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A GENERAL APPROACH TO WEIGHTED BOUNDEDNESS OF OPERATORS OF HARMONIC ANALYSIS IN VARIABLE EXPONENT LEBESGUE SPACES

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1. Introduction

We prove a certain $p(\cdot) \to q(\cdot)$-version of Rubio de Francia's extrapolation theorem [7] within the frameworks of weighted spaces $L^{p(\cdot)}_{\varrho}$ on metric measure spaces. By means of this extrapolation theorem and known theorems on the boundedness with Muckenhoupt weights in the case of constant p, we obtain results on weighted $p(\cdot) \to q(\cdot)$- or $p(\cdot) \to p(\cdot)$-boundedness - in the case of variable exponent $p(x)$ - of the following operators: potential type operators, Fourier multipliers, multipliers of trigonometric Fourier series, singular integral operators on Carleson curves and some others.

2. Definitions and preliminaries

In the sequel, (X, d, μ) denotes a metric space with the (quasi)metric d and non-negative measure μ, Ω is an open set in X. The following conditions are assumed to be satisfied: 1) all the balls $B(x, r)$ are measurable, 2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(\mu)$. In most of the statements we also suppose that 3) the measure μ satisfies the doubling condition: $\mu B(x, 2r) \le C\mu B(x, r)$, where $C > 0$ does not depend on $r > 0$ and $x \in X$.

For a locally μ-integrable function $f : X \to \mathbb{R}^1$ we consider the maximal function

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y).$$

By $A_s = A_s(X)$, where $1 \le s < \infty$, we denote the class of weights $w : X \to \mathbb{R}^1$ which satisfy the Muckenhoupt condition

$$\sup_B \left(\frac{1}{\mu B} \int\limits_B w(y) d\mu(y) \right) \left(\frac{1}{\mu B} \int\limits_B w^{-\frac{1}{s-1}}(y) d\mu(y) \right)^{s-1} < \infty$$

in the case $1 < s < \infty$, and the condition $\mathcal{M}w(x) \le Cw(x)$ with a constant $C > 0$, not depending on $x \in X$, in the case $s = 1$. As is known, the weighted boundedness $\int\limits_X (\mathcal{M}f(x))^s w(x) d\mu(x) \le C \int\limits_X |f(x)|^s w(x) d\mu(x)$ holds, if and only if $w \in A_s$.

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Definition 2.1. A μ-measurable function $p : \Omega \to \mathbb{R}^1$ is said to belong to the class $\mathcal{P}(\Omega)$, if

$$1 < p_- \le p_+ < \infty, \tag{1}$$

where $p_- = p_-(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ = p_+(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. It is said to belong to the class $WL(\Omega)$ (weak Lipshitz), if

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{d(x,y)}}\,, \quad d(x,y) \le \frac{1}{2}\,, \quad x,\, y \in \Omega, \tag{2}$$

where $A > 0$ does not depend on x and y.

Definition 2.2. By $L^{p(\cdot)}_{\varrho}(\Omega)$ we denote the weighted Banach function space of μ-measurable functions $f : \Omega \to \mathbb{R}^+_1$, such that

$$\|f\|_{L^{p(\cdot)}_{\varrho}} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{\varrho(x) f(x)}{\lambda}\right|^{p(x)} d\mu(x) \le 1\right\} < \infty. \tag{3}$$

Definition 2.3. We say that a weight ϱ belongs to the class $\mathfrak{A}_{p(\cdot)}(\Omega)$, if the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}_{\varrho}(\Omega)$.

For lower and upper local dimensions of X at a point x, we use an approach different from known in the fractal geometry and used in the variable exponent analysis on metric measure spaces in [3]. To this end, we use Matuzewska-Orlicz indices of measures of balls. This idea to introduce local dimensions in terms of these indices by the following definition was borrowed from [9].

Definition 2.4. The numbers

$$\underline{\mathfrak{dim}}(X;x) = \sup_{r>1} \frac{\ln\left(\underline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln\ r}\,, \quad \overline{\mathfrak{dim}}(X;x) = \inf_{r>1} \frac{\ln\left(\overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln\ r} \tag{4}$$

will be referred to as local lower and upper dimensions.

The "dimension" $\underline{\mathfrak{dim}}(X;x)$ may be also rewritten in terms of the upper limit as well:

$$\underline{\mathfrak{dim}}(X;x) = \sup_{0<r<1} \frac{\ln\left(\overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln\ r}. \tag{5}$$

Since the function $\mu_0(x,r) = \overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}$ is semimultiplicative in r, by properties of such functions we obtain that $\underline{\mathfrak{dim}}(X;x) \le \overline{\mathfrak{dim}}(X;x)$ and we may rewrite these dimensions also in the form

$$\underline{\mathfrak{dim}}(X;x) = \lim_{r\to 0} \frac{\ln \mu_0(x,r)}{\ln\ r}, \quad \overline{\mathfrak{dim}}(X;x) = \lim_{r\to\infty} \frac{\ln \mu_0(x,r)}{\ln\ r}. \tag{6}$$

For lower local dimensions we also introduce their lower bound

$$\underline{\mathfrak{dim}}(\Omega) := \operatorname*{ess\,inf}_{x\in X} \underline{\mathfrak{dim}}(\Omega;x).$$

In case where Ω is unbounded, we will also need similar dimensions connected in a sense with the influence of infinity. Let $\mu_\infty(x,r) = \overline{\lim}_{h\to\infty} \frac{\mu B(x,rh)}{\mu B(x,h)}$. We introduce the numbers

$$\underline{\mathfrak{dim}}_\infty(X) = \lim_{r\to 0} \frac{\ln \mu_\infty(x,r)}{\ln\ r}, \quad \overline{\mathfrak{dim}}_\infty(X) = \lim_{r\to\infty} \frac{\ln \mu_\infty(x,r)}{\ln\ r}. \tag{7}$$

As shown in [9], these limits do not depend on the "starting" point x. It is easy to see that they are non-negative. In the sequel, we always assume that $\underline{\mathfrak{dim}}(\Omega)$, $\underline{\mathfrak{dim}}_\infty(\Omega)$, $\overline{\mathfrak{dim}}_\infty(\Omega) \in (0,\infty)$.

We consider, in particular, the weights

$$\varrho(x) = \big[1 + d(x_0, x)\big]^{\beta_\infty} \prod_{k=1}^{N} [d(x, x_k)]^{\beta_k}, \quad x_k \in X, \;\; k = 0, 1, \ldots, N, \tag{8}$$

where $\beta_\infty = 0$ in the case where X is bounded. Let $\Pi = \{x_0, x_1, \ldots, x_N\}$ be a given finite set of points in X. We take $d(x, y) = |x - y|$ in all the cases where $X = \mathbb{R}^n$.

Definition 2.5. A weight function of form (8) is said to belong to the class $V_{p(\cdot)}(\Omega, \Pi)$, where $p(\cdot) \in C(\Omega)$, if

$$-\frac{\underline{\mathfrak{dim}}(\Omega)}{p(x_k)} < \beta_k < \frac{\underline{\mathfrak{dim}}(\Omega)}{p'(x_k)} \tag{9}$$

and, in the case Ω is infinite,

$$-\frac{\underline{\mathfrak{dim}}_{\,\infty}(\Omega)}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \underline{\mathfrak{dim}}_{\,\infty}(\Omega) - \frac{\overline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty}. \tag{10}$$

Note that when the metric space X has a constant dimension s in the sense that $c_1 r^s \le \mu B(x, r) \le c_2 r^s$ with the constants $c_1 > 0$ and $c_2 > 0$, not depending on $x \in X$ and $r > 0$, the inequalities in (9), (10) and (16) turn respectively into

$$-\frac{s}{p(x_k)} < \beta_k < \frac{s}{p'(x_k)}, \quad -\frac{s}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \frac{s}{p'_\infty} \tag{11}$$

and

$$-\frac{s}{p(x_k)} < m(w) \le M(w) < \frac{s}{p'(x_k)} \;\;, \quad k = 1, 2, \ldots, N. \tag{12}$$

We admit also a more general class of weights

$$\varrho(x) = w_0\big[1 + d(x_0, x)\big] \prod_{k=1}^{N} w_k\big[d(x, x_k)\big] \tag{13}$$

with “radial” weights, where the functions $w_k, k = 0, 1, \ldots, N$, belong to a class of Zygmund-Bary-Stechkin type with possible oscillation between two power functions with different exponents.

By $U = U([0, \ell])$ we denote the class of functions $u \in C([0, \ell])$, $0 < \ell \le \infty$, such that $u(0) = 0$, $u(t) > 0$ for $t > 0$ and u is an almost increasing function on $[0, \ell]$. By $\widetilde{U}$ we denote the class of function u, such that $t^a u(t) \in U$ for some $a \in \mathbb{R}^1$. Recall that a function $v \in U$ is said to belong to the Zygmund-Bary-Stechkin class Φ^0_δ, if

$$\int_0^h \frac{v(t)}{t} dt \le c v(h) \quad \text{and} \quad \int_h^\ell \frac{v(t)}{t^{1+\delta}} dt \le c \frac{v(h)}{h^\delta},$$

where $c = c(v) > 0$ does not depend on $h \in (0, \ell]$. It is known (see [8]) that $v \in \Phi^0_\delta$, if and only if $0 < m(v) \le M(v) < \delta$, where

$$m(w) = \sup_{t>1} \frac{\ln\left(\underline{\lim}_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \quad \text{and} \quad M(w) = \sup_{t>1} \frac{\ln\left(\overline{\lim}_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t}. \tag{14}$$

For functions w defined in the neighborhood of infinity and such that $w\left(\frac{\ell^2}{r}\right) \in \widetilde{U}([0, \ell])$, we introduce also

$$m_\infty(w) = \sup_{x>1} \frac{\ln\left[\underline{\lim}_{\,h\to\infty} \frac{w(xh)}{w(h)}\right]}{\ln\, x} \;\;, \quad M_\infty(w) = \inf_{x>1} \frac{\ln\left[\overline{\lim}_{h\to\infty} \frac{w(xh)}{w(h)}\right]}{\ln\, x}. \tag{15}$$

Generalizing Definition 2.5, we introduce also the following notion.

Definition 2.6. A weight function ϱ of form (13) is said to belong to the class $V^{osc}_{p(\cdot)}(\Omega,\Pi)$, where $p(\cdot)\in C(\Omega)$, if

$$w_k(r)\in \widetilde{U}([0,\ell]),\ \ell=\operatorname{diam}\Omega \quad\text{and}\quad -\frac{\underline{\mathfrak{dim}}(\Omega)}{p(x_k)}<m(w_k)\le M(w_k)<\frac{\underline{\mathfrak{dim}}(\Omega)}{p'(x_k)}, \tag{16}$$

$k=1,2,\dots,N$, and (in the case Ω is infinite) $w_0\left(\frac{1}{r}\right)\in \widetilde{U}([0,\delta])$ for some $\delta>0$, and

$$-\frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty}<\sum_{k=0}^N m_\infty(w_k)\le \sum_{k=0}^N M_\infty(w_k)<\frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p'_\infty}-\Delta_{p_\infty}, \tag{17}$$

where $\Delta_{p_\infty}=\frac{\overline{\mathfrak{dim}}_\infty(\Omega)-\underline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty}$.

Observe that in the case $\Omega=X=\mathbb{R}^n$ conditions (16) and (17) take the form

$$w_k(r)\in \widetilde{U}(\mathbf{R}^1_+):=\left\{w:\ w\left(r\right),w\left(\frac{1}{r}\right)\in \widetilde{U}([0,1])\right\} \tag{18}$$

and

$$\begin{aligned} -\frac{n}{p(x_k)}&<m(w_k)\le M(w_k)<\frac{n}{p'(x_k)},\\ -\frac{n}{p_\infty}&<\sum_{k=0}^N m_\infty(w_k)\le \sum_{k=0}^N M_\infty(w_k)<\frac{n}{p'_\infty}. \end{aligned} \tag{19}$$

Remark 2.7. For every $p_0\in(1,p_-)$ there hold the implications $\varrho\in V_{p(\cdot)}(\Omega,\Pi)\implies \varrho^{-p_0}\in V_{(\widetilde{p})'(\cdot)}(\Omega,\Pi)$ *and* $\varrho\in V^{ocs}_{p(\cdot)}(\Omega,\Pi)\implies \varrho^{-p_0}\in V^{ocs}_{(\widetilde{p})'(\cdot)}(\Omega,\Pi)$, where $\widetilde{p}(x)=\frac{p(x)}{p_0}$.

Theorem 2.8. *Let X be a metric space with doubling measure and let Ω be bounded. If $p\in\mathcal{P}(\Omega)\cap WL(\Omega)$ and $\varrho\in V^{osc}_{p(\cdot)}(\Omega,\Pi)$, then $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}_\varrho(\Omega)$.*

Theorem 2.9. *Let X be a metric space with doubling measure and let Ω be unbounded. Let $p\in\mathcal{P}(\Omega)\cap WL(\Omega)$ and let there exist $R>0$ such that $p(x)\equiv p_\infty=const$ for $x\in\Omega\backslash B(x_0,R)$. If $\varrho\in V^{osc}_{p(\cdot)}(\Omega,\Pi)$, then $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}_\varrho(\Omega)$.*

The Euclidean version of Theorems 2.8 and 2.9 was proved in [4], [5]; in [5] there were also proved the corresponding versions of these theorems for the maximal operator on Carleson curves.

Theorem 2.10. *Let Ω be a bounded open set in a doubling measure metric space X, let the exponent $p(x)$ satisfy conditions* (1), (2). *Then the operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}_\varrho(\Omega)$, if*

$$[\varrho(x)]^{p(x)}\in A_{p_-}(\Omega).$$

We refer to [6] for Theorem 2.9, its detailed proof for the case where X is a Carleson curve is given in [5], the proof for a doubling measure metric space being in fact the same.

3. Extrapolation theorem on metric measure spaces

In the sequel $\mathcal{F}=\mathcal{F}(\Omega)$ denotes a family of ordered pairs (f,g) of non-negative μ-measurable functions, defined on an open set $\Omega\subset X$. When saying that there holds an inequality of type (2) for all pairs $(f,g)\in\mathcal{F}$ and weights $w\in A_1$, we always mean that it is valid for all the pairs, for which the left-hand side is finite, and that the constant c depends only on p_0,q_0 and the A_1-constant of the weight. In the sequel, the numbers p_0 and q_0 are arbitrary such that

$$0<p_0\le q_0<\infty,\quad p_0<p_-\quad\text{and}\quad \frac{1}{p_0}-\frac{1}{p_+}<\frac{1}{q_0}\,. \tag{1}$$

We use the notation $\widetilde{p}(x) = \frac{p(x)}{p_0}$, $\widetilde{q}(x) = \frac{q(x)}{q_0}$.

Remark 3.1. The extrapolation Theorem 3.2 with variable exponents in the non-weighted case $\varrho(x) \equiv 1$ and in the Euclidean setting was proved in [1].

Observe that the measure μ in Theorem 3.2 is not assumed to be doubling.

Theorem 3.2. *Let X be a metric measure space and Ω an open set in X. Assume that for some p_0 and q_0, satisfying conditions* (1) *and every weight $w \in A_1(\Omega)$ there holds the inequality*

$$\Bigg(\int\limits_\Omega f^{q_0}(x)w(x)d\mu(x)\Bigg)^{\frac{1}{q_0}} \leq c_0\Bigg(\int\limits_\Omega g^{p_0}(x)[w(x)]^{\frac{p_0}{q_0}}d\mu(x)\Bigg)^{\frac{1}{p_0}} \tag{2}$$

for all f, g in a given family $\mathcal{F}$. Let the variable exponent $q(x)$ be defined by $\frac{1}{q(x)} = \frac{1}{p(x)} - \left(\frac{1}{p_0} - \frac{1}{q_0}\right)$, let the exponent $p(x)$ and the weight $\varrho(x)$ satisfy the conditions

$$p \in \mathcal{P}(\Omega) \quad and \quad \varrho^{-q_0} \in \mathfrak{A}_{(\widetilde{q})'}(\Omega). \tag{3}$$

Then for all $(f, g) \in \mathcal{F}$ with $f \in L^{p(\cdot)}_\varrho(\Omega)$ the inequality

$$\|f\|_{L^{q(\cdot)}_\varrho} \leq C\|g\|_{L^{p(\cdot)}_\varrho} \tag{4}$$

is valid with a constant $C > 0$, not depending on f and g.

4. Application to problems of the boundedness in $L^{p(\cdot)}_\varrho$ of classical operators of harmonic analysis

Let

$$I^\alpha_X f(x) = \int\limits_X \frac{f(y)\,d\mu(y)}{\mu B(x, d(x,y))^{1-\gamma}}, \tag{1}$$

where $0 < \gamma < 1$. We suppose that

$$\text{there exists a point } x_0 \in X \text{ such that } \quad \mu(x_0) = 0 \tag{2}$$

and

$$\mu(B(x_0)\backslash B(x_0, r)) > 0 \quad \text{for all} \quad 0 < r < R < \infty. \tag{3}$$

By means of the known results for constant p_0, q_0 ([2], p. 412) and extrapolation Theorem 3.2 we obtain the following statement.

Theorem 4.1. *Let X be a metric measure space with doubling measure satisfying conditions* (2)–(3)*, $\mu X = \infty$, let $p \in \mathcal{P}$, $0 < \gamma < 1$ and $p_+ < \frac{1}{\gamma}$. The weighted estimate $\left\|I^\gamma_X f\right\|_{L^{q(\cdot)}_\rho} \leq C\left\|f\right\|_{L^{p(\cdot)}_\rho}$ with the limiting exponent $q(\cdot)$ defined by $\frac{1}{q(x)} = \frac{1}{p(x)} - \gamma$, holds if $\varrho^{-q_0} \in \mathfrak{A}_{\left(\frac{q(\cdot)}{q_0}\right)'}(X)$ under any choice of $q_0 > \frac{p_-}{1-\gamma p_-}$.*

Remark 4.2. With the help of Theorems 2.8 and 2.9, one can write down the corresponding statements on the validity of the Sobolev inequality in terms of the weights used in Theorems 2.8 and 2.9. For potential operators in the case $\Omega = \mathbb{R}^n$ one can find more general statements of such a kind in [11] and [10] for power weights of the class $V_{p(\cdot)}(\mathbb{R}^n, \Pi)$ and for radial oscillating weights of the class $V^{osc}_{p(\cdot)}(\mathbb{R}^n, \Pi)$, respectively.

The following theorems on multipliers are direct consequences of Theorem 4.1 and may be given for weights of the class $V^{osc}_{p(\cdot)}(\Omega, \Pi)$, but for simplicity of formulation we give the theorems of this subsection for power type weights of the class $V_{p(\cdot)}(\Omega, \Pi)$.

Theorem 4.3. *Let a function $m(x)$ be continuous everywhere in $\mathbb{R}^n$, except for probably the origin, have the mixed distributional derivative $\frac{\partial^n m}{\partial x_1 x_2 \cdots x_n}$ and the derivatives $D^\alpha m = \frac{\partial^{|\alpha|} m}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ of orders $|\alpha| = \alpha_1 + \cdots + \alpha_n \le n-1$ continuous beyond the origin and $|x|^{|\alpha|}|D^\alpha m(x)| \le C$, $|\alpha| \le n-1$, where the constant $C > 0$ does not depend on x. Then under conditions (3) and (1) with $\Omega = \mathbb{R}^n$, m is a Fourier multiplier in $L_\varrho^{p(\cdot)}(\mathbb{R}^n)$.*

Corollary 4.4. *Let m satisfy the assumptions of Theorem 4.3 and let the exponent p and the weight ϱ satisfy the assumptions $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$, $p(x) = p_\infty = const$ for $|x| \ge R$ with some $R > 0$, $\varrho \in V_{p(\cdot)}^{osc}(\mathbb{R}^n, \Pi), \Pi = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$.*

Corollary 4.5. *Let a function $m : \mathbb{R}^n \to \mathbb{R}^1$ satisfy the assumptions of Theorem 4.3 and let p and ρ satisfy conditions i) and ii) of Corollary 4.4. Then m is a Fourier multiplier in $L_\varrho^{p(\cdot)}(\mathbb{R}^n)$.*

Let $\Delta_j = \Delta_j = [2^j, 2^{j+1}]$ or $\Delta_j = [-2^{j+1}, -2^j]$, $j \in \mathbb{Z}$. By T_m, we denote the operator defined on the Schwartz space by $\widehat{T_m f} = m\widehat{f}$. We obtain a generalization of theorems on Marcinkiewicz multipliers and Littlewood-Paley decompositions for trigonometric Fourier series to the case of weighted spaces with variable exponent. Let $\mathbb{T} = [\pi, \pi]$ and $f(x) \sim \frac{a_0}{2} + \sum\limits_{k=0}^{\infty} (a_k \cos kx + b_k sinkx)$.

Theorem 4.6. *Let a sequence λ_k satisfy the conditions $|\lambda_k| \le A$ and $\sum_{k=2^{j-1}}^{2^j-1} |\lambda_k - \lambda_{k+1}| \le A$, where $A > 0$ does not depend on k and j. Suppose that*

$$p \in \mathcal{P}(\mathbb{T}) \quad and \quad \varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\mathbb{T}), \quad where \quad \widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0} \tag{4}$$

with some $p_0 \in (1, p_-(\mathbb{T}))$. Then there exists a function $F(x) \in L_\varrho^{p(\cdot)}(\mathbb{T})$ such that the series $\frac{\lambda_0 a_0}{2} + \sum\limits_{k=0}^{\infty} \lambda_k (a_k \cos kx + b_k \sin kx)$ is Fourier series for F and $\|F\|_{L_\varrho^{p(\cdot)}} \le cA\|f\|_{L_\varrho^{p(\cdot)}}$, where $c > 0$ does not depend on $f \in L_\varrho^{p(\cdot)}(\mathbb{T})$.

Corollary 4.7. *The statement of Theorem 4.6 remains valid in particular, if (4) is replaced by the assumption that $p \in \mathcal{P}(\mathbb{T}) \cap WL(\mathbb{T})$ and*

$$\varrho(x) = \prod_{k=1}^{N} w_k(|x - x_k|), \quad x_k \in \mathbb{T} \tag{5}$$

where

$$w_k \in \widetilde{U}([0, 2\pi]) \quad and \quad -\frac{1}{p(x_k)} < m(w_k) \le M(w_k) < \frac{1}{p'(x_k)}. \tag{6}$$

Theorem 4.8. *Let $A_k(x) = a_k \cos kx + b_k \sin kx$, $k = 0, 1, 2, \dots$, $A_{2^{-1}} = 0$. Under conditions (4) there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \|f\|_{L_\varrho^{p(\cdot)}} \le \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{k=2^{j-1}}^{2^j-1} A_k(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\varrho^{p(\cdot)}} \le c_2 \|f\|_{L_\varrho^{p(\cdot)}}. \tag{7}$$

In particular, inequalities (7) hold for $p \in \mathcal{P}(\mathbb{T}) \cap WL(\mathbb{T})$ and weights ϱ of form (5)–(6).

Let $S_*(f) = S_*(f, x) = \sup\limits_{k \ge 0} |S_k(f, x)|$, where $S_k(f, x) = \sum\limits_{j=0}^{k} A_j(x)$.

Theorem 4.9. *Under conditions* (4), $\|S_*(f)\|_{L^{p(\cdot)}_\varrho} \le c\|f\|_{L^{p(\cdot)}_\varrho}$. *In particular, this inequality is valid, if* $p \in \mathcal{P}(\mathbb{T}) \cap WL(\mathbb{T})$ *and* ϱ *has form* (5)–(6).

Let now Γ be a simple finite Carleson curve and ν the arc length and $S_\Gamma f(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{f(\tau)\,d\nu(\tau)}{\tau - t}$.

Theorem 4.10. *Let* $p \in \mathcal{P}(\Gamma)$ *and* $\varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\Gamma)$, *where* $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$. *Then the operator* S_Γ *is bounded in the space* $L^{p(\cdot)}_\varrho(\Gamma)$. *In particular,* S_Γ *is bounded, if* $p \in \mathcal{P}(\Gamma) \cap WL(\Gamma)$ *and* $\varrho(t) = \prod_{k=1}^N w_k(|t - t_k|)$, $t_k \in \Gamma$, *where*

$$w_k \in \widetilde{U}([0, \nu(\Gamma)]) \quad and \quad -\frac{1}{p(t_k)} < m(w_k) \le M(w_k) < \frac{1}{p'(t_k)}. \tag{8}$$

Let $[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$, $x \in \mathbb{R}^n$ be the commutator, generated by the operator $Tf(x) = \int\limits_{\mathbb{R}^n} K(x, y)f(y)dy$ and a function $b \in BMO(\mathbb{R}^n)$.

Theorem 4.11. *Let the kernel* $K(x, y)$ *fulfill assumptions:* $\exists \lim\limits_{\varepsilon \to 0} \int\limits_{y \in \Omega: |x-y| > \varepsilon} K(x, y)\,dy$ *and* T *is bounded in* $L^2(\Omega)$, $|K(x', y) - K(x, y)| \le C\frac{|x'-x|^\alpha}{|x-y|^{n+\alpha}}$, $|x' - x| < \frac{1}{2}|x - y|$, $|K(x, y') - K(x, y)| \le C\frac{|y'-y|^\alpha}{|x-y|^{n+\alpha}}$, $|y' - y| < \frac{1}{2}|x - y|$, $\alpha > 0$, *and let* $b \in BMO(\mathbb{R}^n)$. *Then under the conditions*

$$p \in \mathcal{P}(\mathbb{R}^n) \quad and \quad \varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\mathbb{R}^n) \quad with \quad \widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0} \tag{9}$$

the commutator $[b, T]$ *is bounded in the space* $L^{p(\cdot)}_\varrho(\mathbb{R}^n)$. *In particular, the commutator is bounded, if* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$ *and* $p(x) \equiv p_\infty = const$ *outside some ball* $|x| < R$, *and the weight* ϱ *has the form* $\varrho(x) = w_0(1 + |x|)\prod_{k=1}^N w_k(|x - x_k|)$, $x_k \in \mathbb{R}^n$, *with the factors* w_k, *satisfying conditions* (18)–(19).

Let $f_B = \frac{1}{|B|}\int\limits_B f(x)\,dx$ and $\mathcal{M}^\# f(x) = \sup\limits_{B \in X} \frac{1}{|B|}\int\limits_B |f(x) - f_B|\,dx$ be the Fefferman-Stein maximal function.

Theorem 4.12. *Under condition* (9), *the inequality*

$$\|\mathcal{M}f\|_{L^{p(\cdot)}_\varrho(\mathbb{R}^n)} \le C\|\mathcal{M}^\# f\|_{L^{p(\cdot)}_\varrho(\mathbb{R}^n)} \tag{10}$$

is valid. In particular, inequality (10) *is valid, if* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$ *and* $p(x) \equiv p_\infty = const$ *outside some ball* $|x| < R$, $\varrho \in V^{osc}_{p(\cdot)}(\mathbb{R}^n, \Pi)$.

Let $f = (f_1, \cdots, f_k, \cdots)$, where $f_i : \mathbb{R}^n \to \mathbb{R}^1$ are locally integrable functions.

Theorem 4.13. *Let* $0 < \theta < \infty$. *Under conditions* (9), *the inequality*

$$\left\| \left(\sum_{j=1}^\infty (\mathcal{M}f_j)^\theta \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}_\varrho(\mathbb{R}^n)} \le C \left\| \left(\sum_{j=1}^\infty |f_j|^\theta \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}_\varrho(\mathbb{R}^n)} \tag{11}$$

is valid. In particular, inequality (11) *is valid, if* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$ *and* $p(x) \equiv p_\infty = const$ *outside some ball* $|x| < R$, $\varrho \in V^{osc}_{p(\cdot)}(\Omega, \Pi)$.

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