The maximal function on variable L^p -spaces. *Ann. Acad. Scient. Fennicae, Math.*, 28:223–238, 2003.
- [7] L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.
- [8] L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.
- [9] N.K. Karapetiants and N.G. Samko. Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω . *Fract. Calc. Appl. Anal.*, 7(4), 2004.
- [10] V. Kokilashvili, N. Samko, and S. Samko. The maximal operator in variable spaces $L^{p(\cdot)}(\Omega, \rho)$. *Georgian Math. J.*, 13(1):109–125, 2006.
- [11] V. Kokilashvili and S. Samko. Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Revista Matematica Iberoamericana*, 20(2):495–517, 2004.
- [12] V. Kokilashvili and S. Samko. Boundedness in Lebesgue spaces with variable exponent of maximal, singular and potential operators. *Izvestija VUZov. Severo-Kavkazskii region. Estestvennie nauki, Special issue "Pseudodifferential equations and some problems of mathematical physics", dedicated to 70th birthday of Prof. I.B.Simonenko*, (Special Issue):152–158, 2005.
- [13] V. Kokilashvili and S. Samko. Weighted boundedness in Lebesgue spaces with variable exponent of classical operators on Carleson curves. *Proc. A. Razmadze Math. Inst.*, 138:106–110, 2005.
- [14] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116):592–618, 1991.
- [15] S.G. Krein, Yu.I. Petunin, and E.M. Semenov. *Interpolation of linear operators*. Moscow: Nauka, 1978. 499 pages.
- [16] S.G. Krein, Yu.I. Petunin, and E.M. Semenov. *Interpolation of linear operators*, volume 54 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1982.
- [17] K. Lindberg. On subspaces of Orlicz sequence spaces. *Studia Math.*, 45:119–146, 1973.

- [18] L. Maligranda. Indices and interpolation. *Dissertationes Math. (Rozprawy Mat.)*, 234:49, 1985.
- [19] Kh. M. Murdaev and S.G. Samko. Fractional integro-differentiation in the weighted generalized Hölder spaces $H_0^\omega(\rho)$ with the weight $\rho(x) = (x - a)^\mu(b - x)^\nu$ (Russian) given continuity modulus of continuity (Russian). Deposited in VINITI, Moscow.
- [20] Kh. M. Murdaev and S.G. Samko. Weighted estimates of the modulus of continuity for fractional integrals of function with a given continuity modulus of continuity (Russian). Deposited in VINITI, Moscow.
- [21] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Math. Inequal. and Appl.* (submitted).
- [22] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Mathematical Preprints: Faculty of Civil Engineering, CTU, Prague*, (02/ May 2002), 2002.
- [23] N.G. Samko. Singular integral operators in weighted spaces with generalized Hölder condition. *Proc. A. Razmadze Math. Inst*, 120:107–134, 1999.
- [24] N.G. Samko. Criterion for the Fredholmness of singular operators with piecewise continuous coefficients in generalized Hölder spaces with weight. In *Singular integral operators, factorization and applications*, volume 142 of *Oper. Theory Adv. Appl.*, pages 345–356. Birkhäuser, Basel, 2003.
- [25] N.G. Samko. On compactness of Integral Operators with a Generalized Weak Singularity in Weighted Spaces of Continuous Functions with a Given Continuity Modulus. *Proc. A. Razmadze Math. Inst*, 136:91, 2004.
- [26] N.G. Samko. On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class. *Real Anal. Exch.*, 30(2), 2005.
- [27] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives. Theory and Applications*. London-New-York: Gordon & Breach. Sci. Publ., (Russian edition - *Fractional Integrals and Derivatives and some of their Applications*, Minsk: Nauka i Tekhnika, 1987.), 1993. 1012 pages.
- [28] S.G. Samko and Kh.M. Murdaev. Weighted Zygmund estimates for fractional differentiation and integration, and their applications. *Trudy Matem. Ibst. Steklov*, 180:197–198, 1987. Translated in *Proc. Steklov Inst. Math.*, AMS, 1989, issue 3, 233-235.
- [29] E.M. Stein. *Harmonic Analysis: real-variable methods, orthogonality and oscillatory integrals*. Princeton Univ. Press, Princeton, NJ, 1993.