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Boundedness of Maximal Operators and Potential Operators on Carleson Curves in Lebesgue Spaces with Variable Exponent

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Abstract We prove the boundedness of the maximal operator $\mathscr{M}_\Gamma$ in the spaces $L^{p(\cdot)}(\Gamma,\rho)$ with variable exponent $p(t)$ and power weight ρ on an arbitrary Carleson curve under the assumption that $p(t)$ satisfies the log-condition on Γ.

We prove also weighted Sobolev type $L^{p(\cdot)}(\Gamma,\rho) \to L^{q(\cdot)}(\Gamma,\rho)$-theorem for potential operators on Carleson curves.

Keywords weighted generalized Lebesgue spaces, variable exponent, singular operator, fractional integrals, Sobolev theorem

MR(2000) Subject Classification 42B20, 47B38, 45P05

1 Introduction

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell \le \infty\}$ be a simple rectifiable curve with arc-length measure $\nu(t) = s$. In the sequel we denote

$$\Gamma(t,r) := \Gamma \cap B(t,r), \qquad t \in \Gamma, \quad r > 0, \tag{1.1}$$

where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$.

Everywhere below we assume that Γ is a Carleson curve. We remind that a curve is called a Carleson curve (regular curve), if there exists a constant $c_0 > 0$ not depending on t and r, such that

$$\nu\{\Gamma(t,r)\} \le c_0 r. \tag{1.2}$$

We consider — along Carleson curves Γ — the following operators within the frameworks of weighted spaces $L^{p(\cdot)}(\Gamma,w),\ w(t) = \prod_{k=1}^n |t-t_k|^{\beta_k},\ t_k \in \Gamma$ with variable exponent $p(t)$ (see definitions in Section 4): the maximal operator

$$Mf(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau) \tag{1.3}$$

and the potential type operator

$$I^{\alpha(\cdot)} f(t) = \int_\Gamma \frac{f(\tau)\, d\nu(\tau)}{|t-\tau|^{1-\alpha(t)}}, \tag{1.4}$$

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where it is supposed that

$$\alpha_- := \inf_{t\in\Gamma} \alpha(t) > 0, \qquad \alpha_+ := \sup_{t\in\Gamma} \alpha(t) < 1 \tag{1.5}$$

(see for instance [1] on Riesz potentials with respect to an arbitrary measure or [2] in case of homogeneous type spaces). By

$$M^\beta f(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)} \frac{|t-t_0|^\beta}{|\tau-t_0|^\beta} |f(\tau)| d\nu(\tau), \tag{1.6}$$

we denote the weighted version of the maximal operator.

The results we obtain here for these classical operators are valid not only on Carleson curves, but also in a more general context of metric spaces or homogeneous type spaces (HTS) at least under the condition $\mu(B(x,r)) \sim r^d$ (see [2] on maximal and potential operators over HTS in case of constant p). However, in this paper we develop our results specially in the context of Carleson curves because of applications to the singular operator

$$S_\Gamma f(t) = \frac{1}{\pi \mathrm{i}} \int_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{1.7}$$

over Carleson curves and singular integral equations over such curves. For example, the following boundedness result is valid, where $w(t) = \prod_{k=1}^n |t-t_k|^{\beta_k},\ t_k \in \Gamma$ in the case Γ is a finite curve and $w(t) = |t-z_0|^\beta \prod_{k=1}^n |t-t_k|^{\beta_k},\ t_k \in \Gamma,\ z_0 \notin \Gamma$ in the case it is infinite.

Theorem *Let*

i) Γ *be a simple Carleson curve*;

ii) p *satisfy conditions*

$$1 < p_- \le p(t) \le p_+ < \infty, \qquad |p(t)-p(\tau)| \le \frac{A}{\ln \frac{1}{|t-\tau|}}, \qquad t\in\Gamma,\ \tau\in\Gamma,\ |t-\tau| \le \frac{1}{2};$$

iii) *In the case* Γ *is an infinite curve, let* p *also satisfy the following condition at infinity*

$$|p(t)-p(\tau)| \le \frac{A_\infty}{\ln \frac{1}{\left|\frac{1}{t}-\frac{1}{\tau}\right|}}, \qquad \left|\frac{1}{t}-\frac{1}{\tau}\right| \le \frac{1}{2}, \qquad |t| \ge L,\ |\tau| \ge L \tag{1.8}$$

for some $L > 0$.

Then the singular operator S_Γ *is bounded in the space* $L^{p(\cdot)}(\Gamma, w)$ *, if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, \dots, n, \tag{1.9}$$

and also

$$-\frac{1}{p_\infty} < \beta + \sum_{k=1}^n \beta_k < \frac{1}{p'(\infty)} \tag{1.10}$$

in the case Γ *is infinite.*

This theorem will be proved in another paper. In the context of a general scheme of investigation of Fredholmness of singular integral equations (see [3, Section 4]), it is clear that the boundedness of the singular operator S_Γ immediately yields the Fredholmness statement for an arbitrary Carleson curve in its sufficiency part and also in its necessity part for Carleson curves without curling points. These questions will also be treated in another publication.

2 Definitions

The theory of generalized Lebesgue spaces with variable exponent may be found for instance in [4–7], where it was presented for the Euclidean case. This theory is known to develop rapidly last years in connection with various applications, see for instance [8–10], where other references may be also found. We give below the necessary definitions for the case of spaces on Carleson curves.

Let p be a measurable function on Γ such that $p:\Gamma\to(1,\infty)$. In what follows we assume that p satisfies the conditions

$$1<p_- := \operatorname*{ess\,inf}_{t\in\Gamma} p(t) \le \operatorname*{ess\,sup}_{t\in\Gamma} p(t) =: p_+ < \infty, \tag{2.1}$$

$$|p(t)-p(\tau)| \le \frac{A}{\ln\frac{1}{|t-\tau|}},\ t\in\Gamma,\ \tau\in\Gamma,\ |t-\tau|\le\frac{1}{2}. \tag{2.2}$$

Definition 2.1 *By $\mathscr{P}=\mathscr{P}(\Gamma)$ we denote the class of exponents p satisfying condition* (2.1) *and by $\mathbb{P}=\mathbb{P}(\Gamma)$ the class of those p for which the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma)$.*

Observe that condition (2.2) may be also written in the form

$$|p(t)-p(\tau)| \le \frac{2\ell A}{\ln\frac{2\ell}{|t-\tau|}}, \qquad t,\tau\in\Gamma, \tag{2.3}$$

where ℓ is the length of the curve.

The generalized Lebesgue space with variable exponent is defined via the modular

$$\mathfrak{I}^p_\Gamma(f) := \int_\Gamma |f(t)|^{p(t)}\, d\nu(t)$$

by the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda>0 : \mathfrak{I}^p_\Gamma\left(\frac{f}{\lambda}\right)\le 1\right\}.$$

Observe that

$$\|f\|_{p(\cdot)} = \|f^a\|^{\frac{1}{a}}_{\frac{p(\cdot)}{a}} \tag{2.4}$$

for any $0<a\le\inf p(t)$.

By $L^{p(\cdot)}(\Gamma,w)$ we denote the weighted Banach space of all measurable functions $f:\Gamma\to\mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma,w)} := \|wf\|_{p(\cdot)} = \inf\left\{\lambda>0 : \int_\Gamma \left|\frac{w(t)f(t)}{\lambda}\right|^{p(t)} d\nu(t)\le 1\right\}<\infty. \tag{2.5}$$

We denote $p'(t)=\frac{p(t)}{p(t)-1}$.

From the Hölder inequality for the $L^{p(\cdot)}$-spaces

$$\left|\int_\Gamma u(\tau)v(\tau)\,d\nu(\tau)\right| \le k\|u\|_{L^{p(\cdot)}(\Gamma)}\|v\|_{L^{p'(\cdot)}(\Gamma)}, \qquad \frac{1}{p(\tau)}+\frac{1}{p'(\tau)}\equiv 1,$$

it follows that

$$\left|\int_\Gamma u(t)v(t)\,d\nu(t)\right| \le k\|u\|_{L^{p'}\left(\Gamma,\frac{1}{w}\right)}\|v\|_{L^p(\Gamma,w)}, \tag{2.6}$$

and for the conjugate space $\left[L^{p(\cdot)}(\Gamma, w)\right]^*$ we have

$$[L^{p(\cdot)}(\Gamma, w)]^* = L^{p'(\cdot)}(\Gamma, 1/w) \tag{2.7}$$

which is an immediate consequence of the fact that $\left[L^{p(\cdot)}(\Gamma)\right]^* = L^{p'(\cdot)}(\Gamma)$ under condition (2.1), see [6–7].

The following value

$$\frac{1}{p_\gamma} = \frac{1}{\nu(\gamma)} \int_\gamma \frac{d\nu(t)}{p(t)}, \quad \gamma \subset \Gamma \tag{2.8}$$

will be used, introduced for balls in $\mathbb{R}^n$ by Diening [11]. Here $\gamma = \Gamma(t, r),\ t \in \Gamma,\ r > 0$, is any portion of the curve Γ.

By $\chi_\gamma(\tau) = \begin{cases} 1, & \tau\in\gamma \\ 0, & \tau\in\Gamma\backslash\gamma \end{cases}$ we denote the characteristic function of a portion γ of the curve Γ.

3 The Main Statements

In the sequel we consider the power weights of the form

$$w(t) = \prod_{k=1}^n |t - t_k|^{\beta_k}, \qquad t_k \in \Gamma \tag{3.1}$$

in the case of finite curve and the weights

$$w(t) = |t - z_0|^\beta \prod_{k=1}^n |t - t_k|^{\beta_k}, \qquad t_k \in \Gamma,\ z_0 \notin \Gamma \tag{3.2}$$

in the case of infinite curve.

3.1 Theorems on Maximal Operator

Theorem A *Let*

i) Γ *be a simple Carleson curve of a finite length*;

ii) p *satisfy conditions* (2.1)–(2.2).

Then the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma, w)$ with weight (3.1), *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, \ldots, n. \tag{3.3}$$

Theorem B *Let*

i) Γ *be an infinite simple Carleson curve*;

ii) p *satisfy conditions* (2.1)–(2.2) *and let there exist a circle $B(0, R)$ such that $p(t) \equiv p_\infty =$ const for $t \in \Gamma\backslash(\Gamma \cap B(0, R))$.*

Then the maximal operator M is bounded in the space $L^{p(\cdot)}(\Gamma, w)$, with weight (3.2), *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)} \qquad \text{and} \qquad -\frac{1}{p_\infty} < \beta + \sum_{k=1}^n \beta_k < \frac{1}{p'_\infty}. \tag{3.4}$$

The Euclidean space versions of Theorems A and B for variable exponents were proved in [12] and [13], respectively.

3.2 Theorems on Potential Operators

Theorem C *Let*

i) Γ *be a simple Carleson curve of a finite length*;

ii) p *satisfy conditions* (2.1)–(2.2);

iii) $\alpha(t)$ *satisfy assumptions* (1.5) *and the condition*

$$\sup_{t\in\Gamma}\alpha(t)p(t) < 1. \tag{3.5}$$

Then the operator $I^{\alpha(\cdot)}$ *is bounded from the space* $L^{p(\cdot)}(\Gamma)$ *into the space* $L^{q(\cdot)}(\Gamma)$ *with* $\frac{1}{q(t)} = \frac{1}{p(t)} - \alpha(t)$. *This statement remains valid for infinite Carleson curves if, in addition to conditions* i)–iii), $p(t) = p_\infty = const$ *and* $\alpha(t) = \alpha_\infty = const$ *outside some circle* $B(t_0, R), t_0 \in \Gamma$.

The next theorem is a weighted generalization of Theorem C for finite curves.

Theorem D *Under assumptions* i)–iii) *of Theorem* C *and the condition*

$$|\alpha(t) - \alpha(t_k)| \le \frac{A}{|\ln|t - t_k||}, \quad k = 1, \dots, n,$$

the operator $I^{\alpha(\cdot)}$ *is bounded from the space* $L^{p(\cdot)}(\Gamma, w)$ *into the space* $L^{q(\cdot)}(\Gamma, w)$ *where* $\frac{1}{q(t)} = \frac{1}{p(t)} - \alpha(t)$, *and* w *is the weight* (3.1), *if*

$$\alpha(t_k) - \frac{1}{p(t_k)} < \beta_k < 1 - \frac{1}{p(t_k)}, \qquad k = 1, \dots, n. \tag{3.6}$$

Corollary *Under the assumptions of Theorem D, the fractional maximal operator*

$$M_{\alpha(\cdot)}f(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}^{n-\alpha(t)}} \int_{\Gamma(t,r)} |f(\tau)|\, d\nu(\tau)$$

is bounded from the space $L^{p(\cdot)}(\Gamma, w)$ *into the space* $L^{q(\cdot)}(\Gamma, w)$.

4 Preliminaries

Lemma 4.1 *Let* Γ *be a Carleson curve and let* $\beta = \beta(t)$ *be a function defined on* Γ. *If* $\sup_{t\in\Gamma}\beta(t) < 1$, *then*

$$c_1 r^{1-\beta(t)} \le \int_{\Gamma(t,r)} \frac{d\nu(\tau)}{|\tau - t|^{\beta(t)}} \le c_2 r^{1-\beta(t)}, \tag{4.1}$$

where positive constants c_1 *and* c_2 *do not depend on* $t \in \Gamma$ *and* $r > 0$. *For fixed* t, *the condition* $\beta(t) < 1$ *is necessary for the convergence of the integral.*

Proof The proof is standard:

$$\int_{\Gamma(t,r)} \frac{d\nu(\tau)}{|t-\tau|^{\beta(t)}} = \sum_{k=0}^{\infty} \int_{\substack{\tau\in\Gamma:\\ 2^{-(k+1)}r<|t-\tau|<2^{-k}r}} \frac{d\nu(\tau)}{|t-\tau|^{\beta(t)}}$$
$$\le \sum_{k=0}^{\infty} \frac{\nu\left(\Gamma(t, 2^{-k}r)\right))}{\left(2^{-(k+1)}r\right)^{\beta(t)}} \le c r^{1-\beta(t)} 2^{\beta(t)} \sum_{k=0}^{\infty} \frac{1}{2^{k[1-\beta(t)]}} \le c_2 r^{1-\beta(t)},$$

where the property (1.2) has been used. Similarly, the lower bound in (4.1) and the necessity of the condition $\beta(t) < 1$ are obtained. □

Similarly, the following statement is proved.

Lemma 4.2 *Let Γ be a Carleson curve and let $\lambda = \lambda(t)$ be a function defined on Γ. If* $\inf_{t\in\Gamma} \lambda(t) > 1$, *then*

$$c_1 r^{1-\lambda(t)} \le \int_{\Gamma\setminus\Gamma(t,r)} \frac{d\nu(\tau)}{|\tau - t|^{\lambda(t)}} \le c_2 r^{1-\lambda(t)} \tag{4.2}$$

with positive constants c_1 and c_2 not depending on $t \in \Gamma$ and $r > 0$. For fixed t, the condition $\lambda(t) > 1$ is necessary for the convergence of the integral in case Γ is an infinite curve.

Proposition 4.3 *Let $p(t)$ satisfy condition* (2.1) *and the maximal operator M be bounded in $L^{p(\cdot)}(\Gamma)$. Then there exists a constant $C > 0$ such that*

$$\|\chi_\gamma\|_{p(\cdot)} \le C[\nu(\gamma)]^{\frac{1}{p_\gamma}} \qquad \textit{for all} \qquad \gamma = \Gamma(t,r) \subset \Gamma, \tag{4.3}$$

where p_γ is the mean value (2.8).

Proposition 4.3 was proved in [11], Lemma 3.4, for balls in the Euclidean space and remains the same for arcs γ on Carleson curves. For completeness of presentation we expose this proof in the Appendix.

5 Auxiliary Statements

5.1 Estimation of $\int_{\Gamma(t,r)} |\tau - t_0|^{-\beta} d\nu(\tau)$

Lemma 5.1 *Let $t_0 \in \Gamma$ and $0 \le \beta < 1$. Then*

$$J_\beta(t, t_0; r) := \frac{|t - t_0|^\beta}{\nu(\Gamma(t,r))} \int_{\Gamma(t,r)} \frac{d\nu(\tau)}{|\tau - t_0|^\beta} \le c < \infty, \tag{5.1}$$

where $c > 0$ does not depend on $t, t_0 \in \Gamma$ and $r > 0$.

Proof 1) The case $|t - t_0| \ge 2r$. In this case $|\tau - t_0| \ge |t - t_0| - |\tau - t| \ge |t - t_0| - r \ge \frac{1}{2}|t - t_0|$. Therefore,

$$J_\beta(t, t_0; r) \le \frac{2^\beta}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)} d\nu(\tau) = 2^\beta. \tag{5.2}$$

2) The case $|t - t_0| \le 2r$. Observe that the ball $B(t,r)$ is embedded into the ball $B(t_0, 3r)$. Indeed, if $|\tau - t| < r$, then $|\tau - t_0| \le |\tau - t| + |t - t_0| < r + 2r = 3r$. Hence

$$J_\beta(t, t_0; r) \le \frac{|t - t_0|^\beta}{\nu(\Gamma(t,r))} \int_{\Gamma(t_0,3r)} \frac{d\nu(\tau)}{|\tau - t_0|^\beta}.$$

Making use of the right-hand side estimate in (4.1), we get

$$J_\beta(t, t_0; r) \le c \left(\frac{|t - t_0|}{r}\right)^\beta \frac{r}{\nu\{\Gamma(t,r)\}} \le \frac{c2^\beta r}{\nu\{\Gamma(t,r)\}} \le c_1 < \infty.$$

5.2 An Auxiliary Statement on Maximal Functions

We will need the following technical lemma.

Lemma 5.2 *Let Γ be a finite or infinite Carleson curve, $p : \Gamma \to [1, \infty)$ be a bounded function satisfying condition* (2.2), *$\gamma = \Gamma(t,r)$ and p_γ be the mean value defined in* (2.8). *Then for any bounded range of r, $0 < r \le \ell < \infty$, there exists a constant $C > 1$ not dependent on $t \in \Gamma$ and $r \in (0, \ell]$ such that*

$$\frac{1}{C} \le r^{\frac{1}{p(t)} - \frac{1}{p_\gamma}} \le C. \tag{5.3}$$

Proof The inequality in (5.3) is equivalent to

$$\frac{1}{C_1} \leq \left(\frac{r}{2\ell}\right)^{\frac{1}{p(t)}-\frac{1}{p_\gamma}} \leq C_1$$

or

$$\left|\frac{1}{p(t)} - \frac{1}{p_\gamma}\right| \ln\frac{2\ell}{r} \leq c, \qquad c = \ln C_1,$$

that is,

$$\left|\frac{1}{\nu(\gamma)} \int_{\Gamma(t,r)} \left(\frac{1}{p(t)} - \frac{1}{p(\tau)}\right) d\nu(\tau)\right| \ln\frac{2\ell}{r} \leq c,$$

which is true since by (2.3) we have

$$\left|\frac{1}{p(t)} - \frac{1}{p(\tau)}\right| \leq c|p(t)-p(\tau)| \leq \frac{2\ell A}{\ln\frac{2\ell}{|t-\tau|}} \leq \frac{2\ell A}{\ln\frac{2\ell}{r}}. \qquad \square$$

6 Proof of Theorem A

6.1 General Remark

Remark 6.1 It suffices to prove Theorem A (as well as Theorems B, D, E and F) for a single weight $|t-t_0|^\beta$ where $t_0 \in \Gamma$ in the case Γ is finite and t_0 may belong or not belong to Γ when Γ is infinite.

Indeed, in the case of a finite curve let $\Gamma = \bigcup_{k=1}^n \Gamma_k$ where Γ_k contains the point t_k in its interior and does not contain $t_j, j \neq k$ in its closure. Then

$$\|f\|_{L^{p(\cdot)}\left(\Gamma, \prod\limits_{k=1}^n |t-t_k|^{\beta_k}\right)} \sim \sum_{k=1}^n \|f\|_{L^{p(\cdot)}\left(\Gamma_k, |t-t_k|^{\beta_k}\right)}, \tag{6.1}$$

whenever $1 \leq p_- \leq p_+ < \infty$. This equivalence follows from the easily checked modular equivalence

$$\Im_\Gamma^p\left(f(t)\prod_{k=1}^n |t-t_k|^{\beta_k}\right) \sim \sum_{k=1}^n \Im_{\Gamma_k}^p\left(f(t)|t-t_k|^{\beta_k}\right),$$

since

$$c_1 \leq \|f\|_{p(\cdot)} \leq c_2 \implies c_3 \leq \Im_\Gamma^p(f) \leq c_4 \ \text{ and } \ C_1 \leq \Im_\Gamma^p(f) \leq C_2 \implies C_3 \leq \|f\|_{p(\cdot)} \leq C_4 \tag{6.2}$$

with $c_3 = \min(c_1^{p_-}, c_1^{p_+}), c_4 = \max(c_2^{p_-}, c_2^{p_+}), C_3 = \min(C_1^{\frac{1}{p_-}}, C_1^{\frac{1}{p_+}})$ and $C_4 = \max(C_2^{\frac{1}{p_-}}, C_2^{\frac{1}{p_+}})$.

Similarly, in the case of an infinite curve

$$\|f\|_{L^{p(\cdot)}\left(\Gamma, |t-z_0|^\beta \prod\limits_{k=1}^n |t-t_k|^{\beta_k}\right)} \sim \|f\|_{L^{p(\cdot)}(\Gamma_\infty, |t-z_0|^\beta)} + \sum_{k=1}^n \|f\|_{L^{p(\cdot)}\left(\Gamma_k, |t-t_k|^{\beta_k}\right)}, \tag{6.3}$$

where Γ_∞ is a portion of the curve outside some large circle, so that Γ_∞ does not contain the points $t_k, k = 1, \dots, n$.

Then, because of (6.1) and (6.3), the statement of Remark 6.1 is obtained by introduction of the standard partition of unity $1 = \sum_{k=1}^n a_k(t)$, where $a_k(t)$ are smooth functions equal to 1 in a neighborhood $\Gamma(t_k, \varepsilon)$ of the point t_k and equal to 0 outside its neighborhood $\Gamma(t_k, 2\varepsilon)$

(and similarly in a neighborhood of infinity in the case Γ is infinite), so that $a_k(t)|t-t_j|^{\pm\beta_j} \equiv 0$ in a neighborhood of the point t_k, if $k \neq j$.

6.2 A pointwise estimate for the weighted means

We follow the main ideas in [12]. First we prove the following auxiliary result on a pointwise estimate for the weighted means

$$M_r^\beta f(t) = \frac{1}{r}\int_{\Gamma(t,r)} \left(\frac{|t-t_0|}{|\tau-t_0|}\right)^\beta |f(\tau)|\, d\nu(\tau). \tag{6.4}$$

We denote also $M_r f(t) = M_r^\beta f(t)\big|_{\beta=0}$.

Theorem 6.2 *Let $p(t)$ satisfy conditions* (2.1)–(2.2). *If $0 \le \beta < \frac{1}{p'(t_0)}$, then*

$$\left[M_r^\beta f(t)\right]^{p(t)} \le c\left(1 + \frac{1}{r}\int_{\Gamma(t,r)} |f(\tau)|^{p(\tau)}\, d\nu(\tau)\right) \tag{6.5}$$

for all $f \in L^{(p(\cdot)}(\Gamma)$ such that $\|f\|_{p(\cdot)} \le 1$, where $c = c(p,\beta)$ is a constant not depending on $t, t_0 \in \Gamma$ and $r > 0$.

Proof Since $\beta < \frac{1}{p'(t_0)}$, we conclude that there exists a $d > 0$ such that

$$\beta p'(t) < 1 \ \text{ for all } \ t \in \Gamma(t_0, d) \tag{6.6}$$

where we may assume that $d \le 1$. Let

$$p_r(t) = \min_{\tau\in\Gamma(t,r)} p(\tau)$$

and $\frac{1}{p_r'(t)} = 1 - \frac{1}{p_r(t)}$. From (6.6) we see that

$$\beta p_r'(t) < 1 \ \text{ if } \ t \in \Gamma\left(t_0, \frac{d}{2}\right) \ \text{ and } \ 0 < r \le \frac{d}{4}\,. \tag{6.7}$$

1^0 The case $\mathbf{t} \in \Gamma\left(\mathbf{t_0}, \frac{\mathbf{d}}{\mathbf{2}}\right)$ and $\mathbf{0 < r \le \frac{d}{4}}$ (the main case). Applying the Hölder inequality with the exponents $p_r(t)$ and $p_r'(t)$ to the integral on the right-hand side of the equality

$$\left|M_r\left(\frac{f(\cdot)}{|\cdot - t_0|^\beta}\right)(t)\right|^{p(t)} = \frac{c}{r^{p(t)}}\left(\int_{\Gamma(t,r)} \frac{|f(\tau)|}{|\tau-t_0|^\beta}\, d\nu(\tau)\right)^{p(t)}$$

and taking (6.7) into account, we get

$$\left|M_r\left(\frac{f(\cdot)}{|\cdot - t_0|^\beta}\right)(t)\right|^{p(t)} \le \frac{c}{r^{p(t)}}\left(\int_{\Gamma(t,r)} |f(\tau)|^{p_r(t)}\, d\nu(\tau)\right)^{\frac{p(t)}{p_r(t)}} \cdot \left(\int_{\Gamma(t,r)} \frac{d\nu(\tau)}{|\tau-t_0|^{\beta p_r'(t)}}\right)^{\frac{p(t)}{p_r'(t)}}. \tag{6.8}$$

By (6.7), estimate (5.1) is applicable which yields

$$\left|M_r\left(\frac{f(\cdot)}{|\cdot - t_0|^\beta}\right)(t)\right|^{p(t)} \le c\,\frac{|t-t_0|^{-\beta p(t)}}{r^{\frac{p(t)}{p_r(t)}}}\left(\int_{\Gamma(t,r)} |f(\tau)|^{p_r(t)}\, d\nu(\tau)\right)^{\frac{p(t)}{p_r(t)}}.$$

Here

$$\int_{\Gamma(t,r)} |f(\tau)|^{p_r(t)}\, d\nu(\tau) \le \int_{\Gamma(t,r)} d\nu(\tau) + \int_{\tau\in\Gamma(t,r):|f(\tau)|\ge 1} |f(\tau)|^{p(\tau)}\, d\nu(\tau),$$

since $p_r(t) \le p(\tau)$ for $\tau \in B(t,r)$. Since $p(t)$ is bounded, we see that

$$\left|M_r\left(\frac{f(\cdot))}{|\cdot - t_0|^\beta}\right)(t)\right|^{p(t)} \le c_1\,\frac{|t-t_0|^{-\beta p(t)}}{r^{\frac{p(t)}{p_r(t)}}}\left[r + \frac{1}{2}\int_{\Gamma(t,r)} |f(\tau)|^{p(\tau)}\, d\nu(\tau)\right]^{\frac{p(t)}{p_r(t)}}.$$

Since $r \le \frac{d}{2} \le \frac{1}{2}$ and the second term in the brackets is also less than or equal to $\frac{1}{2}$, we arrive at the estimate

$$|M_r^\beta f(t)|^{p(t)} \le \frac{c}{r^{\frac{p(t)}{p_r(t)}}}\left[r + \int_{\Gamma(t,r)} |f(\tau)|^{p(\tau)}\, d\nu(\tau)\right]$$
$$\le c\, r^{\frac{p_r(t)-p(t)}{p_r(t)}}\left[1 + \frac{1}{r}\int_{\Gamma(t,r)} |f(\tau)|^{p(\tau)}\, d\nu(\tau)\right].$$

From here (6.5) follows, since $r^{\frac{p_r(t)-p(t)}{p_r(t)}} \le c$. Indeed, $r^{\frac{p_r(t)-p(t)}{p_r(t)}} = e^{\frac{1}{p_r}[p(t)-p_r(t)]\ln\frac{1}{r}}$, where

$$\left|\frac{1}{p_r}[p(t)-p_r(t)]\ln\frac{1}{r}\right| \le |p(t)-p(\xi_r)|\ln\frac{1}{r}$$

with $\xi_r \in \Gamma(t,r)$, and then by (2.2),

$$\left|\frac{1}{p_r}[p(t)-p_r(t)]\ln\frac{1}{r}\right| \le A\frac{\ln\frac{1}{r}}{\ln\frac{1}{|t-\xi_r|}} \le A,$$

since $|t-\xi_r| \le r$.

$\mathbf{2^0}$ **The case** $|\mathbf{t}-\mathbf{t_0}| \ge \frac{\mathbf{d}}{\mathbf{2}}, \mathbf{0} < \mathbf{r} \le \frac{\mathbf{d}}{\mathbf{4}}$**.** This case is trivial, because $|\tau-t_0| \ge |t-t_0|-|\tau-t| \ge \frac{d}{2}-\frac{d}{4} = \frac{d}{4}$. Thus $|\tau-t_0|^\beta \ge \left(\frac{d}{4}\right)^\beta$. Since $|t-t_0|^\beta \le \ell^\beta$, it follows that

$$M_r^\beta f(x) \le cM_r f(x),$$

and one may proceed as above for the case $\beta = 0$ (the condition $|t-t_0| \le \frac{d}{2}$ is not needed in this case).

$\mathbf{3^0}$ **The case** $\mathbf{r} \ge \frac{\mathbf{d}}{\mathbf{4}}$. This case is also easy. It suffices to show that the left-hand side of (6.5) is bounded. We have

$$M_r^\beta f(t) \le \frac{c\ell^\beta}{\left(\frac{d}{4}\right)^n}\left[\int_{\Gamma\left(t_0,\frac{d}{8}\right)} \frac{|f(\tau)|}{|\tau-t_0|^\beta}\, d\nu(\tau) + \int_{\Gamma\setminus\Gamma\left(t_0,\frac{d}{8}\right)} \frac{|f(\tau)|}{|\tau-t_0|^\beta}\, d\nu(\tau)\right].$$

Here the first integral is estimated via the Hölder inequality with the exponents $p_{\frac{d}{8}} = \min_{|\tau-t_0|\le\frac{d}{8}} p(\tau)$ and $q_{\frac{d}{8}} = p'_{\frac{d}{8}}$, which is possible since $\beta p'_{\frac{d}{8}} < 1$. The estimate of the second integral is trivial since $|t-t_0| \ge \frac{d}{8}$.

Corollary *Let* $0 \le \beta < \frac{1}{p'(t_0)}$ *. If conditions* (2.1)–(2.2) *are satisfied, then*

$$|M^\beta f(t)|^{p(t)} \le c + cM[|f(\cdot)|^{p(\cdot)}](t) \tag{6.9}$$

for all $f \in L^{p(\cdot)}(\Gamma)$ *such that* $\|f\|_{p(\cdot)} \le 1$ *.*

6.3 Proof of Theorem A Itself

We have to show that $\left\|M^\beta f\right\|_{p(\cdot)} \le c$ in some ball $\|f\|_{p(\cdot)} \le R$, which is equivalent to the inequality

$$\mathfrak{I}_\Gamma^p(M^\beta f) \le c \text{ for } \left\|f\right\|_{p(\cdot)} \le R.$$

We observe that

$$|t-t_0|^{\beta p(t)} \sim |t-t_0|^{\beta p(t_0)} \tag{6.10}$$

since $p(t)$ satisfies condition (2.2).

I Sufficiency part By (6.10) we obtain

$$\mathfrak{I}_\Gamma^p(M^\beta f) \le c\int_\Gamma |t-t_0|^{\beta p(t_0)}\left|M\left(\frac{f(\tau)}{|\tau-t_0|^\beta}\right)(t)\right|^{p(t)} d\nu(t).$$

Following the idea in [14–15], we represent this as

$$\mathfrak{I}_\Gamma^p(M^\beta f) \le c\int_\Gamma \left(|t-t_0|^{\beta r(t_0)}\left|M\left(\frac{f(\tau)}{|\tau-t_0|^\beta}\right)(t)\right|^{r(t)}\right)^{p_-} dt, \tag{6.11}$$

where $r(t)=\frac{p(t)}{p_-}$. In the further estimations we distinguish the cases $\beta\le 0$ and $\beta\ge 0$.

1^0 The case $-\frac{1}{p(t_0)}<\beta\le 0$

Estimate (6.9) with $\beta=0$ says that

$$|M\psi(t)|^{r(t)} \le c(1+M[\psi^{r(\cdot)}](t)) \tag{6.12}$$

for all $\psi\in L^{r(\cdot)}(\Gamma)$ with $\|\psi\|_{r(\cdot)}\le 1$. For $\psi(t)=\frac{f(t)}{|t-t_0|^\beta}$ we have $\|\psi\|_{r(\cdot)}\le a_0\|f\|_{r(\cdot)}$, $a_0=\ell^{|\beta|}$, where we took into account that $\beta\le 0$. Hence $\|\psi\|_{r(\cdot)}\le a_0\cdot k\|f\|_{p(\cdot)}\le a_0kR$. Therefore we choose $R=\frac{1}{a_0k}$. Then $\|\psi\|_{r(\cdot)}\le 1$, so that (6.12) is applicable. From (6.11) we now get

$$\mathfrak{I}_\Gamma^p(M^\beta f) \le c\int_\Gamma \left(|t-t_0|^{\beta r(t_0)}\left[1+M\left(\left|\frac{f(\tau)}{|\tau-t_0|^\beta}\right|^{r(\tau)}\right)\right]\right)^{p_-} d\nu(\tau).$$

By property (6.10), this yields

$$\begin{aligned}\mathfrak{I}_\Gamma^p(M^\beta f) &\le c\int_\Gamma \left\{|t-t_0|^{\beta p(t_0)}+\left(|t-t_0|^{\beta r(t_0)}M\left(\frac{f(\tau)|^{r(\tau)}}{|\tau-t_0|^{\beta r(t_0)}}\right)\right)^{p_-}\right\}dt\\ &\le c+c\int_\Gamma (M^{\beta_1}(|f(\cdot)|^{r(\cdot)})(t))^{p_-}\,d\nu(t),\end{aligned}$$

where

$$\beta_1=\beta r(t_0)=\frac{\beta p(t_0)}{p_-}.$$

As is known [16, p. 149, Corollary 5.3], the weighted maximal operator M^{β_1} on Carleson curves is bounded in L^{p_-} with a constant p_- if $-\frac{1}{p_-}<\beta_1<\frac{1}{p'_-}$, which is satisfied since $-\frac{1}{p(t_0)}<\beta\le 0$. Therefore,

$$\begin{aligned}\mathfrak{I}_\Gamma^p(M^\beta f) &\le c+c\int_\Gamma |f(\tau)|^{r(\tau)\cdot p_-}\,d\nu(\tau)\\ &= c+c\int_\Gamma |f(\tau)|^{p(\tau)}\,d\tau \le c_1<\infty.\end{aligned}$$

2^0 The case $0\le\beta<\frac{1}{p'(t_0)}$.

We represent the functional $\mathfrak{I}_\Gamma^p(M^\beta f)$ in the form

$$\mathfrak{I}_\Gamma^p(M^\beta f)=\int_\Gamma (|M^\beta f(t)|^{r(t)})^\lambda\,d\nu(t) \tag{6.13}$$

with $r(t)=\frac{p(t)}{\lambda}>1$, $\lambda>1$, where λ will be chosen in the interval $1<\lambda<p_-$.

In (6.13), we wish to use the pointwise weighted estimate (6.9):

$$|M^\beta f(t)|^{r(t)} \le c\big[1+M(f^{r(\cdot)})(t)\big]. \tag{6.14}$$

This estimate is applicable according to Corollary of Theorem if $\|f\|_{r(\cdot)} \le c$ and $\beta < \frac{1}{[r(t_0)]'}$. The condition $\|f\|_{r(\cdot)} \le c$ is satisfied since $r(x) \le p(t)$. Condition $\beta < \frac{1}{[r(t_0)]'}$ is fulfilled if $\lambda < (1-\beta)\, p(t_0)$. Therefore, under the choice

$$1 < \lambda < \min(p_-, (1-\beta)\, p(t_0)),$$

we may apply (6.14) to (6.13). This yields

$$\begin{aligned}\mathfrak{I}^p_\Gamma(M^\beta f) &\le c + c \int_\Gamma \left|M(|f|^{r(\cdot)})(t)\right|^\lambda d\nu(t)\\ &\le c + c \int_\Gamma \left(|f(t)|^{r(t)}\right)^\lambda d\nu(t)\end{aligned}$$

by the boundedness of the maximal operator M in $L^\lambda(\Gamma)$, $\lambda > 1$. Hence

$$\mathfrak{I}^p_\Gamma(M^\beta f) \le c + c \int_\Gamma |f(x)|^{p(t)}\, dt \le c.$$

II Necessity Part Suppose that M^β is bounded in $L^{p(\cdot)}(\Gamma)$. Then, given a function $f(x)$ such that

$$\mathfrak{I}^p_\Gamma(wf) \le c_1, \quad w(t) = |t-t_0|^\beta, \tag{6.15}$$

we have

$$\mathfrak{I}^p_\Gamma(wMf) \le c. \tag{6.16}$$

1) We choose $f(t) = |t-t_0|^\mu$ with $\mu > -\beta - \frac{1}{p(t_0)}$. Then

$$\mathfrak{I}^p_\Gamma(wf) \le c + c\int_{\Gamma\cap B(t_0,r)} |t-t_0|^{(\beta+\mu)p(t)} d\nu(t) \le c + c\int_{\Gamma\cap B(t_0,r)} |t-t_0|^{(\beta+\mu)p(t_0)} d\nu(t)$$

for some $r > 0$,where the integral converges by (4.2), so that we are in the situation (6.15). However,

$$\mathfrak{I}^p_\Gamma(w\, Mf) \ge c \int_{\Gamma\cap B(t_0,r)} |t-t_0|^{\beta p(t_0)} d\nu(t)\ ,$$

which diverges if $\beta p(t_0) < -1$ according to Lemma 4.1. Therefore, from (6.16) it follows that $\beta > -\frac{1}{p(t_0)}$.

2) To show the necessity of the right-hand side bound in (3.3), suppose that, on the contrary, $\beta \ge \frac{1}{p'(t_0)}$. Let first $\beta > \frac{1}{p'(t_0)}$. We choose $f(t) = \frac{1}{|t-t_0|}$, for which $\mathfrak{I}^p_\Gamma(wf)$ converges but Mf just does not exist. Let now $\beta = \frac{1}{p'(t_0)}$. We choose

$$f(t) = \frac{1}{|t-t_0|}\left(\ln\frac{1}{|t-t_0|}\right)^a, \quad t \in \Gamma \cap B\left(t_0, \frac{1}{2}\right).$$

Then $\mathfrak{I}^p_\Gamma(wf)$ exists under the choice $a < -\frac{1}{p(t_0)}$, but Mf does not exist when $\beta_1 > -1$. Thus, taking $a \in \left(-1, -\frac{1}{p(t_0)}\right)$, we arrive at a contradiction.

7 Proof of Theorem B

According to Remark 6.1 it suffices to prove Theorem B for a single power weight $|t-t_0|^\beta$ where t_0 belongs or does not belong to Γ. The arguments below are given for $t_0 \in \Gamma$, the case where $t_0 \notin \Gamma$ being easier.

I Sufficiency Part

We have to show that $\Im^p_\Gamma(wMf) \le C < \infty$ provided that $\|wf\|_{p(\cdot)} \le 1,\ w = |t-t_0|^\beta$.

Let

$$\Gamma_R = \Gamma \cap B(t_0, R), \qquad \Gamma_{2R} = \Gamma \cap B(t_0, 2R) \qquad \text{and} \qquad \Gamma_{4R} = \Gamma \cap B(t_0, 4R).$$

We split the function f as

$$f = f \cdot \chi_{\Gamma_{2R}} + f \cdot \chi_{\Gamma\backslash\Gamma_{2R}} = \varphi + \psi,$$

so that

$$\Im^p_\Gamma(wMf) \le \Im^p_\Gamma(wM\varphi) + \Im^p_\Gamma(wM\psi).$$

When estimating $\Im^p_\Gamma(wM\varphi)$, we distinguish the cases $t \in \Gamma_{4R}$ and $t \in \Gamma\backslash\Gamma_{4R}$.

Let first $t \in \Gamma_{4R}$. We find it convenient to introduce a notation for the maximal function with respect to the portion Γ_{4R} of Γ, that is,

$$M_{\Gamma_{4R}} f(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r) \cap \Gamma_{4R}\}} \int_{\Gamma(t,r)\cap\Gamma_{4R}} |f(\tau)| d\nu(\tau), \qquad t \in \gamma \subseteq \Gamma.$$

For $Mf(t) = M_\Gamma f(t)$ we have

$$M\varphi(t) \le C M_{\Gamma_{4R}} f(t), \qquad t \in \Gamma_{4R}. \tag{7.1}$$

Indeed,

$$M\varphi(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)\cap\Gamma_{2R}} |f(\tau)| d\nu(\tau) \le \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)\cap\Gamma_{4R}} |f(\tau)| d\nu(\tau).$$

When $r \ge 2R$ we have $\Gamma(t,r) = \Gamma(t,r) \cap \Gamma_{4R}$, so that the right hand side above is $M^{\Gamma_{4R}} f(t)$ and we get (7.1) with $C = 1$; when $r \le 2R$, we observe that $\nu\{\Gamma(t,r)\} \sim \nu\{\Gamma(t,r) \cap \Gamma_{4R}\} \sim R$ for $t \in \Gamma_{2R}$ and we get (7.1) with some $C > 1$.

Then by (7.1) and Theorem A,

$$\int_{\Gamma_{4R}} [w(t)M\varphi(t)]^{p(t)} d\nu(t) \le C \int_{\Gamma_{4R}} [w(t) M_{\Gamma_{4R}} f(t)]^{p(t)} d\nu(t) \le C \tag{7.2}$$

since $\|wf\|_{L^{p(\cdot)}(\Gamma_{4R})} \le \|wf\|_{L^{p(\cdot)}(\Gamma)} \le 1$.

Let $t \in \Gamma\backslash\Gamma_{4R}$. If $r < 2R$, then $\Gamma(t,r) \cap \Gamma_{2R} = \emptyset$ and $M_r\varphi(t) = 0$, so we consider $r \ge 2R$. It can be also easily seen then that whenever the set $\Gamma(t,r) \cap \Gamma_{2R}$ is non-empty, we have $|t - t_0| \le 2R + r \le 2r$. Consequently

$$M_r\varphi(t) = \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)} |\varphi(\tau)| d\nu(\tau) \le \frac{1}{\nu\{\Gamma(t, \frac{|t-t_0|}{2})\}} \int_{\Gamma_{2R}} |f(\tau)| d\nu(\tau).$$

Hence $M_r\varphi(t) \le \frac{C}{|t-t_0|} \left[1 + \|wf\|_{p(\tau)}\right]$ and then

$$M\varphi(t) \le \frac{C}{|t-t_0|} \qquad \text{for} \qquad t \in \Gamma\backslash\Gamma_{4R}. \tag{7.3}$$

Therefore,

$$\int_{\Gamma\backslash\Gamma_{4R}} [w(t)M\varphi(t)]^{p(t)}\, d\nu(t) \le C \int_{\Gamma\backslash\Gamma(t_0,4R)} |t-t_0|^{(\beta-1)p_\infty} d\nu(t) \le C_1 < \infty, \tag{7.4}$$

where we have made use of Lemma 4.2 and the fact that $\beta < \frac{1}{p'_\infty}$. Combining (7.2) and (7.4), we get

$$\mathfrak{I}^p_\Gamma (wM\varphi) \le C < \infty. \tag{7.5}$$

Now we pass to the function ψ. Let first $t \in \Gamma_R$. If $r < R$, then $\Gamma(t,r) \cap \Gamma\backslash\Gamma_{2R} = \emptyset$ and $M_r\psi(t) = 0, t \in \Gamma_R$. Therefore, we have to consider only $r \ge R$ and then

$$M_r\psi(t) = \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)\cap\Gamma_{2R}} |f(\tau)| d\nu(\tau) \le \frac{1}{\nu\{\Gamma(t,r)\}} \int_{\Gamma(t,r)\cap\Gamma_{2R}} (1 + |f(\tau)|^{p_\infty})\, d\nu(\tau).$$

Hence

$$M_r\psi(t) = \frac{1}{\nu\{\Gamma(t,r)\}} (+1) \le C < \infty.$$

Thus, $M\psi(t) \le C$ for $t \in \Gamma_R$ and then

$$\begin{aligned}\int_{\Gamma_R} [w(t)M\psi(t)]^{p(t)} d\nu(t) &\le C \int_{\Gamma_R} [w(t)]^{p(t)} d\nu(t)\\ &= C \int_{\Gamma_R} [w(t)]^{p(t)-p(t_0)} [w(t)]^{p(t_0)} d\nu(t)\\ &\le C \int_{\Gamma_R} |t-t_0|^{\beta p(t_0)} d\nu(t) \le C < \infty, \end{aligned} \tag{7.6}$$

where we have taken into account the property (6.10) and the fact that $\beta > -\frac{1}{p(t_0)}$.

It remains to estimate

$$\int_{\Gamma\backslash\Gamma_R} [w(t)M\psi(t)]^{p(t)} d\nu(t) = \int_{\Gamma\backslash\Gamma_R} [w(t)M\psi(t)]^{p_\infty} d\nu(t).$$

To this end, it suffices to make use of the known boundedness of the maximal operator in the Lebesgue space with constant $p_\infty > 1$ (the fact valid in general for maximal functions on weighted spaces of homogeneous type, in particular, on Carleson curves, see [2, Theorem 2.3.1]); and the power function $w(t) = |t-t_0|^\beta$ on an infinite curve $\Gamma\backslash\Gamma_{2R}, t_0 \notin \Gamma\backslash\Gamma_{2R}$, is a Muckenhoupt weight for $\beta \in \left(-\frac{1}{p_\infty}, \frac{1}{p'_\infty}\right)$, see [16, p. 32].

Therefore,

$$\int_{\Gamma\backslash\Gamma_R} [w(t)M\psi(t)]^{p(t)} d\nu(t) \le C \int_{\Gamma\backslash\Gamma_R} [w(t)\psi(t)]^{p_\infty} d\nu(t) \le C \int_{\Gamma} [w(t)f(t)]^{p(t)} d\nu(t) \le C,$$

which together with (7.6) yields

$$\mathfrak{I}^p_\Gamma (M\psi) \le C < \infty$$

and proves the "if part" of the theorem.

II Necessity Part

Suppose that the maximal operator M is bounded in $L^{p(\cdot)}(\Gamma, w)$. Necessity of the conditions $-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)}$ follows from Theorem A. The necessity of the conditions $-\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty}$ is a consequence of their necessity for the boundedness of the maximal operator in the case of constant exponent p. Indeed, the boundedness

$$\mathfrak{I}^p_\Gamma (wf) \le 1 \Longrightarrow \mathfrak{I}^p_\Gamma (wMf) \le C$$

implies the boundedness

$$\int_{\Gamma\backslash\Gamma_{2R}} |w(t)f(t)|^{p_\infty} d\nu(t)) \leq 1 \Longrightarrow \int_{\Gamma\backslash\Gamma_{2R}} |w(t)Mf(t)|^{p_\infty} d\nu(t)) \leq C$$

with constant exponent. For the latter, as is known, it is necessary that the weight w satisfies the Muckenhoupt condition ([2, Theorem 2.3.1]) and the power function $w(t) = |t - t_0|^\beta$ on an infinite curve $\Gamma\backslash\Gamma_{2R}, t_0 \notin \Gamma\backslash\Gamma_{2R}$, is a Muckenhoupt weight if and only if $\beta \in (-\frac{1}{p_\infty}, \frac{1}{p'_\infty})$.

Anyhow, to make the presentation more self-contained, we independently prove the necessity of the condition $\beta < \frac{1}{p'_\infty}$ in Appendix 2.

8 Proof of Theorem C

We have to show that

$$\mathfrak{I}^p_\Gamma(I^{\alpha(\cdot)}f) \leq c < \infty, \tag{8.1}$$

when $\|f\|_{p(\cdot)} \leq 1$. We simultaneously treat the cases of finite or infinite curve and use some ideas of the proof of the corresponding theorem for the Euclidean space in [11]. For $f \in L^{p(\cdot)}(\Gamma), f(t) \geq 0$, with $\|f\|_{p(\cdot)} \leq 1$ we prove the following pointwise estimate

$$[I^{\alpha(\cdot)}f(t)]^{q(t)} \leq C([Mf(t)]^{p(t)} + 1), \tag{8.2}$$

valid independently of the fact whether the curve is finite or infinite, and give its improvement

$$[I^{\alpha(\cdot)}f(t)]^{q(t)} \leq C\left([Mf(t)]^{p(t)} + \chi_{\Gamma(t_0,2R)}(t) + \frac{\chi_{\Gamma\backslash\Gamma(t_0,2R)}(t)}{|t-t_0|^{p_-}}\right), \tag{8.3}$$

when Γ is an infinite curve. The required statement (8.1) will follow from (8.2), (8.3).

Proof of (8.2) We make use of the standard splitting

$$I^{\alpha(\cdot)}f(t) = \int_{\Gamma(t,r)} \frac{f(\tau)\,d\tau}{|t-\tau|^{1-\alpha(t)}} + \int_{\Gamma\backslash\Gamma(t,r)} \frac{f(\tau)\,d\tau}{|t-\tau|^{1-\alpha(t)}} =: A_r(t) + B_r(t), \tag{8.4}$$

where $0 < r < \infty$, and the well-known pointwise inequality

$$A_r(t) \leq C\frac{r^{\alpha(t)}}{2^{\alpha(t)}-1}Mf(t), \qquad t \in \Gamma, \tag{8.5}$$

where $C > 0$ does not depend on t and r, see [17, p. 54] for Euclidean spaces; since $\nu(\Gamma(t,r)) \sim r$, the proof for Carleson curves is the same under the condition $\inf_{t\in\Gamma}\alpha(t) > 0$:

$$A_r(t) = \sum_{k=0}^{\infty}\int_{\gamma_k(t)} \frac{f(\tau)\,d\nu(\tau)}{|t-\tau|^{1-\alpha(t)}} \leq C2^{\alpha(t)}r^{\alpha(t)}\sum_{k=0}^{\infty}2^{-\alpha(t)k}Mf(t),$$

where $\gamma_k(t) = \Gamma(t, 2^{-k}r)\backslash\Gamma(t, 2^{-k-1}r)$.

By (8.5) and the first condition in (1.5) we have

$$A_r(t) \leq cr^{\alpha(t)}Mf(t), \qquad t \in \Gamma \tag{8.6}$$

with some absolute constant $c > 0$ not depending on t and r.

For the term $B_r(t)$ we make use of the Hölder inequality and obtain

$$B_r(t) \leq \|f\|_{p(\cdot)}\|\chi_{\Gamma\backslash\Gamma(t,r)}(\tau)|t-\tau|^{\alpha(t)-1}\|_{p'(\cdot)} \leq \|\chi_{\Gamma\backslash\Gamma(t,r)}(\tau)|t-\tau|^{\alpha(t)-1}\|_{p'(\cdot)}$$

the norm being taken with respect to τ. Then by (2.4)

$$B_r(t) \le \left\| \frac{\chi_{\Gamma\setminus\Gamma(t,r)}(\tau)}{|t-\tau|} \right\|^{1-\alpha(t)}_{(1-\alpha(\cdot))p'(\cdot)}. \tag{8.7}$$

Now we show that

$$\frac{\chi_{\Gamma\setminus\Gamma(t,r)}(\tau)}{|t-\tau|} \le CM\left(\frac{\chi_{\Gamma(t,r)}}{\nu(\Gamma(t,r))}\right)(\tau), \tag{8.8}$$

where $C > 0$ does not depend on t, τ and r. Inequality (8.8) should be checked for $\tau \in \Gamma\setminus\Gamma(t,r)$. Indeed, we have

$$M\left(\frac{\chi_{\Gamma(t,r)}}{\nu(\Gamma(t,r))}\right)(\tau) \ge \sup_{\delta>0} \frac{\nu\{\Gamma(t,r)\cup\Gamma(\tau,\delta)\}}{\nu(\Gamma(t,r))\nu(\Gamma(\tau,\delta))} \ge \frac{\nu\{\Gamma(t,r)\cup\Gamma(\tau,\delta_0)\}}{\nu(\Gamma(t,r))\nu(\Gamma(\tau,\delta_0))}$$

with an arbitrary $\delta_0 > 0$. We choose it so that $2|t-\tau| \le \delta_0 \le 3|t-\tau|$. Then $\Gamma(t,r) \subseteq \Gamma(\tau,\delta)$ and consequently $\nu\{\Gamma(t,r)\cup\Gamma(\tau,\delta)\} = \nu\{\Gamma(t,r)\}$ and then

$$M\left(\frac{\chi_{\Gamma(t,r)}}{\nu(\Gamma(t,r))}\right)(\tau) \ge \frac{1}{\nu(\Gamma(\tau,\delta_0))} \ge \frac{c}{\delta_0} \ge \frac{c/3}{|t-\tau|}, \qquad \tau \in \Gamma\setminus\Gamma(t,r)$$

which proves (8.8).

From (8.7) and (8.8) we obtain

$$B_r(t) \le C[\nu(\Gamma(t,r))]^{\alpha(t)-1} \left\| M\left(\chi_{\Gamma(t,r)}\right)\right\|^{1-\alpha(t)}_{(1-\alpha(\cdot))p'(\cdot)}.$$

By the boundedness of the maximal operator in the space $L^{p(\cdot)}(\Gamma)$ provided by Theorem A, we conclude that

$$B_r(t) \le C[\nu(\Gamma(t,r))]^{\alpha(t)-1} \left\|\chi_{\Gamma(t,r)}\right\|^{1-\alpha(t)}_{(1-\alpha(\cdot))p'(\cdot)} = C[\nu(\Gamma(t,r))]^{\alpha(t)-1} \left\|\chi_{\Gamma(t,r)}\right\|_{p'(\cdot)}.$$

By Proposition 4.3 we then obtain

$$B_r(t) \le C[\nu\{\Gamma(t,r)\})]^{\alpha(t)-\frac{1}{p_\gamma}}. \tag{8.9}$$

Therefore, from (8.4), (8.6) and (8.9) we get

$$I^{\alpha(\cdot)}f(t) \le Cr^{\alpha(t)}Mf(t) + C[\nu\{\Gamma(t,r)\}]^{\alpha(t)-\frac{1}{p_\gamma}}. \tag{8.10}$$

Observe that $\alpha(t) - \frac{1}{p_\gamma}$ is negative and $\sup_{t\in\Gamma}\left[\alpha(t) - \frac{1}{p_\gamma}\right] < 0$ according to (3.5). Since $\nu\{\Gamma(t,r)\} \ge 2r$, we obtain

$$I^{\alpha(\cdot)}f(t) \le Cr^{\alpha(t)}Mf(t) + Cr^{\alpha(t)-\frac{1}{p_\gamma}}, \qquad t \in \Gamma, \quad 0 < r < \infty. \tag{8.11}$$

Observe that when $\alpha(t) = \text{const}$ and $p(t) = \text{const}$, estimate (8.11) with the standard choice $r = [Mf(t)]^{-p}$ yields the Hedberg-type pointwise estimate

$$I^{\alpha(\cdot)}f(t) \le C[Mf(t)]^{\frac{p}{q}}, \qquad t \in \Gamma \tag{8.12}$$

for any Carleson curve, bounded or unbounded.

In view of (5.3), estimate (8.11) takes the form

$$I^{\alpha(\cdot)}f(t) \le Cr^{\alpha(t)}Mf(t) + Cr^{\alpha(t)-\frac{1}{p(t)}}, \qquad t \in \Gamma, \quad 0 < r \le \ell \tag{8.13}$$

for any finite $\ell < \infty$. Let $\ell \ge 1$. From (8.13) the required estimate in (8.2) follows. Indeed, for those $t \in \Gamma$ for which $Mf(t) \ge 1$, we choose $r = [Mf(t)]^{-p(t)} \le 1 \le \ell$ and obtain

$$I^{\alpha(\cdot)}f(t) \le C[Mf(t)]^{1-\alpha(t)p(t)} = C[Mf(t)]^{\frac{p(t)}{q(t)}} \qquad \text{when} \ \ Mf(t) \ge 1. \tag{8.14}$$

If $Mf(t) < 1$, we take $r = 1$ and from (8.13) get $I^{\alpha(\cdot)}f(t) \leq C$ which together with (8.14) gives (8.2) for all $t \in \Gamma$.

2 *Proof of* (8.3) To prove (8.3), we proceed as follows. For $t \in \Gamma(t_0, 2R)$ in any case we have estimate (8.2):

$$[I^{\alpha(\cdot)}f(t)]^{q(t)} \leq C([Mf(t)]^{p(t)} + 1), \qquad t \in \Gamma(t_0, 2R). \tag{8.15}$$

Let now $t \in \Gamma\backslash\Gamma(t_0, 2R)$. We put

$$f = f_0 + f_1, \qquad \text{where} \quad f_0 = \chi_{\Gamma(t_0,R)}f \quad \text{and} \quad f_1 = \chi_{\Gamma\backslash\Gamma(t_0,R)}f.$$

For $\tau \in \Gamma(t_0, R)$ and $t \in \Gamma\backslash\Gamma(t_0, 2R)$ we have

$$|t - \tau| \geq \frac{1}{2}|t - t_0|.$$

Therefore, for $t \in \Gamma\backslash\Gamma(t_0, 2R)$ we obtain

$$(I^{\alpha(\cdot)}f_0(t))^{\frac{q(t)}{p_-}} \leq c\left(\frac{1}{|t - t_0|^{1-\alpha(t)}} \int_{\Gamma(t_0,R)} f_0(t)d\nu(t)\right)^{\frac{q(t)}{p_-}}.$$

But

$$\int_{\Gamma(t_0,R)} f_0(t)d\nu(t) \leq \int_{\Gamma(t_0,R)} [|f(t)|^{p(t)} + 1]d\nu(t) \leq C$$

since $\|f\|_{p(\cdot)} \leq 1$. Consequently,

$$(I^{\alpha(\cdot)}f_0(t))^{\frac{q(t)}{p_-}} \leq \frac{c}{|t - t_0|^{\frac{1-\alpha(t)}{p_-}q(t)}}, \qquad t \in \Gamma\backslash\Gamma(t_0, 2R). \tag{8.16}$$

Observe that $q(t) = \frac{p(t)}{1-\alpha(t)p(t)} \geq \frac{p_-}{1-\alpha(t)p_-}$. We may assume that $R \geq 1$, so that $|t - t_0| \geq 1$ and then from (8.16) we get

$$(I^{\alpha(\cdot)}f_0(t))^{\frac{q(t)}{p_-}} \leq \frac{c}{|t - t_0|^{\frac{1-\alpha(t)}{1-\alpha(t)p_-}}} \leq \frac{c}{|t - t_0|}, \qquad t \in \Gamma\backslash\Gamma(t_0, 2R). \tag{8.17}$$

Passing to the function $f_1(t)$, we observe that supp $f_1(t) \subseteq \Gamma\backslash\Gamma(t_0, R)$ and in this set $p(t) = p_\infty = \text{const}$. Therefore, for

$$[I^{\alpha(\cdot)}f_1(t)]^{q(t)} = [I^{\alpha_\infty}f_1(t)]^{q_\infty}, \qquad t \in \Gamma\backslash\Gamma(t_0, 2R),$$

where $\frac{1}{q_\infty} = \frac{1}{p_\infty} - \alpha_\infty = \text{const}$. Then by the Hedberg pointwise estimate (8.12) for constant exponents we have

$$[I^{\alpha(\cdot)}f_1(t)]^{q(t)} \leq C[Mf_1(t)]^{p_\infty} = C[Mf_1(t)]^{p(t)}, \qquad t \in \Gamma\backslash\Gamma(t_0, 2R). \tag{8.18}$$

Then from (8.17) and (8.18) it follows that

$$[I^{\alpha(\cdot)}f(t)]^{q(t)} \leq C\left(\frac{1}{|t - t_0|^{p_-}} + [Mf(t]^{p(t)}\right)$$

which together with (8.15) yields (8.3).

Proof of (8.1) Let Γ be a finite curve. Then by (8.2)

$$\int_\Gamma |I^{\alpha(\cdot)}f(t)|^{q(t)}\, d\nu(t) \leq \int_\Gamma |Mf(t)|^{p(t)}\, d\nu(t) + C,$$

which proves the theorem, since the maximal operator is bounded in $L^{p(\cdot)}(\Gamma)$ according to the non-weighted case of Theorem A. Let Γ be infinite. Then by (8.3) we have

$$\int_\Gamma |I^{\alpha(\cdot)}f(t)|^{q(t)}\,d\nu(t) \le \int_\Gamma |Mf(t)|^{p(t)}\,d\nu(t) + C + \int_{\Gamma\setminus\Gamma(t_0,2R)} \frac{d\nu(t)}{|t-t_0|^{p_-}}$$

where it remains to refer to the non-weighted case of Theorem B and the fact that the last integral is finite according to Lemma 4.2.

9 Proof of Theorem D

9.1 An Auxiliary Estimate

Lemma 9.1 *Let Γ be a bounded Carleson curve of the length $\ell, 0<r<\ell,\ t,t_0\in\Gamma,\ \sigma>-1$ and a bounded measurable function $h(t)$ defined on Γ satisfy the conditions*

$$\sup_{x\in\Gamma}|h(t)| := H < \infty, \tag{9.1}$$

$$\sup_{t\in\Gamma}[h(t)+1] := -d_0 < 0, \tag{9.2}$$

and

$$\sup_{t\in\Gamma}[h(t)+1+\sigma] := -d_1 < 0. \tag{9.3}$$

Then

$$A(t,t_0;r) := \int_{\Gamma\setminus\Gamma(t,r)} |t-\tau|^{h(t)}|\tau-t_0|^\sigma\,d\nu(\tau) \ \le\ Cr^{h(t)+1}(r+|t-t_0|)^\sigma, \qquad t\in\Gamma, \tag{9.4}$$

where $C>0$ does not depend on t and r.

Proof We consider separately the cases $|t-t_0|\le\frac{r}{2},\ \frac{r}{2}\le|t-t_0|\le 2r,\ |t-t_0|\ge 2r$.

The case $|t-t_0|\le\frac{r}{2}$. We have

$$\frac{|\tau-t_0|}{|\tau-t|} \le \frac{|\tau-t|+|t-t_0|}{|\tau-t|} \le 1+\frac{t-t_0}{r} \le 2$$

and similarly

$$\frac{|\tau-t_0|}{|\tau-t|} \ge 1-\frac{|t-t_0|}{r} \ge \frac{1}{2}.$$

Hence $\frac{1}{2}\le\frac{|\tau-t_0|}{|\tau-t|}\le 2$ and therefore, $(\frac{|\tau-t_0|}{|\tau-t|})^\sigma \le 2^{|\sigma|}$. Then

$$A(t,t_0;r) \le 2^{|\sigma|}\int_{\Gamma\setminus\Gamma(t,r)} |t-\tau|^{h(t)+\sigma}\,d\nu(\tau).$$

It remains to make use of Lemma 4.2, applicable by condition (9.3), which yields

$$A(t,t_0;r) \le r^{h(t)+\sigma+1}, \qquad |t-t_0|\le\frac{r}{2}. \tag{9.5}$$

The case $\frac{r}{2}\le|t-t_0|\le 2r$. We split the integration in $A(t,t_0;r)$ as follows:

$$A(t,t_0;r) = \int_{\substack{\tau\in\Gamma\\ r<|\tau-t|<2\rho}} |t-\tau|^{h(t)}|\tau-t_0|^\sigma\,d\nu(\tau) + \int_{\Gamma\setminus\Gamma(t,2\rho)} |t-\tau|^{h(t)}|\tau-t_0|^\sigma\,d\nu(\tau) =: \mathfrak{I}_1+\mathfrak{I}_2,$$

where we denote $\rho = |t - t_0|$ for brevity. Since $h(t) < 0$, for the integral $\mathfrak{I}_1$ we have

$$\mathfrak{I}_1 \le r^{h(t)} \int\limits_{\substack{\tau\in\Gamma \\ r<|\tau-t|<2\rho}} |\tau - t_0|^\sigma \, d\tau.$$

Observe that

$$|\tau - t| > r \quad \Longrightarrow \quad |\tau - t_0| \le |\tau - t| + |t - t_0| \le |\tau - t| + 2r \le 3|\tau - t|.$$

Consequently,

$$\mathfrak{I}_1 \le r^{h(t)} \int\limits_{\substack{|\tau-t|<2\rho \\ |\tau-t_0|\le 3|\tau-t|}} |\tau - t_0|^\sigma \, d\nu(\tau) \le r^{h(t)} \int_{|\tau-t_0|<6\rho} |\tau - t_0|^\sigma \, d\nu(\tau)$$

and Lemma 4.1 yields

$$A(t, t_0; r) \le C r^{h(t)} |t - t_0|^{\sigma+1}, \qquad \frac{r}{2} \le |t - t_0| \le 2r, \tag{9.6}$$

the application of Lemma 4.1 being possible by condition (9.2).

As regards the integral $\mathfrak{I}_2$, this is nothing else, but $A(t, t_0, 2\rho)$, $\rho = |t - t_0|$ and its estimate is contained in (9.5) under $r = 2|t - t_0|$.

The case $\rho \ge 2r$, $\rho = |t - t_0|$. We have

$$A(t, t_0; r) = \int\limits_{\substack{\tau\in\Gamma \\ r<|\tau-t|<\frac{1}{2}\rho}} |t - \tau|^{h(t)} |\tau - t_0|^\sigma \, d\nu(\tau) + \int_{\Gamma\setminus\Gamma\left(t, \frac{1}{2}\rho\right)} |t - \tau|^{h(t)} |\tau - t_0|^\sigma \, d\nu(\tau)$$
$$=: \mathfrak{I}_3 + \mathfrak{I}_4.$$

For the term $\mathfrak{I}_3$ we have $\frac{1}{2}\rho \le |\tau - t_0| \le 2\rho$, so that $|\tau - t_0|^\sigma \le 2^{|\sigma|}\rho^\sigma$. Therefore,

$$\mathfrak{I}_3 \le 2^{|\sigma|}\rho^\sigma \int\limits_{\substack{\tau\in\Gamma \\ r<|\tau-t|<\frac{1}{2}\rho}} |t - \tau|^{h(t)} \, d\nu(\tau)$$

and condition (9.2) and Lemma 4.2 yield

$$A(t, t_0; r) \le C r^{h(t)+1} |t - t_0|^\sigma, \qquad |t - t_0| \ge 2r. \tag{9.7}$$

Gathering estimates (9.5), (9.6) and (9.7), we arrive at (9.4). □

9.2 Estimation of the Norms $n_{\delta,\varkappa,p}(t, r)$

Let $\chi_r(\rho) = \begin{cases} 1, & \text{if } \rho>r \\ 0, & \text{if } \rho<r \end{cases}$ and let

$$g_\delta(t, \tau, r) = |t - \tau|^{\delta(t)} \chi_r(|t - \tau|), \tag{9.8}$$

where $\delta(t)$ in future will be chosen as $\delta(t) = \alpha(t) - 1$.

We are interested in estimation of the weighted norms

$$n_{\delta,\varkappa,p}(t, r) = \|g_\delta(t, \tau, r)\|_{L^{p(\cdot)}\left(\Gamma, |\tau - t_0|^{\varkappa(\tau)}\right)} \tag{9.9}$$

(taken with respect to τ) as $r \to 0$, where we suppose that $t_0 \in \Gamma$ and $\varkappa(\tau)$ is some variable exponent. Later, in the proof of Theorem D, we will need this norm with $p(\cdot)$ replaced by $p'(\cdot)$ and $\varkappa(t)$ chosen as $\varkappa(t) = -\beta p'(t)$.

Theorem 9.2 *Let Γ be a bounded Carleson curve, $t_0 \in \Gamma$, p satisfy conditions* (2.1)–(2.2), *$\varkappa \in L^\infty(\Gamma)$ and $\delta \in L^\infty(\Gamma)$ and also $\varkappa(t)$ satisfy the logarithmic condition at the point t_0*

$$|\varkappa(\tau) - \varkappa(t_0)| \le \frac{A_2}{\ln \frac{1}{|\tau - t_0|}}, \qquad \tau \in \Gamma, \quad |\tau - t_0| \le \frac{1}{2} \tag{9.10}$$

and let $\varkappa(t_0)p(t_0) > -1$. *If*

$$\sup_{t\in\Gamma}[\delta(t)p(t)+1] := -d_0 < 0, \tag{9.11}$$

$$\sup_{t\in\Gamma}\{1+[\delta(t)+\varkappa(t)]p(t)\} := -d_1 < 0, \tag{9.12}$$

and

$$\sup_{t\in\Gamma}\{1+[\delta(t)+\varkappa(t_0)]p(t)\} := -d_2 < 0, \tag{9.13}$$

then

$$n_{\delta,\varkappa,p}(t,r) \le Cr^{\delta(t)+\frac{1}{p(t)}}(r+|t-t_0|)^{\varkappa(t)}, \tag{9.14}$$

for all $t\in\Gamma$, $0<r<\ell$, *where* $C>0$ *does not depend on* t *and* r.

Proof For the norm $n_{\delta,\varkappa,p} = n_{\delta,\varkappa,p}(t,r)$ as defined in (9.9) we have

$$\int\limits_{\substack{\tau\in\Gamma\\ |\tau-t|>r}} \left(\frac{|\tau-t|^{\delta(t)}|\tau-t_0|^{\varkappa(\tau)}}{n_{\delta,\varkappa,p}}\right)^{p(\tau)} d\tau = 1 \tag{9.15}$$

by definition (2.5).

1*st step*: *Values* $n_{\delta,\varkappa,p}(t,r)\ge 1$ *are only of interest.* First we observe that the right-hand side of (9.14) is bounded from below:

$$\inf_{\substack{t\in\Gamma\\ 0<r<\ell}} r^{\delta(t)+\frac{1}{p(t)}}(r+|t-t_0|)^{\varkappa(t)} := c_1 > 0. \tag{9.16}$$

To verify (9.16), suppose first that $\varkappa(t)\ge 0$. Then by (9.12)

$$r^{\delta(t)+\frac{1}{p(t)}}(r+|t-t_0|)^{\varkappa(t)} \ge r^{\delta(t)+\frac{1}{p(t)}+\varkappa(t)} = r^{-\frac{|\delta(t)p(t)+\varkappa(t)p(t)+1|}{p(t)}} \ge \ell^{-\frac{|\delta(t)p(t)+\varkappa(t)p(t)+1|}{p(t)}}.$$

The right hand side here is bounded from below since $\frac{|\delta(t)p(t)+\varkappa(t)p(t)+1|}{p(t)} \in L^\infty(\Gamma)$. When $\varkappa(t)\le 0$, we observe that

$$r^{\delta(t)+\frac{1}{p(t)}}(r+|t-t_0|)^{\varkappa(t)} \ge r^{\delta(t)+\frac{1}{p(t)}}\ell^{\varkappa(t)} = r^{-\left|\delta(t)+\frac{1}{p(t)}\right|}\ell^{\varkappa(t)},$$

where (9.11) was taken into account. The right hand side here is also bounded from below.

From (9.16) we conclude that to prove (9.14), we may suppose that

$$n_{\delta,\varkappa,p}(t,r)\ge 1.$$

2*nd step*: *Small values of* r *are only of interest.* We assume that r is small enough, $0<r<\varepsilon_0$. To show that this assumption is possible, we have to check that the right-hand side of (9.14) is bounded from below and $n_{\delta,\varkappa,p}(t,r)$ is bounded from above when $r\ge\varepsilon_0$. The former is obvious; to verify the latter, we observe that from (9.15) it follows that

$$1 \le \int\limits_{\substack{\tau\in\Gamma\\ |\tau-t|>\varepsilon_0}} \frac{|\tau-t|^{\delta(t)p(\tau)}|\tau-t_0|^{\varkappa(\tau)p(\tau)}}{n_{\delta,\varkappa,p}}\, d\nu(\tau)$$

whence

$$n_{\delta,\varkappa,p}(t,r) \le \int_{\Gamma\setminus\Gamma(t,\varepsilon_0)} |\tau-t|^{\delta(t)p(t)}\, |\tau-t_0|^{\varkappa(t_0)p(t_0)} u(t,\tau)v(\tau)\, d\nu(\tau),$$

where $u(t,\tau) = |\tau-t|^{\delta(t)[p(\tau)-p(t)]}$ and $v(\tau) = |\tau-t_0|^{\varkappa(\tau)p(\tau)-\varkappa(t_0)p(t_0)}$. By direct estimation of $\ln u(t,\tau)$ and $\ln v(\tau)$ we obtain that

$$\mathrm{e}^{-2\ell AB} \le u(t,\tau) \le \mathrm{e}^{2\ell AB}, \qquad t,\tau\in\Gamma \tag{9.17}$$

where ℓ and A are the constants from (2.3) and $B=\sup_{t\in\Gamma}|\delta(t)|$, and

$$\mathrm{e}^{-c}\le|\tau-t_0|^{\varkappa(\tau)p(\tau)-\varkappa(t_0)p(t_0)})\le\mathrm{e}^{c},\qquad t,\tau\in\Gamma \tag{9.18}$$

with some constant $c>0$ (one may take $c=2\max\{A_2,\ell\sup_{t\in\Gamma}|\varkappa(t)p(t)|\}$, where A_2 is the constant from (9.10)).

Therefore,

$$\begin{aligned} n_{\delta,\varkappa,p}(t,r)&\le \mathrm{e}^{c+2\ell AB}\int_{\Gamma\backslash\Gamma(t,\varepsilon_0)}|\tau-t|^{\delta(t)p(t)}\,|\tau-t_0|^{\varkappa(t_0)p(t_0)}\,d\nu(\tau)\\ &\le \mathrm{e}^{c+2\ell AB}\varepsilon_0^{-Bp_+}\int_\Gamma|\tau-t_0|^{\varkappa(t_0)p(t_0)}d\nu(\tau)=\mathrm{const}\end{aligned}$$

which proves the boundedness of $n_{\delta,\varkappa,p}(t,r)$ from above.

The value of ε_0 will be chosen later.

3*rd step*: *A rough estimate.* First, we derive a weaker estimate

$$n_{\delta,\varkappa,p}(t,r)\le Cr^{\delta(t)} \tag{9.19}$$

which will be used later to obtain the final estimate (9.14). To this end, we note that always $\lambda^{p(\tau)}\le\lambda^{\inf p(\tau)}+\lambda^{\sup p(\tau)}$, so that from (9.15) and (9.18) we have

$$1\le\int_{\Gamma\backslash\Gamma(t,r)}\left[\left(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p_-}+\left(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p_+}\right]|\tau-t_0|^{\varkappa(t_0)p(t_0)}\,d\nu(\tau).$$

Since $|\tau-t|>r$ and $\delta(t)<0$, we obtain

$$1\le\left[\left(\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p_-}+\left(\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p_+}\right]\int_{\tau\in\Gamma}|\tau-t_0|^{\varkappa(t_0)p(t_0)}\,d\nu(\tau).$$

Hence $(\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}})^{p_-}+(\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}})^{p_+}\ge c$ which yields $\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}}\ge C$ and we arrive at the estimate in (9.19).

4*rd step.* We split integration in (9.15) as follows

$$\begin{aligned}1&=\left(\int_{\Gamma_1(\varepsilon_0)}+\int_{\Gamma_2(\varepsilon_0)}+\int_{\Gamma\backslash\Gamma(t,\varepsilon_0)}\right)\left(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p(\tau)}|\tau-t_0|^{\varkappa(\tau)p(\tau)}\,d\nu(\tau)\\ &:=\mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3,\end{aligned} \tag{9.20}$$

where

$$\Gamma_1(\varepsilon_0)=\left\{\tau\in\Gamma:\ r<|\tau-t|<\varepsilon_0,\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}>1\right\},$$

$$\Gamma_2(\varepsilon_0)=\left\{\tau\in\Gamma:\ r<|\tau-t|<\varepsilon_0,\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}<1\right\}.$$

5*th step*: *Estimation of* $\mathscr{I}_1$. We have

$$\mathscr{I}_1=\int_{\Gamma_1(\varepsilon_0)}\left(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p(t)}|\tau-t_0|^{\varkappa(\tau)p(\tau)}\,u_r(t,\tau)\,d\nu(\tau), \tag{9.21}$$

where

$$u_r(t,\tau)=\left(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}\right)^{p(\tau)-p(t)}.$$

The estimate

$$e^{-2\ell A} \le u_r(t,\tau) \le e^{2\ell A} \tag{9.22}$$

is valid. In its proof below we follow a similar estimation in [18, p. 266]. We have

$$|\ln u_r(t,\tau)| \le A \left| \frac{\ln \big(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}}\big)}{\ln \frac{2\ell}{|t-\tau|}} \right|.$$

Since $\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}} \ge 1$, we obtain

$$|\ln u_r(t,\tau)| \le A \frac{|\delta(t)| \ln \frac{1}{|\tau-t|} - \ln n_{\delta,\varkappa,p}}{\ln \frac{2\ell}{|t-\tau|}} \le A \frac{|\delta(t)| \ln \frac{1}{|\tau-t|}}{\ln \frac{2\ell}{|t-\tau|}} \le AB,$$

where $B = \sup_{t\in\Gamma} |\delta(t)|$ (without loss of generality we may assume that $2\ell \ge 1$). Hence (9.22) follows.

By (9.22) and (9.18) we obtain from (9.21)

$$\begin{aligned} \mathscr{I}_1 &\le \frac{C}{n_{\delta,\varkappa,p}^{p(t)}} \int_{\Gamma_1(\varepsilon_0)} |\tau-t|^{\delta(t)p(t)} \, |\tau-t_0|^{\varkappa(t_0)p(t_0)} \, d\nu(\tau) \\ &\le \frac{C}{n_{\delta,\varkappa,p}^{p(t)}} \int_{\Gamma\backslash\Gamma(t,r)} |\tau-t|^{\delta(t)p(t)} \, |\tau-t_0|^{\varkappa(t_0)p(t_0)} \, d\nu(\tau). \end{aligned} \tag{9.23}$$

Now we make use of the estimate obtained in (9.4) which gives

$$\mathscr{I}_1 \le \frac{C}{n_{\delta,\varkappa,p}^{p(t)}} r^{\delta(t)p(t)+1} (r + |t-t_0|)^{\varkappa(t_0)p(t_0)}. \tag{9.24}$$

The validity of conditions (9.1)–(9.3) under which the estimate (9.4) was obtained, follows from assumptions of our theorem.

6*th step*: *Estimation of* $\mathscr{I}_2$ *and the choice of* ε_0. In the integral $\mathscr{I}_2$ we have

$$\mathscr{I}_2 \le C \int_{\Gamma_2(\varepsilon_0)} \left(\frac{|\tau-t|^{\delta(t)}}{n_{\delta,\varkappa,p}} \right)^{p_{\varepsilon_0}(t)} |\tau-t_0|^{\varkappa(t_0)p(t_0)} \, d\nu(\tau), \tag{9.25}$$

where

$$p_{\varepsilon_0}(t) = \min_{|\tau-t|<\varepsilon_0} p(\tau)$$

and (9.18) was taken into account. Then

$$\mathscr{I}_2 \le \frac{C}{n_{\delta,\varkappa,p}^{p_{\varepsilon_0}(t)}} \int_{\Gamma_2(\varepsilon_0)} |\tau-t|^{\delta(t)p_{\varepsilon_0}(t)} \, |\tau-t_0|^{\varkappa(t_0)p(t_0)} \, d\nu(\tau)$$

and consequently

$$\mathscr{I}_2 \le \frac{C}{n_{\delta,\varkappa,p}^{p_{\varepsilon_0}(t)}} \int_{\Gamma\backslash\Gamma(t,r)} |\tau-t|^{\delta(t)p_{\varepsilon_0}(t)} \, |\tau-t_0|^{\varkappa(t_0)p(t_0)} \, d\nu(\tau). \tag{9.26}$$

We wish to apply estimate (9.4), but to this end we have to guarantee the validity of conditions (9.1)–(9.3). This may be achieved by a choice of ε_0 sufficiently small so that

$$\delta(t_0)p_{\varepsilon_0}(t_0) + 1 \le -\delta_1 < 0 \quad \text{and} \quad \delta(t)p_{\varepsilon_0}(t) + 1 + \varkappa(t_0) \le -\delta_2 < 0$$

which is easily derived from conditions (9.11)–(9.13) and continuity of $p(t)$ (compare with Lemma 1.7 from [18]). Conditions (9.1)–(9.3) being satisfied, we make use of (9.4) and get

$$\mathscr{I}_2 \leq \frac{C}{n_{\delta,\varkappa,p}^{p_{\varepsilon_0}(t)}} r^{\delta(t)p_{\varepsilon_0}(t)+1}(r+|t-t_0|)^{\varkappa(t_0)p(t_0)}, \tag{9.27}$$

where C does not depend on x and r.

7*th step*: *Estimation of* $\mathscr{I}_3$. We have

$$\mathscr{I}_3 \leq \frac{C}{n_{\delta,\varkappa,p}^{p_-}}\mathscr{I}_4, \ \mathscr{I}_4 = \mathscr{I}_4(t) = \int\limits_{\substack{\tau\in\Gamma \\ |\tau-t|>\varepsilon_0}} |\tau-t|^{\delta(t)p(\tau)}|\tau-t_0|^{\varkappa(\tau)p(\tau)}\, d\nu(\tau).$$

The integral $\mathscr{I}_4(t)$ is a bounded function of t. Indeed, by (9.17)–(9.18) we obtain

$$\begin{aligned}\mathscr{I}_4(t) &\leq C\int_{\Gamma\setminus\Gamma(t,\varepsilon_0)} |\tau-t|^{\delta(t)p(\tau)}|\tau-t_0|^{\varkappa(t_0)p(t_0)}\, d\nu(\tau)\\ &\leq C\int_{\Gamma\setminus\Gamma(t,\varepsilon_0)} |\tau-t|^{\delta(t)p(t)}|\tau-t_0|^{\varkappa(t_0)p(t_0)}\, d\nu(\tau)\end{aligned}$$

which is bounded by (9.4). Therefore,

$$\mathscr{I}_3 \leq \frac{C}{n_{\delta,\varkappa,p}^{p_-}}. \tag{9.28}$$

8*th step*. Gathering estimates (9.4), (9.27) and (9.28), we have from (9.20)

$$1 \leq C_0\left(\frac{r^{\delta(t)p(t)+1}}{n_{\delta,\varkappa,p}^{p(t)}}(r+|t-t_0|)^{\varkappa(t_0)p(t_0)} + \frac{r^{\delta(t)p_{\varepsilon_0}(t)+1}}{n_{\delta,\varkappa,p}^{p_{\varepsilon_0}(t)}}(r+|t-t_0|)^{\varkappa(t_0)p(t_0)} + \frac{1}{n_{\delta,\varkappa,p}^{p_-}}\right) \tag{9.29}$$

with a certain constant C_0 not depending on t and r. We may assume that

$$n_{\delta,\varkappa,p}(t,r) \geq \left(\frac{1}{2C_0}\right)^{\frac{1}{p_-}} := C_1 \tag{9.30}$$

because for those t and r where $n_{\delta,\varkappa,p}(t,r) \leq C_1$ there is nothing to prove, the right-hand side of (9.14) being bounded from below according to (9.16). In the situation (9.30) we derive from (9.29) the inequality

$$1 \leq C_0\left(\frac{r^{\delta(t)p(t)+1}}{n_{\delta,\varkappa,p}^{p(t)}} + \frac{r^{\delta(t)p_{\varepsilon_0}(t)+1}}{n_{\delta,\varkappa,p}^{p_{\varepsilon_0}(t)}}\right)(r+|t-t_0|)^{\varkappa(t_0)p(t_0)}. \tag{9.31}$$

Since $n_{\delta,\varkappa,p}(t,r) \geq 1$ we observe that $(\frac{1}{n_{\delta,\varkappa,p}})^{p_{\varepsilon_0}(x)} \leq (\frac{1}{n_{\delta,\varkappa,p}})^{p(t)}$ and $(\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}})^{p_{\varepsilon_0}(t)} \leq C(\frac{r^{\delta(t)}}{n_{\delta,\varkappa,p}})^{p(t)}$ by (9.19). Hence,

$$\frac{r^{\delta(t)p_{\varepsilon_0}(t)+1}}{n_{\delta,\varkappa,p}^{p_{\varepsilon_0}(t)}} \leq \frac{r^{\delta(t)p(t)+1}}{n_{\delta,\varkappa,p}^{p(t)}}.$$

Therefore, from (9.31) we derive the estimate

$$\frac{r^{\delta(t)p(t)+1}}{n_{\delta,\varkappa,p}^{p(t)}}(r+|t-t_0|)^{\varkappa(t_0)p(t_0)} \geq C,$$

where $C>0$ does not depend on t and r, which yields (9.14), because

$$\mathrm{e}^{-c} \leq (r+|t-t_0|)^{1-\varkappa(t_0)\frac{p(t_0)}{p(t)}} \leq \mathrm{e}^{C} \tag{9.32}$$

with $C > 0$ not depending on t and r. Inequality (9.32) is easily obtained by the direct estimation of $\ln\,(r+|t-t_0|)^{1-\varkappa(t_0)\frac{p(t_0)}{p(t)}}$ and taking (9.10) into account. □

9.3 Proof of Theorem D Itself

Basing on Remark 6.1, we consider the case of a single power weight $|t-t_0|^\beta, t_0 \in \Gamma$.

The case $\beta \geq 0$ The starting point is the same as in the proof of Theorem C: we base ourselves on (8.4) and (8.5). By (8.4) and the first condition in (1.5) we have

$$A_r(t) \leq c r^{\alpha(t)} Mf(t) \tag{9.33}$$

with some absolute constant $c > 0$ not depending on t and r.

Let $f(t) \geq 0$ and $\|f\|_{L^{p(\cdot)}(\Gamma,\rho^\beta)} \leq 1,\ \rho^\beta = |t-t_0|^\beta$. Applying the Hölder inequality (2.6) in the integral $B_r(t)$, we obtain

$$|B_r(t)| \leq k\ n_{\delta,\varkappa,\overline{p}}(t,r)\|f\|_{L^{p(\cdot)}(\Gamma,\rho^\beta)} \leq n_{\delta,\varkappa,\overline{p}}(t,r), \tag{9.34}$$

where

$$\delta(t) = \alpha(t) - 1 \qquad \text{and} \qquad \varkappa(t) = -\beta p'(t).$$

We make use of our estimate (9.14) and obtain

$$|B_r(t)| \leq C\ r^{-\frac{1}{q(t)}}(r+|t-t_0|)^{-\beta}, \tag{9.35}$$

the assumptions of Theorem 9.2 being satisfied by (1.5) and the fact that $\beta \geq 0$. From (9.35) we obtain

$$|B_r(t)| \leq C\ |t-t_0|^{-\beta}\ r^{-\frac{1}{q(t)}}, \tag{9.36}$$

since $\beta \geq 0$.

Therefore, taking into account (9.33) and (9.36) in (8.4), we arrive at

$$I^{\alpha(\cdot)}f(t) \leq C\{r^{\alpha(t)}Mf(t) + |t-t_0|^{-\beta}\ r^{-\frac{1}{q(t)}}\}. \tag{9.37}$$

It remains to choose the value of r which minimizes the right-hand side. A direct calculation provides

$$r = \left[\frac{1}{q(t)\alpha(t)}\right]^{p(t)} \left(|t-t_0|^\beta Mf(t)\right)^{-p(t)}.$$

Substituting this into (9.37), after easy evaluations we get

$$I^{\alpha(\cdot)}f(t) \leq C|t-t_0|^{-\beta\alpha(t)p(t)}[Mf(t)]^{\frac{p(t)}{q(t)}}.$$

Hence,

$$\int_\Gamma (|t-t_0|^\beta |I^{\alpha(\cdot)}f(t)|)^{q(t)}\ d\nu(t) \leq C \int_\Gamma (|t-t_0|^\beta |Mf(t)|)^{p(t)}\ d\nu(t). \tag{9.38}$$

It remains to make use of Theorem A. Condition (3.3) of that theorem is satisfied by (3.6). By Theorem A we have $\|Mf\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^\beta)} \leq C\,\|f\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^\beta)} \leq C$. Then also $\int_\Gamma \left(|t-t_0|^\beta|Mf(t)|\right)^{p(t)}\ d\nu(t) \leq C$. Consequently, by (9.38) we obtain that

$$\int_\Gamma (|t-t_0|^\beta |I^{\alpha(\cdot)}f(t)|)^{q(t)}\ d\nu(t) \leq C$$

for all $f \in L^{p(\cdot)}(\Gamma, |t-t_0|^{\beta})$ with $|f\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^{\beta})} \leq 1$ which completes the proof in the case $\beta \geq 0$.

The case $\beta < 0$ This case is reduced to the previous one by the duality arguments. First we observe that the operator conjugate to $I^{\alpha(\cdot)}$ has the form

$$(I^{\alpha(\cdot)})^* g(t) = \mathfrak{I}_{\alpha(\cdot)} g(t) := \int_\Gamma \frac{g(\tau)\, d\nu(\tau)}{|t-\tau)|^{1-\alpha(\tau))}} \sim \int_\Gamma \frac{g(\tau))\, d\nu(\tau)}{|t-\tau)|^{1-\alpha(t)}} = I^{\alpha(\cdot)} g(t), \tag{9.39}$$

where the equivalence

$$C_1 |t-\tau)|^{1-\alpha(t)} \leq |t-\tau)|^{1-\alpha(\tau)} \leq C_2 |t-\tau)|^{1-\alpha(t)}$$

follows from the logarithmic condition for $\alpha(t)$.

We pass to the duality statement in Theorem A. We take it as already proved with $\beta \geq 0$ to arrive at the statement with negative exponent. By (9.39) we obtain from the already proved part of the theorem with non-negative $bt(t_0)$ that

$$\|I^{\alpha(\cdot)} g\|_{\left(L^{p(\cdot)}(\Gamma,|t-t_0|^{\beta}\right)^*} \leq C \|g\|_{\left(L^{q(\cdot)}(\Gamma,|t-t_0|^{\beta}\right)^*}.$$

In view of (2.7) this yields

$$\|I^{\alpha(\cdot)} g\|_{L^{p'(\cdot)}(\Gamma,|t-t_0|^{-\beta})} \leq C \|g\|_{L^{q'(\cdot)}(\Gamma,|t-t_0|^{-\beta})}, \tag{9.40}$$

where the the exponent $-\beta$ is already positive. Now it makes sense to redenote $-\beta =: \beta_1, \quad q'(t) =: p_1(t)$ so that the Sobolev exponent for $p_1(t)$ is $q_1(t) = \frac{p_1(t)}{1-\alpha p_1(t)} = p'(t)$ and then (9.40) takes the form

$$\|I^{\alpha(\cdot)} g\|_{L^{q_1(\cdot)}\left(\Gamma,|t-t_0|^{\beta_1}\right)} \leq C \|g\|_{L^{p_1(\cdot)}\left(\Gamma,|t-t_0|^{\beta_1}\right)}. \tag{9.41}$$

Since

$$1-\alpha(t)p_1(t) = \frac{p(t)-1}{p(t)-1+\alpha(t)p(t)} \geq c > 0$$

and

$$0 \leq \beta < \frac{1}{p'(t_0)} \quad \Longleftrightarrow \quad \alpha(t_0) - \frac{1}{p_1(t_0)} < \beta_1 \leq 0$$

the estimate in (9.41) is nothing else but our Theorem A for the negative subinterval of possible values of the exponent.

Proof of Corollary to Theorem D It suffices to refer to the well-known pointwise estimate

$$M_{\alpha(\cdot)} f(t) \leq c I^{\alpha(\cdot)} |f|(t), \tag{9.42}$$

where c does not depend on f and t. The proof of (9.42) on Carleson curves is the same as in the case of functions in $\mathbb{R}^n$ (see for instance [19, p. 909]): for any $t \in \Gamma$ there exists an $r = r_t$ such that

$$M_{\alpha(\cdot)} f(t) \leq \frac{2}{\nu\{\Gamma(t,r_t)\}^{1-\alpha(t)}} \int_{\Gamma(t,r_t)} |f(\tau)|\, d\nu(\tau)$$

and on the other hand

$$I^{\alpha(\cdot)} f(t) \geq \int_{\Gamma(t,r_t)} \frac{f(\tau)\, d\nu(\tau)}{|t-\tau|^{n-\alpha(t)}} \geq \frac{c}{\nu\{B(t,r_t)\}^{1-\alpha(t)}} \int_{\Gamma(t,r_t)} |f(\tau|\, d\nu(\tau).$$

10 Appendices

10.1 Appendix 1: Proof of Proposition 4.3

Let $f(\tau) = \chi_\gamma(\tau)[\nu(\gamma)]^{-\frac{1}{p(\tau)}}$, $\gamma = \Gamma(t, r)$, so that $\|f\|_{p(\cdot)} = 1$. For all $z \in \gamma$ we have

$$CMf(z) \geq \frac{1}{\nu(\gamma)} \int_\gamma f(\tau)\, d\nu(\tau) = \frac{1}{\nu(\gamma)} \int_\gamma [\nu(\gamma)]^{-\frac{1}{p(\tau)}}\, d\nu(\tau) \qquad \text{for any} \quad \gamma = \Gamma(t, r). \tag{10.1}$$

Since the function $\Phi(x) = a^{-x}, x \in \mathbb{R}^1_+$, is convex for any $a > 0$, by Jensen's inequality

$$\Phi\left(\frac{1}{\nu(\gamma)} \int_\gamma |f(\tau)| d\nu(\tau)\right) \leq \frac{1}{\nu(\gamma)} \int_\gamma \Phi(|f(\tau)|)\, d\nu(\tau), \tag{10.2}$$

we obtain

$$CMf(z) \geq [\nu(\gamma)]^{-\frac{1}{\nu(\gamma)} \int_\gamma \frac{d\nu(\tau)}{p(\tau)}} = [\nu(\gamma)]^{-\frac{1}{p_\gamma}}, \qquad z \in \gamma = [\nu(\gamma)]^{-\frac{1}{p_\gamma}}.$$

Hence $\|\chi_\gamma(z)[\nu(\gamma)]^{-\frac{1}{p_\gamma}}\|_{p(\cdot)} \leq C\|Mf\|_{p(\cdot)}$ and by the boundedness of the maximal operator we obtain that $\|\chi_\gamma(z)[\nu(\gamma)]^{-\frac{1}{p_\gamma}}\|_{p(\cdot)} \leq C$, which yields (4.3).

10.2 Appendix 2: Proof of the Necessity of the Condition $\beta < \frac{1}{p'_\infty}$ of Theorem B

To show the necessity of the condition $\beta < \frac{1}{p'_\infty}$, we choose $f_0(t) = \chi_{\Gamma_{2R}}(t)$. Then $f_0 \in L^{p(\cdot)}(\Gamma, w)$ according to Lemma 4.1 since $\beta > -\frac{1}{p(t_0)}$. By the boundedness of the maximal operator we have

$$\mathfrak{I}^p_\Gamma(wMf_0) < \infty. \tag{10.3}$$

On the other hand,

$$Mf_0(t) \geq \frac{C}{|t - t_0|}, \qquad t \in \Gamma \backslash \Gamma_{2R} \tag{10.4}$$

with $C > 0$ not depending on t. Indeed, to prove (10.4), observe that since $\nu\{\Gamma(t, r)\} \leq cr$, we have

$$Mf_0(t) \geq \frac{C}{r} \int_{\Gamma(t,r)\cap\Gamma_{2R}} d\nu(\tau) \tag{10.5}$$

for any $r > 0$. We choose $r = 2|t - t_0|$. Then for $\tau \in \Gamma_{2R}$ and $t \in \Gamma\backslash\Gamma_{2R}$ we have $|\tau - t| \leq |\tau - t_0| + |t - t_0| \leq 2R + |t - t_0| \leq 2|t - t_0| = r$, that is, $\Gamma(t, r) \cap \Gamma_{2R} = \Gamma_{2R}$ for $r = 2|t - t_0|$ and then from (10.5) we obtain $Mf_0(t) \geq \frac{C}{|t-t_0|} \int_{\Gamma_{2R}} d\nu(\tau)$, which is (10.4).

In view of (10.4) we get

$$\mathfrak{I}^p_\Gamma(wMf_0) \geq \int_{\Gamma\backslash\Gamma_{2R}} [w(t)Mf_0(t)]^{p_\infty} d\nu(t) \geq \int_{\Gamma\backslash\Gamma_{2R}} |t - t_0|^{(\beta-1)p_\infty} d\nu(t).$$

Since the condition $(\beta - 1)p_\infty < -1$ is necessary for the convergence of the last integral, see Lemma 4.2, we conclude that (10.3) implies the condition $\beta < \frac{1}{p'_\infty}$.

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