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Essential Spectra of Pseudodifferential Operators in Sobolev Spaces with Variable Smoothness and Variable Lebesgue Indices

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During the past decade, there has been a substantial increase of interest in various classes of linear operators on generalized Lebesgue spaces with variable indices (see surveys [1, 7, 6] and the references therein) and on Sobolev spaces of variable order (see, e.g., [4]).

The Lebesgue spaces with variable indices play an important role in applications, in particular, in studying electrorheological fields [6], and the Sobolev spaces of variable order are very important for studying general boundary value problems on compact manifolds [3].

In this paper, we state a theorem about the boundedness of pseudodifferential operators (PDOs) from the Hörmander class $OPS^0_{1,\delta}$, where $0 \le \delta < 1$, in the Lebesgue spaces $L^{p(\cdot)}(\mathbf{R}^n)$ with variable indices, which follows from a more general theorem about the boundedness of integral operators with Calderon–Zygmund–type kernels in the spaces $L^{p(\cdot)}(\mathbf{R}^n)$. The boundedness of PDOs in the spaces $L^{p(\cdot)}(\mathbf{R}^n)$ implies the boundedness of PDOs of class $OPS^0_{1,\delta}$ in the Sobolev spaces $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$ with variable smoothness s and variable Lebesgue index (degree of integrability) p.

We also obtain a criterion for PDOs of class $OPS^m_{1,\delta}$ with symbols slowly oscillating at infinity to be Fredholm in the spaces $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$. This criterion implies, in particular, that the essential spectrum of a PDO $Op(a) \in OPS^m_{1,\delta}$ with $m > 0$ acting from $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$ in $H^{s(\cdot)-m,p(\cdot)}(\mathbf{R}^n)$ does not depend on the variable smoothness s and the variable index p.

Finally, we consider PDOs whose symbols can be extended in the momentum variable over some tubular domain in C^n. We give a criterion for such PDOs acting on the spaces $H^{s(\cdot),p(\cdot)}(\mathbf{R}^{n,w})$ with exponential weights w to be Fredholm and describe their essential spectra, which does not depend on the functions s and p but essentially depends on the weight w.

The Fredholm theory of PDOs in classes $OPS^m_{1,0}$ with symbols slowly oscillating at infinity on the Sobolev spaces $H^s(\mathbf{R}^n)$ was first constructed in [2]. The Fredholmness of PDOs belonging to $OPS^m_{0,0}$ on the Sobolev spaces was studied in [5, Chapter 4] by using the method of limit operators.

PDOs in the spaces $L^{p(\cdot)}(\mathbf{R}^n)$ and $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$. Let $p: \mathbf{R}^n \to (1,\infty)$ be a measurable function. The generalized Lebesgue space with variable index p consists of measurable functions for which

$$I^p(f) := \int_{\mathbf{R}^n} |f(x)|^{p(x)} dx < \infty.$$

The norm on $L^{p(\cdot)}(\mathbf{R}^n)$ is defined by

$$\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} := \inf\left\{\lambda > 0 \colon I^p\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{1}$$

Hereafter, we assume that the variable index p satisfies the conditions

$$1 < \inf_{x\in\mathbf{R}^n} p(x) \le \sup_{x\in\mathbf{R}^n} p(x) < \infty,$$

$$|p(x)-p(y)| \le A(-\ln|x-y|)^{-1},$$

$$x, y \in \mathbf{R}^n, \quad |x-y| \le \frac{1}{2},$$

$$|p(x)-p(\infty)| \le A(-\ln(1+|x|))^{-1}, \quad x \in \mathbf{R}^n.$$

Consider the integral operator

$$Au(x) = \int_{\mathbf{R}^n} k_A(x, x-y)u(y)dy$$

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with kernel $k_A(x, z) \in C^1(\mathbf{R}^n \times (\mathbf{R}^n \backslash 0))$. We suppose that

$$\lambda_1(A) := \sup_{|\alpha| = 1} \sup_{(x, z) \in \mathbf{R}^n \times \mathbf{R}^n} |z|^{n+1} \left|\partial_x^\alpha k(x, z)\right| < \infty, \quad (2)$$

$$\lambda_2(A) := \sup_{|\alpha| = 1} \sup_{(x, z) \in \mathbf{R}^n \times \mathbf{R}^n} |z|^{n+1} \left|\partial_z^\alpha k(x, z)\right| < \infty. \quad (3)$$

Theorem 1. *Suppose that A is an operator of weak* (1, 1)*-type, i.e., for any function* $f \in L^1(\mathbf{R}^n)$,

$$\operatorname{mes}\{x \in \mathbf{R}^n : |Af(x)| > t\} \le \frac{\nu(A)}{t} \int_{\mathbf{R}^n} |f(x)| dx, \quad (4)$$

and the kernel k_A *of the operator A satisfies conditions* (2) *and* (3).

Then, A is bounded in the space $L^{p(\cdot)}(\mathbf{R}^n)$ *and*

$$\|A\|_{L^{p(\cdot)}(\mathbf{R}^n) \to L^{p(\cdot)}(\mathbf{R}^n)} \le c(n, p)(\lambda_1(A) + \lambda_2(A) + \nu(A)), \quad (5)$$

where the constant $c(n, p)$ *depends only on the dimension n and the variable index p.*

We say that a symbol a belongs to the Hörmander class $S^m_{\rho,\delta}$, where $0 \le \delta \le \rho \le 1$ and $\delta < 1$, if $a \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and

$$|a|_{l_1, l_2} = \sum_{|\alpha| \le l_1, |\beta| \le l_2} \sup_{(x, \xi)} \left|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\right| \langle\xi\rangle^{-m + \rho|\alpha| - \delta|\beta|} < \infty$$

for all nonnegative integers l_1, l_2. The class of PDOs of the form

$$Op(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} d\xi \int_{\mathbf{R}^n} a(x, \xi) u(y) e^{i(x-y, \xi)} dy,$$

$$u \in C_0^\infty(\mathbf{R}^n)$$

with symbols belonging to $S^m_{\rho,\delta}$ is denoted by $OPS^m_{\rho,\delta}$.

Theorem 2. *The PDO* $Op(a) \in OPS^0_{1,\delta}$, *where* $0 \le \delta < 1$, *is bounded in the space* $L^{p(\cdot)}(\mathbf{R}^n)$, *and there exists a constant* $\mu = \mu(\{|a|_{l_1, l_2}\}_{l_1 \le N, l_2 \le N})$ *such that*

$$\|Op(a)\|_{L^{p(\cdot)}(\mathbf{R}^n) \to L^{p(\cdot)}(\mathbf{R}^n)} \le C\mu(\{|a|_{r, t}\}_{r \le N, t \le N}),$$

where C does not depend on the symbol a.

Theorem 2 follows from Theorem 1 and well-known estimates for kernels of PDOs of class $OPS^m_{1,\delta}$ (see, e.g., [8, p. 241]), which imply that $\lambda_1(Op(a))$, $\lambda_2(Op(a))$ depend on finitely many seminorms $|a|_{l_1, l_2}$. Moreover, the operator $Op(a) \in OPS^0_{1,\delta}$ is operator of weak (1, 1)-type (see, e.g., [8, pp. 16–23, 250]), and the constant $\nu(Op(a))$ also depends on finitely many seminorms $|a|_{l_1, l_2}$.

We set $\Lambda(s(\cdot), q) = Op((|\xi|^2 + q^2)^{s(x)/2})$ for $s \in C_b^\infty(\mathbf{R}^n)$, where $q > 0$ and s is a real-valued function from the space $C_b^\infty(\mathbf{R}^n)$ of functions bounded together with all of their derivatives. It is easy to show that $\Lambda(s(\cdot), q) = Op(|\xi|^2 + q^2)^{s(x)/2}) \in OPS^{s_+}_{1,\delta}$, where $s_+ = \sup_{x \in \mathbf{R}^n} s(x)$ and $\delta > 0$ is arbitrary. Moreover, $\Lambda(s(\cdot), q)$ is a hypoelliptic PDO, which depends on the parameter $q > 0$ and is continuous on the Schwartz spaces $S(\mathbf{R}^n)$ and $S'(\mathbf{R}^n)$. As is known (see, e.g., [9]), for some $q > 0$, there exists an inverse PDO $\Lambda^{-1}(s(\cdot), q) \in OPS^{s_-}_{1,\delta}$, where $s_- = \inf_{x \in \mathbf{R}^n} s(x)$. In what follows, we assume that, for the function s under consideration, the parameter q is chosen so that there exists an inverse operator. Consider the space $H^{s(\cdot), p(\cdot)}(\mathbf{R}^n)$ defined as the closure in the norm

$$\|u\|_{H^{s(\cdot), p(\cdot)}(\mathbf{R}^n)} := u\|\Lambda(s(\cdot), q)\|_{L^{p(\cdot)}(\mathbf{R}^n)}$$

of the space $S(\mathbf{R}^n)$. Obviously, the operator $\Lambda(s(\cdot), q)$: $H^{s(\cdot), p(\cdot)}(\mathbf{R}^n) \to L^{p(\cdot)}(\mathbf{R}^n)$ is an isometric isomorphism.

Theorem 3. *If* $s_+ - s_- < 1$, *then the PDO* $OPS^m_{1,\delta}$, *where* $0 \le \delta < 1$, *is bounded from* $H^{s(\cdot), p(\cdot)}(\mathbf{R}^n)$ *to* $H^{s(\cdot) - m, p(\cdot)}(\mathbf{R}^n)$, *and there exists a constant* $\mu = \mu(\{|a|_{l_1, l_2}\}_{l_1 \le N, l_2 \le N})$ *such that*

$$\|Op(a)\|_{H^{s(\cdot), p(\cdot)}(\mathbf{R}^n \to H^{s(\cdot) - m, p(\cdot)}(\mathbf{R}^n)} \le C\mu(\{|a|_{l_1, l_2}\}_{l_1 \le N, l_2 \le N}),$$

where $C > 0$ *does not depend on a.*

Theorem 3 follows from Theorem 2 and the fact that

$$\Lambda(s(\cdot) - m, q) Op(a) \Lambda^{-1}(s(\cdot), q)$$
$$= Op(b) \in OPS^0_{1,\delta}, \quad 0 < \delta < 1,$$

provided that $s_+ - s_- < 1$.

We say that a function $a \in C_b^\infty(\mathbf{R}^n)$ slowly oscillates at infinity if $\lim_{x \to \infty} \partial_{x_j} a(x, \xi) = 0$ for $j = 1, \ldots, n$. We denote the class of slowly oscillating functions by $SO(\mathbf{R}^n)$. In turn, we say that a symbol $S^m_{1,\delta}$ slowly oscillates at infinity if, for any multi-indices α and β, we have

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\right| \le C_{\alpha\beta}(x) \langle\xi\rangle^{m - |\alpha| + \delta|\beta|},$$

where $\lim_{x \to \infty} C_{\alpha\beta}(x) = 0$ if $\beta \ne 0$. We denote the class of symbols slowly oscillating at infinity by $SO^m_{1,\delta}$ and the corresponding class of PDOs by $OPSO^m_{1,\delta}$.

Theorem 4. *If* $s \in SO(\mathbf{R}^n)$ *and* $s_+ - s_- < 1$, *then the PDO* $Op(a) \in OPSO^m_{1,\delta}$ *is a Fredholm operator from* $H^{s(\cdot), p(\cdot)}(\mathbf{R}^n)$ *to* $H^{s(\cdot) - m, p(\cdot)}(\mathbf{R}^n)$ *if and only if*

$$\liminf_{(x, \xi) \to \infty} |a(x, \xi)| \langle\xi\rangle^{-m} > 0.$$

Remark. The sufficiency of the condition that the operator $Op(a)$ is Fredholm is proved in a fairly standard way by using composition formulas for PDOs, while the proof of necessity involves essential difficulties; the usual proof technique in the spaces $L^p(\mathbf{R}^n)$ with constant index p, which is based on the possibility of mapping any compact set into a given neighborhood of any point $x \in \mathbf{R}^n$ by isometric operators of translation and homothety, does not apply to the spaces $L^{p(\cdot)}(\mathbf{R}^n)$ with variable index, because the translation and homothety operators are not generally bounded in the spaces $L^{p(\cdot)}(\mathbf{R}^n)$.

Let $A: X \to Y$ be a bounded linear operator, and let $Y \subseteq X$. We say that $\lambda \in C$ is a point of essential spectrum of $A: X \to Y$ if the operator $A - \lambda I: X \to Y$ is not Fredholm. We denote the essential spectrum of $A: X \to Y$ by $\mathrm{sp}_{\mathrm{ess}}(A: X \to Y)$. The following theorem, which is a corollary of Theorem 4, describes the essential spectrum of the operator $Op(a) \in OPSO^m_{1,\delta}$, where $m \geq 0$.

Theorem 5. *If $A = Op(a) \in OPSO^m_{1,\delta}$, where $m \geq 0$, then*

$$\mathrm{sp}_{\mathrm{ess}}(A: H^{s(\cdot),p(\cdot)}(\mathbf{R}^n) \to H^{s(\cdot)-m,p(\cdot)}(\mathbf{R}^n))$$
$$= \bigcup_{(x_k,\xi_k)} \left\{ \lambda \in C: \lim_{k\to\infty} (a(x_k,\xi_k) - \lambda)\langle\xi_k\rangle^{-m} = 0 \right\}, \quad (6)$$

where the union is over all sequences $(x_k, \xi_k) \to \infty$ for which the limit on the right-hand side of (6) *exists.*

Example 1. Consider the Schrödinger operator $-\Delta + \Phi I$ with real potential $\Phi \in SO(\mathbf{R}^n)$. The operator $-\Delta + \Phi I$ belongs to $OPSO^2_{1,0}$ and, therefore, is bounded from $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$ to $H^{s(\cdot)-m,p(\cdot)}(\mathbf{R}^n)$. Using Theorem 5, we obtain

$$\mathrm{sp}_{\mathrm{ess}}(-\Delta + \Phi I: H^{s(\cdot),p(\cdot)}(\mathbf{R}^n) \to H^{s(\cdot)-m,p(\cdot)}(\mathbf{R}^n))$$
$$= [\Phi_-, \infty), \quad (7)$$

where $\Phi_- = \liminf_{x\to\infty} \Phi(x)$.

PDOs with analytic symbols. Let B be a convex bounded open domain in $\mathbf{R}^n$ containing the origin. By $S^m_{1,\delta}(B)$ we denote the subclass in $S^m_{1,\delta}$ consisting of those symbols $a(x, \xi)$ which can be extended analytically in the variable ξ over a tubular domain $\mathbf{R}^n + iB$ and, for any numbers l_1 and l_2, satisfy the condition

$$|a|_{l_1,l_2} = \sup_{x\in\mathbf{R}^n,\, \xi+i\eta\in\mathbf{R}^n+iB} \sum_{|\alpha|\le l_1,\, |\beta|\le l_2} \left|\partial_\xi^\alpha \partial_x^\beta a(x, \xi + i\eta)\right| \langle\xi\rangle^{-m+|\alpha|-\delta|\beta|} < \infty.$$

As above, we assign a pseudodifferential operator to each symbol $a \in S^m_{1,\delta}(B)$ and denote the class of such operators by $OPS^m_{1,\delta}(B)$. By $\mathcal{R}(B)$ we denote the class of positive weight functions w such that (i) $w = \exp v$, where $\partial_{x_j} v \in C_b^\infty(\mathbf{R}^n)$ for all $j = 1, 2, \ldots, n$ and $\lim_{x\to\infty} \partial^2_{x_i x_j} v(x) = 0$ for all $i, j = 1, 2, \ldots, n$; (ii) $\nabla v(x) \in B$ for any $x \in \mathbf{R}^n$.

The following proposition plays a key role in studying PDOs in weight spaces.

Proposition 1. *Suppose that $Op(a) \in OPSO^m_{1,\delta}(B)$ $(:= OPS^m_{1,\delta}(B) \cap OPSO^m_{1,\delta})$ and $w \in \mathcal{R}(B)$.*

Then, $wOp(a)w^{-1} \in OPSO^m_{1,\delta}$ and

$$wOp(a)w^{-1} = Op(a(x, \xi + i\nabla v(x))) + Op(t(x, \xi)), \quad (8)$$

where the symbol t satisfies the estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta t(x, \xi)\right| \le C_{\alpha\beta}(x)\langle\xi\rangle^{m-1-|\alpha|+\delta|\beta|},$$

in which

$$\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$$

for all multi-indices α and β.

Note that $Op(t)$ is a compact operator from $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$ to $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)$.

Let $H^{s(\cdot),p(\cdot)}(\mathbf{R}^n, w)$ denote the weight space with norm

$$\|u\|_{H^{s(\cdot),p(\cdot)}(\mathbf{R}^n,w)} = \|wu\|_{H^{s(\cdot),p(\cdot)}(\mathbf{R}^n)}.$$

The following theorem is a corollary of Proposition 1 and Theorems 3 and 4.

Theorem 6. *If $Op(a) \in OPSO^m_{1,\delta}(B)$, where $0 \le \delta < 1$, a weight w belongs to $\mathcal{R}(B)$, and $s \in SO(\mathbf{R}^n)$, then $Op(a): H^{s(\cdot),p(\cdot)}(\mathbf{R}^n, w) \to H^{s(\cdot)-m,p(\cdot)}(\mathbf{R}^n, w)$ is a Fredholm operator if and only if*

$$\liminf_{(x,\xi)\to\infty} |a(x, \xi + i\nabla v(x))|\langle\xi\rangle^{-m} > 0.$$

Theorem 6 has the following corollary, which describes the essential spectrum of a uniformly elliptic PDO in a weight space.

Theorem 7. *Suppose that* $Op(a) \in OPSO^m_{1,\delta}(B)$, $0 \le \delta < 1$, *a weight w belongs to* $\mathcal{R}(B)$, $s \in SO(\mathbf{R}^n)$, *and the operator* $Op(a)$ *is uniformly elliptic, i.e.,*

$$\lim_{r \to \infty} \inf_{|\xi| > r} |a(x, \xi)| \langle \xi \rangle^{-m} > 0.$$

Then,

$$\mathrm{sp}_{\mathrm{ess}}(Op(a): H^{s(\cdot), p(\cdot)}(\mathbf{R}^n, w) \to H^{s(\cdot) - m, p(\cdot)}(\mathbf{R}^n, w))$$

$$= \bigcup_g \{\lambda \in \mathbb{C}: \lambda = a_g(\xi + i\mu_g), \xi \in \mathbf{R}^n\},$$

where the union is over all sequences $g = \{x_k\} \to \infty$ *for which the limits* $a_g(\xi + i\mu_g) = \lim_{k \to \infty} a(x_k, \xi + i\nabla v(x_k))$ *exist.*

Thus, the essential spectrum of a PDO does not depend on the functions s and p, but it essentially depends on the exponential weight. However, if $\lim_{k \to \infty} \nabla v(x) = 0$, as in the case of the power weight $w(x) = \langle x \rangle^\rho = \exp(\rho \ln \langle x \rangle)$, $\rho \in \mathbf{R}$, then the essential spectrum does not depend on the weight.

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