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Integral Equations and Operator Theory

Boundedness and Fredholmness of Pseudodifferential Operators in Variable Exponent Spaces

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Abstract. We prove a statement on the boundedness of a certain class of singular type operators in the weighted spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$ with variable exponent $p(x)$ and a power type weight w, from which we derive the boundedness of pseudodifferential operators of Hörmander class $S^0_{1,0}$ in such spaces.

This gives us a possibility to obtain a necessary and sufficient condition for pseudodifferential operators of the class $OPS^m_{1,0}$ with symbols slowly oscillating at infinity, to be Fredholm within the frameworks of weighted Sobolev spaces $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ with constant smoothness s, variable $p(\cdot)$-exponent, and exponential weights w.

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1. Introduction

The main objective of this paper is to investigate the boundedness and Fredholmness of pseudodifferential operators of the Hörmander class $OPS^0_{1,0}$ in weighted Sobolev type spaces $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ with constant smoothness s, variable $p(\cdot)$-exponent, and exponential weights w.

We prove the boundedness of more general singular type integral operators in weighted variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$ with power weights w, from which there follows the boundedness of operators of the class $OPS^0_{1,0}$ in such

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spaces. Making use of the calculus of pseudodifferential operators, we obtain the result on boundedness of pseudodifferential operators in the spaces $H_w^{s,p(\cdot)}(\mathbb{R}^n)$.

The obtained boundedness is the crucial result for the investigation of the Fredholm property of pseudodifferential operators, with symbols slowly oscillating at infinity, in weighted Sobolev spaces, including their essential spectra and behavior of solutions of pseudodifferential equations at infinity.

The paper is arranged as follows. After Section 2, where we give some necessary preliminaries, in Section 3 we study the boundedness of singular type operators in the spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$ with a power type weight w. With the help of the results of Section 3, after a preliminary Section 4 on pseudodifferential operators, in Section 5 we prove the boundedness of pseudodifferential operators in the space $H^{s,p(\cdot)}(\mathbb{R}^n)$. In Section 6 we obtain a necessary and sufficient condition for pseudodifferential operators with slowly oscillating symbols to be Fredholm in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$. In Section 7 we study Fredholmness of pseudodifferential operators with analytical symbols in weighted spaces $H_w^{s,p(\cdot)}(\mathbb{R}^n)$.

We linger more in detail on results of every section and mention the relevant investigations on the subject.

Section 3. In relation to the boundedness results in variable exponent Lebesgue spaces, observe that the last decade there was an evident increase of interest to the operator theory in the generalized Lebesgue spaces with variable exponent $p(x)$, we refer, in particular to surveys L. Diening, P. Hästö and A. Nekvinda [7], P. Harjulehto and P. Hästö [13], V. Kokilashvili [24], S. Samko [42] on the progress in this topic.

Lebesgue and Sobolev spaces with variable exponent proved to be appropriate for studying various applications, including electroreolhogical fluids, see [41]. This raised an enormous increase of interest to such spaces. Both the problem of the boundedness of the main objects of harmonic analysis, such as maximal and singular operators and potential type operators, and Fredholmness of singular integral operators has already been treated in these spaces.

For maximal operators we refer, besides the above mentioned surveys, to L. Diening [6], D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer [5] and A. Nekvinda [31] in the non-weighted case, and to V. Kokilashvili and S. Samko [23] and V. Kokilashvili, N. Samko and S. Samko [19] in the weighted case.

Boundedness of Calderon-Zygmund singular operators was studied by L. Diening and M. Ružička [8], [9] in the non-weighted case and by V. Kokilashvili and S. Samko [21], [22] in the weighted case. Recently, the boundedness of the Cauchy singular operator S_Γ on Carleson curves Γ was proved in V. Kokilashvili and S. Samko [24], [18].

In the proof of the result on boundedness of singular type operators in the spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$, presented in Theorem 3.2, we use the technique of the pointwise estimation of the sharp maximal operator of the power of order $s, 0 < s < 1$ of the singular operator via the maximal operator. In Section 3.2 we develop this technique for variable exponent Lebesgue space.

Section 5**.** We use the results of Section 3 to prove the boundedness of pseudodifferential operators in the space $H^{s,p(\cdot)}(\mathbb{R}^n)$. As a corollary of those results and the formulas of composition of pseudodifferential operators we obtain boundedness of pseudodifferential operators of the class $OPS^m_{1,0}$ from $H^{s,p(\cdot)}(\mathbb{R}^n)$ to $H^{s-m,p(\cdot)}(\mathbb{R}^n)$.

As is known, the boundedness of pseudodifferential operators of the class $OPS^0_{\delta,\delta}, 0 \leq \delta < 1$ in the space L^2 was proved in the well known paper [3] by A.P.Calderón and R.Vaillancourt. For the boundedness of pseudodifferential operators in Lebesgue spaces with constant $p, 1 < p < \infty$, we refer to [48] and references therein.

Section 6**.** Note that the Fredholmness of pseudodifferential operators of the class $OPS^m_{1,0}$ acting in the Sobolev spaces $H^s(\mathbb{R}^n)$ was established by V.V.Grushin [12]. Fredholmness of pseudodifferential operators of the class $OPS^m_{0,0}$ acting in the spaces $H^s(\mathbb{R}^n)$ was considered in the papers V.S. Rabinovich [33], see also the paper [39] and the book [40], Chap. 4, by means of the limit operators method. Fredholmness and exponential estimates of solutions of general pseudodifferential operators acting in general exponential weight classes were considered in [37]. Note also the paper by V.S. Rabinovich [38] where operators of the class $OPS^m_{1,0}$ with symbols slowly oscillating at infinity were considered in weighted Hölder-Zygmund spaces.

Fredholmness of operators in algebras of pseudodifferential operators acting in $L^p(\mathbb{R}^n)$, with constant $p \in (1, \infty)$ with applications to one-dimensional singular integral operators on Carleson curves has been developed in V. Rabinovich [34], see also [36].

As regards Fredholm properties in variable Lebesgue spaces $L^{p(\cdot)}(\Gamma, w)$, it was studied only in the case of one-dimensional singular integral operators in the papers V. Kokilashvili and S. Samko [20] and A. Karlovich [16].

Section 7**.** Finally, in this section we consider boundedness and Fredholmness of operators with analytical symbols acting in weighted spaces $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ with exponential weight w. As a corollary of Fredholmness in weight spaces we consider a Phragmen-Lindelöf principle (see for instance [29], p. 284–286) for solutions of pseudodifferential operators with analytical symbols in $H^{s,p(\cdot)}_w(\mathbb{R}^n)$.

N o t a t i o n :
$Fu(\xi) = \int_{\mathbb{R}^n} u(x)e^{-ix\xi}dx$;
$F^{-1}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi)e^{ix\xi}d\xi$;
$H^{s,p(\cdot)}(\mathbb{R}^n)$, see Definition 2.4;
$I^p(f)$, see (2.1);
$Op(a)$, see (4.2);
$S^m_{1,0}$, see Definition 4.1;
$S(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing test functions;
$\langle\xi\rangle = \sqrt{1 + |\xi|^2}$.

2. Preliminaries

2.1. Variable exponent spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$ and $H^{s,p(\cdot)}(\mathbb{R}^n)$

2.1.1. The spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Let p be a measurable function on $\mathbb{R}^n$ such that $p : \mathbb{R}^n \to (1, \infty)$, $n \geq 1$. The generalized Lebesgue space with variable exponent is defined via the modular

$$I^p(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \tag{2.1}$$

by the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I^p\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

We denote $p'(x) = \frac{p(x)}{p(x)-1}$.

In what follows we assume that p satisfies the conditions

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty, \tag{2.2}$$

$$|p(x) - p(y)| \leq \frac{A}{ln\ \frac{1}{|x-y|}}, \qquad x, y \in \mathbb{R}^n, \qquad |x - y| \leq \frac{1}{2}. \tag{2.3}$$

We shall also use the condition

$$|p(x) - p(\infty)| \leq \frac{A}{ln\ (2 + |x|)}, \qquad x \in \mathbb{R}^n, \tag{2.4}$$

which together with (2.2) and (2.3) guarantees the boundedness of the maximal operator (2.18) in $L^{p(\cdot)}(\mathbb{R}^n)$, see [5].

Note that under the condition

$$1 \leq p_- \leq p(x) \leq p_+ < \infty \tag{2.5}$$

for a function $a(x) \in L^\infty(\mathbb{R}^n)$ we have

$$\|aI\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \leq \|a\|_{L^\infty} \tag{2.6}$$

which follows from the definition of the norm in $L^{p(\cdot)}$. Note also that under the same condition (2.5) the modular boundedness is equivalent to the norm boundedness and the modular convergence is equivalent to the norm convergence, because

$$c_1 \leq \|f\|_p \leq c_2 \quad \Longrightarrow c_3 \leq I^p(f) \leq c_4 \tag{2.7}$$

and

$$C_1 \leq I^p(f) \leq C_2 \quad \Longrightarrow C_3 \leq \|f\|_p \leq C_4 \tag{2.8}$$

with $c_3 = \min\left(c_1^{p_-}, c_1^{p^+}\right), c_4 = \max\left(c_2^{p_-}, c_2^{p^+}\right), C_3 = \min\left(C_1^{1/p_-}, C_1^{1/p^+}\right)$ and $C_4 = \max\left(C_2^{1/p_-}, C_2^{1/p^+}\right)$.

By $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ we denote the class of exponents p satisfying condition (2.2) and by $\mathbb{P} = \mathbb{P}(\mathbb{R}^n)$ the class of those p for which the maximal operator M is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.

The validity of the Riesz-Thorin interpolation theorem for the variable exponent spaces $L^{p(\cdot)}$, stated in Proposition 2.1, was observed by L. Diening [7]; it is known in a more general setting for Musielak-Orlich spaces in [30], Theorem 14.16. Proposition 2.1 follows from the fact that $L^{p_\theta(\cdot)}(\mathbb{R}^n)$ is an interpolation space between $L^{p_1(\cdot)}(\mathbb{R}^n)$ and $L^{p_2(\cdot)}(\mathbb{R}^n)$ under the method of real-valued interpolation. For complex interpolation for $L^{p(\cdot)}$-spaces we refer to [7].

Proposition 2.1. *Let $p_j : \mathbb{R}^n \to [1,\infty), j = 1,2,$ be bounded measurable functions, A a linear operator defined on $L^{p_1(\cdot)}(\mathbb{R}^n) \cup L^{p_2(\cdot)}(\mathbb{R}^n)$ and*

$$\|Au\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \le C_j \|u\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}, \quad j = 1,2. \tag{2.9}$$

Then A is also bounded on $L^{p_\theta(\cdot)}(\mathbb{R}^n)$, where $\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}$, $\theta \in [0,1]$, and

$$\|A\|_{L^{p_\theta(\cdot)} \to L^{p_\theta(\cdot)}} \le \|A\|^{\theta}_{L^{p_1(\cdot)} \to L^{p_1(\cdot)}} \|A\|^{1-\theta}_{L^{p_2(\cdot)} \to L^{p_2(\cdot)}}.$$

The following proposition is an extension of the well-known theorem of M.A. Krasnosel′skii [26] on the interpolation of the compactness property in L^p-spaces with a constant p.

Proposition 2.2. *Let $p_j : \mathbb{R}^n \to [1,\infty), j = 1,2,$ be bounded measurable functions satisfying assumptions* (2.2)–(2.4) *and let a linear operator A defined on $L^{p_1(\cdot)}(\mathbb{R}^n) \cup L^{p_2(\cdot)}(\mathbb{R}^n)$ satisfy assumption* (2.9). *If*

$$A : L^{p_1(\cdot)}(\mathbb{R}^n) \to L^{p_1(\cdot)}(\mathbb{R}^n)$$

is a compact operator, then

$$A : L^{p_\theta(\cdot)}(\mathbb{R}^n) \to L^{p_\theta(\cdot)}(\mathbb{R}^n)$$

is a compact operator for all $\theta \in (0,1]$.

Proof. We derive this proposition from the abstract Banach spaces version of Krasnosel′skii's theorem proved in the paper A. Persson [32]. The crucial condition of Persson's theorem is the existence of a unity approximation in the interpolation couple with some properties. We formulate it with respect to the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ under consideration:

there exists a topological space $\mathcal{E}$ such that $L^{p(\cdot)}(\mathbb{R}^n) \subset \mathcal{E}$, and a sequence P_m of linear operators with the properties:

(i) $P_m : \mathcal{E} \to \mathcal{E}$,

(ii) $P_m(L^{p_j(\cdot)}(\mathbb{R}^n)) \subset L^{p_1(\cdot)}(\mathbb{R}^n) \cap L^{p_2(\cdot)}(\mathbb{R}^n)$ *for every m,*

(iii) *the sequence P_m strongly converges in $L^{p_i(\cdot)}(\mathbb{R}^n), i = 1,2$.*

We take $\mathcal{E} = \mathcal{D}'(\mathbb{R}^n)$ and construct such a sequence P_m in the following way. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a non-negative function such that $\phi(x) = 1$ if $|x| \le 1/2$ and $\phi(x) = 0$ if $|x| \ge 1$, and $\phi_m(x) = \phi(x/m)$. Let $\phi_m I$ be the operators of multiplication by ϕ_m. Then for the sequence $\phi_m I$ conditions (i) and (iii) are satisfied.

Let

$$\varphi(x) = \frac{\phi(x)}{\int_{\mathbb{R}^n} \phi(x)dx} \quad \text{and} \quad \varphi_m(x) = m^n \varphi(mx)$$

and T_m be a sequence of operators

$$T_m u(x) = \int_{\mathbb{R}^n} \varphi_m(x-y)u(y)dy.$$

It is known [6], Corollary 3.6 (see also [4]), that the identity approximation sequence T_m strongly converges to the unit operator in $L^{p(\cdot)}(\mathbb{R}^n)$, under the assumptions on $p(\cdot)$. Hence the sequence $P_m = T_m\phi_m I$ strongly converges to the unit operator in $L^{p(\cdot)}(\mathbb{R}^n)$. Hence condition (iii) holds.

Moreover, it is easily seen that $P_m u \in C_0^\infty(\mathbb{R}^n)$ for every $u \in L^{p(\cdot)}(\mathbb{R}^n)$. Hence condition (ii) is also satisfied, and consequently, Proposition 2.2 follows from A.Persson result [32]. □

Corollary 2.3. *Let $p : \mathbb{R}^n \to (1,\infty)$ $(1 < p_- \le p(x) \le p_+ < \infty)$. Then there exists $q : \mathbb{R}^n \to (1,\infty)$ $(1 < q_- \le q(x) \le q_+ < \infty)$, and $\theta \in [0,1]$ such that $L^{p(\cdot)}(\mathbb{R}^n)$ is an intermediate space between $L^2(\mathbb{R}^n)$ and $L^{q(\cdot)}(\mathbb{R}^n)$ corresponding to the interpolation parameter θ.*

Proof. We will find q and θ from the equality $\frac{1}{p(x)} = \frac{\theta}{2} + \frac{1-\theta}{q(x)}$, $\theta \in [0,1]$ and conditions

$$1 < q_- \le q(x) \le q_+ < \infty. \tag{2.10}$$

Then

$$q(x) = \frac{2(1-\theta)p(x)}{2-\theta p(x)}.$$

If we fix a $\theta \in (0,\theta_0)$ where $\theta_0 = \min\left\{1, \frac{2}{p_+}, 2\left(1-\frac{1}{p_-}\right)\right\}$, then condition (2.10) will be satisfied. □

By $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \in \mathbb{R}^n \backslash E \end{cases}$ we denote the characteristic function of a set $E \subset \mathbb{R}^n$.

2.1.2. The weighted spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$. By $L^{p(\cdot)}(\mathbb{R}^n, w)$ we denote the weighted Banach space of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} := \|wf\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{w(x)f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\} < \infty. \tag{2.11}$$

Observe that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} = \|f^a\|^{\frac{1}{a}}_{L^{\frac{p(\cdot)}{a}}(\mathbb{R}^n,w^a)} \tag{2.12}$$

for any $0 < a \le \inf p(x)$.

From the Hölder inequality for the $L^{p(\cdot)}$-spaces

$$\left|\int_{\mathbb{R}^n} u(x)v(x)\,dx\right| \le k\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|v\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \qquad \frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1,$$

it follows that

$$\left|\int_{\mathbb{R}^n} u(x)v(x)\,dx\right| \le k\|u\|_{L^{p'}\left(\mathbb{R}^n,\frac{1}{w}\right)}\|v\|_{L^p(\mathbb{R}^n,w)}, \tag{2.13}$$

and for the conjugate space $\left[L^{p(\cdot)}(\mathbb{R}^n,w)\right]^*$ we have

$$\left[L^{p(\cdot)}(\mathbb{R}^n,w)\right]^* = L^{p'(\cdot)}(\mathbb{R}^n,1/w) \tag{2.14}$$

which is an immediate consequence of the fact that $\left[L^{p(\cdot)}(\mathbb{R}^n)\right]^* = L^{p'(\cdot)}(\mathbb{R}^n)$ under conditions (2.2), see [25], [43].

In Section 3.3 we will deal with the power weights of the form

$$w(x) = (1+|x|)^\beta \prod_{k=1}^m |x-x_k|^{\beta_k}, \qquad x_k \in \mathbb{R}^n. \tag{2.15}$$

2.1.3. Spaces $H^{s,p(\cdot)}(\mathbb{R}^n)$. Note that Sobolev type spaces $W^{s,p(\cdot)}$ of integer order $s \in \mathbb{N}$ with variable exponent $p(\cdot)$ have already been were investigated, we refer to the original paper [25] and surveys mentioned in the beginning of Section 1. A generalization to fractional values of s, the Bessel potential space, was considered in [1], where a characterization of functions in the Bessel potential space based on $L^{p(\cdot)}(\mathbb{R}^n)$ was in particular given in terms of convergence of certain singular operators. For our purposes we use the following definition of the space $H^{s,p(\cdot)}(\mathbb{R}^n)$.

Definition 2.4. Let $s \in \mathbb{R}$. By $H^{s,p(\cdot)}(\mathbb{R}^n)$ we denote the closure of the set $S(\mathbb{R}^n)$ respect to the norm

$$\|u\|_{H^{s,p(\cdot)}(\mathbb{R}^n)} = \|\langle D\rangle^s u\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\langle D\rangle^s = F^{-1}\langle\xi\rangle^s F$.

In the case $s > 0$ the space $H^{s,p(\cdot)}(\mathbb{R}^n)$ may be characterized as the range $\mathcal{B}^s[L^{p(\cdot)}(\mathbb{R}^n)]$, where

$$\mathcal{B}^s\varphi(x) = \int_{\mathbb{R}^n} G_s(x-y)\,\varphi(y)\,dy. \tag{2.16}$$

is the Bessel potential operator with the kernel

$$G_s(x) = F^{-1}\left[\langle\xi\rangle^{-s/2}\right](x) = c(s)\int_0^\infty e^{-\frac{\pi|x|^2}{t}-\frac{t}{4\pi}}\, t^{\frac{s-n}{2}}\,\frac{dt}{t}, \quad x \in \mathbb{R}^n,$$

and in the case $0 < s < n$ and $p_+ < \frac{n}{s}$ it may be also interpreted in terms of Riesz potentials:

$$H^{s,p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n) \cap I^s[L^{p(\cdot)}(\mathbb{R}^n)] \tag{2.17}$$

where I^s is the Riesz potential operator, see Theorems 4.1 and 5.7 in [1].

2.2. On maximal operators

We will need the following results for the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy. \tag{2.18}$$

Theorem 2.5. ([5]) *Let $p(x)$ satisfy conditions* (2.2)–(2.4)*. Then the maximal operator M is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.*

The following theorem for weighted spaces was proved in [17] for the case of $\mathbb{R}^n$, the case of bounded domains in $\mathbb{R}^n$ being earlier treated in [23].

Theorem 2.6. *Let $p(x)$ satisfy conditions* (2.2)–(2.3) *and let there exist an $R > 0$ such that $p(x) \equiv p_\infty = const$ for $|x| \geq R$. Then the maximal operator M is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, w)$ with weight* (2.15)*, if and only if*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)} \qquad \text{and} \qquad -\frac{n}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{n}{p'_\infty}. \tag{2.19}$$

Remark 2.7. *In* [17] *and* [23] *the case of a single weight $w(x) = |x - x_0|^\beta$ was considered. The validity of Theorem* 2.6 *for weight* (2.15) *is easily obtained from the case of a single weight.*

Indeed, for the weight $w(x) = \prod\limits_{k=1}^{m+1} w_k(x)$, with $w_k(x) = |x - x_k|^{\beta_k}, k = 1, \ldots, m$ and $w_{m+1}(x) = (1 + |x|)^\beta$ we have to prove the boundedness of the operator $wM\frac{1}{w}$ in the space $L^{p(\cdot)}(\mathbb{R}^n)$. We make use of a standard partition of unity $1 = \sum\limits_{k=1}^{m} a_k(t)$, where $a_k(t)$ are smooth functions equal to 1 in a neighborhood of the point x_k and equal to 0 outside some neighborhood of this point $k = 1, \ldots, m$, (and similarly in a neighborhood of infinity for $k = m+1$), so that $a_k(x)|x - x_j|^{\pm\beta_j} \equiv 0$ in a neighborhood of the point x_k, if $k \neq j$. Then

$$\frac{w(x)}{w(y)} = \sum_{\mu=1}^{m+1} \tilde{w}_\mu(x) b_\mu(x) \sum_{\nu=1}^{m+1} \frac{c_\nu(y)}{\tilde{w}_\nu(y)}$$

where $\tilde{w}_k(x) = w_k(x), k = 1, \ldots, m$ and $\tilde{w}_{m+1}(x) = (1 + |x|)^{\beta_0}$, $\beta_0 = \beta + \sum_{k=0}^{\infty} \beta_k$, while $b_\mu(x), \mu = 1, \ldots, m+1$, and $c_\nu(x), \nu = 1, \ldots, m+1$, are bounded functions supported in the same neighborhoods of the points x_k. Then

$$\left| wM\frac{f}{w} \right| \leq C \sum_{\mu=1}^{m+1} \tilde{w}_\mu M \frac{f}{\tilde{w}_\mu} + C \sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{m+1} \tilde{w}_\mu M \frac{f}{\tilde{w}_\nu}.$$

The terms where $\mu \neq \nu$ have separated singularities and are easily treated by means of the Hölder inequality, so that it remains to have the boundedness with the separate weights $w_\mu(x)$.

2.3. On sharp maximal function

Let

$$\mathcal{M}^{\#} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B(x)|\, dy, \qquad x \in \mathbb{R}^n \tag{2.20}$$

where $f_B(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$, be the sharp maximal function.

The following theorem is well known for constant p, see [48], p. 148, where it is given in the non-weighted case. For variable $p(x)$ and the weighted case see [22].

Theorem 2.8. *Let $p(x)$ satisfy conditions (2.2)–(2.3) and $p(x) = p_\infty$ for large $|x| \geq R > 0$, and let $w(x)$ be weight (2.15). Then under condition (2.19) there exists a constant $C_0 > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leq C_0 \left\|\mathcal{M}^{\#} f\right\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \tag{2.21}$$

for every $f \in L^{p(\cdot)}(\mathbb{R}^n, w)$.

3. Boundedness in $L^{p(\cdot)}(\mathbb{R}^n, w)$ of singular type operators

3.1. Formulation of the main result

We consider operators of the form

$$\mathbb{A} f(x) = \int\limits_{\mathbb{R}^n} k(x, x-y) f(y) dy \tag{3.1}$$

with $k(x,z) \in C^1(\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}))$ and assume that the following conditions are satisfied

$$\lambda_1(\mathbb{A}) := \sup_{|\alpha|=1} \sup_{x,z \in \mathbb{R}^n \times \mathbb{R}^n} |z|^{n+1} \left|\partial_x^\alpha k(x,z)\right| < \infty \tag{3.2}$$

and

$$\lambda_2(\mathbb{A}) := \sup_{|\beta|=1} \sup_{x,z \in \mathbb{R}^n \times \mathbb{R}^n} |z|^{n+1} \left|\partial_z^\beta k(x,z)\right| < \infty \tag{3.3}$$

and the operator $\mathbb{A}$ is of weak (1,1)-type:

$$|\{x \in \mathbb{R}^n : |\mathbb{A} f(x)| > t\}| \leq \frac{\nu(\mathbb{A})}{t} \int\limits_{\mathbb{R}^n} |f(x)|\, dx. \tag{3.4}$$

Theorem 3.1. *Let the operator $\mathbb{A}$ satisfy conditions (3.2)–(3.4).*

I. *Let p satisfy conditions (2.2)–(2.4). Then the operator $\mathbb{A}$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.*

II. *Let p satisfy conditions (2.2)–(2.3) and be constant at infinity, that is, there exist $R > 0$ such that $p(x) \equiv const = p_\infty$ for $|x| \geq R$. Then the operator $\mathbb{A}$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, w)$ with weight (2.15), if*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \qquad k = 1, \dots, n, \tag{3.5}$$

and

$$-\frac{n}{p_\infty} < \beta + \sum_{k=1}^{m} \beta_k < \frac{n}{p'_\infty}. \tag{3.6}$$

In both cases I *and* II*,*

$$\|\mathbb{A}\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \le c(n,p,w)\,[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + \nu(\mathbb{A})] \tag{3.7}$$

where the constant $c(n,p,w)$ depends only on n, exponent $p(x)$ and the weight w.

Theorem 3.1 is proved in Subsection 3.3.

In particular, from Theorem 3.1 we have the following corollary (see Definition 4.1 for the class OPS^0).

Corollary 3.2. *Statements of Theorem* 3.1 *are valid for every PDO $\mathbb{A} \in OPS^0_{1,0}(\mathbb{R}^n)$.*

3.2. The crucial step: the pointwise estimate

Following the ideas of the paper T.Alvarez and C.Pérez [2], in this section we prove the following statement.

Theorem 3.3. *For any operator $\mathbb{A}$ of form* (3.1) *with the kernel $k(x,z)$ satisfying conditions* (3.2)–(3.3)*, the following pointwise estimate is valid*

$$\mathcal{M}^{\#}\left(|\mathbb{A}f|^s\right)(x) \le C[Mf(x)]^s, \qquad 0 < s < 1, \tag{3.8}$$

where the constant $C > 0$ has the form $C = c(n,s)[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + \nu(\mathbb{A})]$ with $c(n,s)$ depending only on n and s.

Corollary 3.4. *For any pseudodifferential operator $\mathbb{A} \in OPS^0_{1,0}(\mathbb{R}^n)$ the pointwise estimate* (3.8) *is valid.*

Theorem 3.3 and its corollary are proved in Subsection 3.2.3.

3.2.1. Regularity of the kernel. To prove Theorem 3.3, we need some auxiliary statements and some notions of regularity of the kernel.

Definition 3.5. ([10],[2]) *Let $r > 0$ and $x_0 \in \mathbb{R}^n$. We say that a kernel $k(x,z)$ satisfies the regularity property ($\mathcal{D}_1$), if the inequality holds*

$$|k(u,u-x) - k(v,v-x)| \le \frac{D_1 r}{|x-x_0|^{n+1}} \tag{3.9}$$

for all $u,v,x \in \mathbb{R}^n$ such that

$$|u-x_0| < r, \quad |v-x| < r, \qquad |x-x_0| > 4r, \tag{3.10}$$

where $D_1 > 0$ does not depend on u,v,x,x_0.

Let

$$H_{r,x_0}(x) = \frac{1}{|B(x_0,r)|^2} \int\limits_{B(x_0,r)} \int\limits_{B(x_0,r)} |k(u,u-x) - k(v,v-x)|\, du dv. \tag{3.11}$$

Definition 3.6. *A kernel $k(x,z)$ is said to have the regularity property $(\mathcal{D}_2)$, if for any locally integrable function f (such that $Mf(x_0) < \infty$) the inequality*

$$\sup_{r>0} \int\limits_{B(x_0,4r)} |f(x)| H_{r,x_0}(x)dx \leq D_2 Mf(x_0) \tag{3.12}$$

is valid, where $D_2 > 0$ does not depend on f and x_0.

Lemma 3.7. I. *Let the kernel $k(x,z) \in C^1(\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\})$ satisfy assumptions* (3.2)–(3.3)*. Then $k(x,z)$ has the regularity property $(\mathcal{D}_1)$ with the constant $D_1 = 2^{2n+3}[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A})]$.*

II. *Any kernel $k(x,z)$ with regularity property $(\mathcal{D}_1)$ satisfies also property $(\mathcal{D}_2)$ with the constant $D_2 = \frac{2^{n+1}}{2^n-1} D_1$.*

Proof. I. By the mean value theorem we have

$$k(u, u-x) - k(v, v-x) = [\partial_x k(\xi,\eta) + \partial_z k(\xi,\eta)](v-u)$$

where $\xi = u + \theta(v-u), \eta = u - x + \theta(v-x)$. By (3.2) we get

$$|k(u,u-x) - k(v,v-x)| \leq [\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A})] \frac{2r}{|\eta|^{n+1}}.$$

We have $|\eta| \geq |u-x| - \theta|v-u| \geq |x-x_0| - |u-x_0| - 2r \geq |x-x_0| - 3r \geq \frac{1}{4}|x-x_0|$. Therefore,

$$|k(u,u-x) - k(v,v-x)| \leq \frac{C_1 r}{|x-x_0|^{n+1}}, \qquad C_1 = 2^{2n+3}[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A})],$$

which gives (3.9) and proves the first part of the lemma.

II. Let $k(x,z)$ have property $(\mathcal{D}_1)$. By the definition of this property we obtain

$$H_{r,x_0}(x) \leq \frac{D_1 r}{|x-x_0|^{n+1}} \qquad \text{when} \qquad |x-x_0| > 4r. \tag{3.13}$$

Then

$$\sup_{r>0} \int\limits_{|x-x_0|>4r} |f(x)| H_{r,x_0}(x)dx \leq D_1 \sup_{r>0} \sum_{k=0}^{\infty} \int\limits_{2^k r < |x-x_0| < 2^{k+1} r} \frac{r|f(x)|}{|x-x_0|^{n+1}} dx.$$

Hence

$$\sup_{r>0} \int\limits_{|x-x_0|>4r} |f(x)| H_{r,x_0}(x)dx \leq D_1 \sup_{r>0} \sum_{k=0}^{\infty} \frac{1}{2^{nk-1}} \frac{1}{(2^{k+1}r)^n} \int\limits_{|x-x_0|<2^{k+1}r} |f(x)|dx$$

$$\leq 2D_1 Mf(x_0) \sum_{k=0}^{\infty} \frac{1}{2^{nk}} \leq \frac{2^{n+1}}{2^n - 1} D_1 Mf(x_0). \qquad \square$$

Corollary 3.8. *Every kernel $k(x,z) \in C^1(\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\})$ with properties* (3.2)–(3.3) *has the regularity property $(\mathcal{D}_2)$.*

3.2.2. On Kolmogorov inequality. It is known that any sublinear operator $\mathbb{A}$ of weak (1,1)-type admits the Kolmogorov inequality. Namely, the following lemma is valid, see [10], p. 102.

Lemma 3.9. *Let $\mathbb{A}$ be a sublinear operator of weak $(1,1)$-type and let $E \subset \mathbb{R}^n$ be a measurable set in $\mathbb{R}^n$. Then the Kolmogorov inequality*

$$\int_E |\mathbb{A}f(x)|^s dx \leq \frac{[\nu(\mathbb{A})]^s}{1-s}|E|^{1-s}\|f\|_1^s, \qquad 0 < s < 1, \tag{3.14}$$

is valid, where $\nu(\mathbb{A})$ is the constant from the weak estimate (3.4).

To check that the constant in (3.14) is exactly $\frac{\nu^s(\mathbb{A})}{1-s}$, we reproduce the proof of this lemma from [10], p. 102, in Appendix 8.

3.2.3. Proof of Theorem 3.3. Fix the point $x = x_0$. Observe that for any real-valued function g on $\mathbb{R}^n$ and the ball $B(x_0, r)$, the following is valid

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |g(y) - g_B(x_0)|dy \leq \frac{2}{|B(x_0,r)|}\int_{B(x_0,r)} |g(y) - c|dy \tag{3.15}$$

for any constant c on the right-hand side (which may depend on x_0 and r). The proof of (3.15) is well known:

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |g(y) - f_{B(x_0)}|dy \leq \frac{1}{|B(x_0,r)|^2}\int_{B(x_0,r)}\int_{B(x_0,r)} |g(y) - g(u)|dydu$$
$$\leq \frac{1}{|B(x_0,r)|^2}\int_{B(x_0,r)}\int_{B(x_0,r)} (|g(y) - c| + |c - g(u)|)dydu$$
$$= \frac{2}{|B(x_0,r)|}\int_{B(x_0,r)} |g(y) - c|dy.$$

Hence, for any partition of $g = g_1 + g_2$ we have

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |g(y) - g_B(x_0)|dy \leq \frac{2}{|B(x_0,r)|}\int_{B(x_0,r)} |g_1(y) - c_1|dy + \frac{2}{|B(x_0,r)|}\int_{B(x_0,r)} |g_2(y) - c_2|dy \tag{3.16}$$

whatever the constants c_1 and c_2 are.

To prove estimate (3.8), we split $g = Af$ as $Af = Af_1 + Af_2$ with $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{B(x_0,4r)}$ and $f_2 = f \cdot \chi_{\mathbb{R}^1 \setminus B(x_0,4r)}$.

Then according to (3.16) we have

$$M^{\#}\left(|\mathbb{A}f|^s\right)(x) = \frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)} \left||\mathbb{A}f(y)|^s - (|\mathbb{A}f|^s)_B(x_0)\right| dy$$

$$\le \frac{2}{|B(x_0,r)|}\int\limits_{B(x_0,r)} \left||\mathbb{A}f_1(y)|^s - c_1\right| dy + \frac{2}{|B(x_0,r)|}\int\limits_{B(x_0,r)} \left||\mathbb{A}f_2(y)|^s - c_2\right| dy.$$

We choose now $c_1 = 0$ and

$$c_2 = [(|\mathbb{A}f_2|)_B(x_0)]^s = \left[\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)} |\mathbb{A}f_2(y)|\, dy\right]^s.$$

Then, taking into account that $||a|^s - |b^s|| \le |a-b|^s$ for $0 < s < 1$, we have

$$M^{\#}\left(|\mathbb{A}f|^s\right)(x_0) \le \frac{c}{|B(x_0,r)|}\int\limits_{B(x_0,r)} \left|\mathbb{A}f_1(y)\right|^s dy$$

$$+ \frac{c}{|B(x_0,r)|}\int\limits_{B(x_0,r)} \left||\mathbb{A}f_2(y)| - c_2^{\frac{1}{s}}\right|^s dy =: c(I_1 + I_2).$$

E s t i m a t i o n o f I_1. Since the operator $\mathbb{A}$ is of weak (1,1)-type, from (3.14) we obtain

$$I_1^{\frac{1}{s}} \le \frac{\nu(\mathbb{A})}{(1-s)^{\frac{1}{s}}}\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,4r)} |f_1(y)|dy \le \frac{4^n\nu(\mathbb{A})}{(1-s)^{\frac{1}{s}}} Mf(x_0). \tag{3.17}$$

E s t i m a t i o n o f I_2. By Jensen inequality and Fubini theorem after easy estimations we get

$$I_2^{\frac{1}{s}} \le \frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)} \left|(\mathbb{A}f_2)(y) - \frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)} (\mathbb{A}f_2)(\xi)d\xi\right| dy$$

$$\le \int\limits_{\mathbb{R}^1\setminus B(x_0,4r)} |f(x)| H_{r,x_0}(x)dx,$$

where $H_{r,x_0}(x)$ is the function defined in (3.11).

By Corollary 3.8, the kernel $k(x,z)$ has property $\mathcal{D}_2$. Therefore, according to (3.12), $I_2^{\frac{1}{s}} \le D_2 Mf(x_0)$, which completes the proof. □

3.3. Proof of Theorem 3.1

Let $0 < s < 1$. By (2.12) we have

$$\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} = \||\mathbb{A}f|^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n,w^s)}.$$

Then by Theorem 2.8 we have

$$\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leq C_0 \left\|\mathcal{M}^{\#}(|\mathbb{A}f|^s)\right\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n,w^s)},$$

where C_0 is the constant from (2.21), so it does not depend on the choice of operator $\mathbb{A}$.

Theorem 2.8 was applicable in this case, because $\frac{p(t)}{s}$ satisfies conditions (2.2)-(2.3) and the exponents $s\beta_k$ of the weight w^s automatically satisfy the conditions $-\frac{1}{\frac{p(x_k)}{s}} < s\beta_k < \frac{1}{\frac{p'(x_k)}{s}}$ (and similarly for the exponent β at infinity), required by Theorem 2.8. Therefore, by Theorem 3.3 we get

$$\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leq C_0 C^{\frac{1}{s}} \|(Mf)^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n,w^s)} = C_0 C^{\frac{1}{s}} \|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n,w)}.$$

It remains to apply Theorems 2.5 and 2.6 to obtain $\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leq c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)}$ where the constant c has the form $c = c(n,s,p,w)[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + \nu(\mathbb{A})]$ with $c(n,s,p,w)$ not depending on the operator $\mathbb{A}$. $\square$

4. On calculus of pseudodifferential operators on $\mathbb{R}^n$.

The goal of this section is to give some definitions and summarize (without proof) some basic facts for pseudodifferential operators. Standard references are [15], [14], [27], [44], [48], [50], [49].

We recall that $S(\mathbb{R}^n)$ is the L. Schwartz space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ with the topology defined by the semi-norms

$$|\varphi|_m = \sup_{x\in\mathbb{R}^n} (1+|x|)^m \sum_{|\alpha|\leq m} |\partial^\alpha \varphi(x)|, m \in \mathbb{N}\cup 0$$

and by $S'(\mathbb{R}^n)$ we denote the dual space of distributions.

Definition 4.1. We say that a function a belongs to the L. Hörmander class $S^m_{1,0}$, if $a \in C^\infty\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\right)$, and

$$|a|_{r,t} = \sum_{|\alpha|\leq r, |\beta|\leq t} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \langle\xi\rangle^{-m+|\alpha|} < \infty \tag{4.1}$$

for all the multi-indices α, β.

As usual, with a symbol a we associate the pseudodifferential operator defined on the space $S(\mathbb{R}^n)$ by the formula

$$Op(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi)u(y)e^{i(x-y,\xi)}dy. \tag{4.2}$$

We denote by $S^m_{1,0,0}$ the class of double symbols $a \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\xi)$ satisfying the estimates

$$|a|_{r,t,l} = \sum_{|\alpha|\leq r, |\beta|\leq t, |\gamma|\leq l} \sup_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} \left|\partial^\alpha_\xi \partial^\beta_x \partial^\gamma_y a(x,y,\xi)\right| \langle\xi\rangle^{-m} < \infty. \tag{4.3}$$

With $a \in S^m_{1,0,0}$ we associate the pseudodifferential operator with double symbol

$$Au(x) = Op_d(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x, y, \xi)u(y)e^{i(x-y,\xi)}dy, \tag{4.4}$$

and we denote the class of such operators by $OPS^m_{1,0,0}$.

By $H^s(\mathbb{R}^n)$ we denote the Sobolev space with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)},$$

where $\langle D\rangle^s = Op(\langle\xi\rangle^s), \langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

Proposition 4.2. *Let $Op(a) \in OPS^m_{1,0}$. Then:*

(i) *$Op(a)$ is bounded in the space $S(\mathbb{R}^n)$. Moreover, for every $l_1 \in \mathbb{N}\cup 0$ there exist $l_2, r, t \in \mathbb{N}\cup 0$ such that*

$$|Op(a)\varphi|_{l_1} \le C\,|a|_{r,t}\,|\varphi|_{l_2},$$

where the constant C does not depend on a.

(ii) *$Op(a)$ is bounded from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ and*

$$\|Op(a)\|_{H^s(\mathbb{R}^n)\to H^{s-m}(\mathbb{R}^n)} \le C\,|a|_{r,t},$$

where $C > 0$, $r, t \in \mathbb{N}$ do not depend on a.

Proposition 4.3. (i) *Let $A = Op(a) \in OPS^{m_1}_{1,0}(\mathbb{R}^n)$, $B = Op(b) \in OPS^{m_2}_{1,0}(\mathbb{R}^n)$. Then $AB \in OPS^{m_1+m_2}_{1,0}(\mathbb{R}^n)$ and $AB = Op(c)$, where*

$$c(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, \xi + \eta)b(x + y, \xi)e^{-i(y,\eta)}dyd\eta, \tag{4.5}$$

and

$$|c(x, \xi)|_{l_1,l_2} \le C(l_1, l_2)\,|a|_{2k_1+l_1+m_1,l_2}\,|b|_{l_1,l_2+2k_2}, \tag{4.6}$$

where $2k_1 > n + m_1, 2k_2 > n$.

(ii) *Let $A = Op_d(a) \in OPS^m_{1,0,0}(\mathbb{R}^n)$. Then $A = Op(c) \in OPS^m_{1,0}(\mathbb{R}^n)$, where*

$$c(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x, x + y, \xi + \eta)e^{-i(y,\eta)}dyd\eta, \tag{4.7}$$

and

$$|c(x, \xi)|_{l_1,l_2} \le C(l_1, l_2)\,|a|_{2k_1+l_1,l_2,l_2+2k_2},$$

where $2k_1 > n + m, 2k_2 > n$.

(iii) *Let A^t be a formal adjoint operator for $A = Op(a) \in OPS^m_{1,0}$ defined by the formula*

$$(Au, v)) = (u, A^t v), u, v \in S(\mathbb{R}^n), \left((u, v) = \int_{\mathbb{R}^n} u(x)\bar{v}(x)dx\right). \tag{4.8}$$

Then $A^t = Op(a^t) \in OPS^m_{1,0}(\mathbb{R}^n)$, and

$$a^t(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bar{a}(x + y, \xi + \eta)e^{-i(y,\eta)}dyd\eta. \tag{4.9}$$

The integrals in (4.5), (4.7), (4.9) are understood in the oscillatory sense.

Notice that formula (4.8) allows us to extend pseudodifferential operators to the space of distributions $S'(\mathbb{R}^n)$.

Proposition 4.4. (see [48], p. 241). *Let $A = Op(a) \in OPS^m_{1,0}$. Then*

$$Au(x) = \int_{\mathbb{R}^n} k_A(x,z)u(x-z)dz, u \in S(\mathbb{R}^n),$$

where

$$k_A(x,z) = F^{-1}_{\xi\to z}a(x,\xi).$$

($F^{-1}_{\xi\to z}$ is the inverse Fourier transform in the sense of distributions.)

The kernel $k_A(x,z) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n\backslash 0)$, and satisfies

$$\left|\partial_x^\beta \partial_z^\alpha k_A(x,z)\right| \leq C_{\alpha,\beta,N}(a)\,|z|^{-n-m-|\alpha|-N}, z \neq 0 \tag{4.10}$$

for all the multi-indices α, β, and all $N \geq 0$ so that $n + m + |\alpha| + N > 0$, where $C_{\alpha,\beta,N}(a)$ depends on the finite set of the seminorms $|a|^m_{r,t}$ of the symbol a.

4.1. Operators with slowly oscillating symbols

Below we set up some facts (without proof) on calculus of pseudodifferential operators with slowly oscillating symbols following [35], see also [40], Chap. 4.

Definition 4.5. A symbol a is called slowly oscillating at infinity if $a \in S^m_{1,0}$, and

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha\beta}(x)\,\langle\xi\rangle^{m-|\alpha|}, \tag{4.11}$$

where $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ for every α and $\beta \neq 0$. We denote by SO^m the class of slowly oscillating symbols, and by SO^m_0 the subclass in SO^m of symbols such that the $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ for every α and β. We use the notations $OPSO^m$, $OPSO^m_0$ for the classes of operators with symbols in SO^m, SO^m_0 respectively.

A double symbol $a \in S^m_{1,0,0}$ is called slowly oscillating if for every compact set $K \subset \mathbb{R}^n$

$$\sup_{y\in K}\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,x+y,\xi)\right| \leq C^K_{\alpha\beta\gamma}(x)\,\langle\xi\rangle^m,$$

where

$$\lim_{x\to\infty} C^K_{\alpha\beta\gamma}(x) = 0$$

for every α and $|\beta+\gamma| \neq 0$. We denote by SO^m_d the class of slowly oscillating double symbols, and by $OPSO^m_d$ the corresponding class of pseudodifferential operators.

Proposition 4.6. (i) *Let $A = Op(a) \in OPSO^{m_1}, B = Op(b) \in OPSO^{m_2}$. Then $AB \in OPSO^{m_1+m_2}$, and*

$$AB = Op(a)Op(b) + Op(t(x,\xi)),$$

where $t(x,\xi) \in SO_0^{m_1+m_2-1}$.

(ii) *Let* $A = Op_d(a) \in OPSO_d^m(\mathbb{R}^n)$*. Then*

$$A = Op(a(x,x,\xi)) + Op(t(x,\xi)),$$

where $t(x,\xi) \in SO_0^{m-1}$.

5. Boundedness of pseudodifferential operators in $H^{s,p(\cdot)}(\mathbb{R}^n)$

Theorem 5.1. *Let a variable exponent* p *satisfy conditions* (2.2)–(2.4)*. Then the operator* $A = Op(a) \left(\in OPS^0_{1,0}\right)$ *is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^n)$*, and*

$$\|A\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)} \le c(n,p)\,[\lambda_1(A) + \lambda_2(A) + \nu(A)] \tag{5.1}$$

where the constant $c(n,p)$ *depends only on* n *and the exponent* $p(x)$*. The constants* $\lambda_1(A), \lambda_2(A), \nu(A)$ *are defined by formulas* (3.2)–(3.4)*, and they depend on the finite set of the semi-norms* $|a|_{r,t}$ *of the symbol* a*.*

Proof. We have to check that the pseudodifferential operator $A = Op(a) \in OPS^0_{1,0}$ satisfy conditions (3.2)–(3.4). We obtain estimate (3.2), if in (4.10) we take $|\alpha| = 1, \beta = 0, N = 1$, and we obtain estimate (3.3) if in (4.10) we take $\alpha = 0, |\beta| = 1, N = 0$. It is well known that a pseudodifferential operator $A = Op\,(a) \in OPS^0_{1,0}$ is of weak $(1,1)$-type (see for instance [48], p. 16–23, and p. 250), hence condition (3.4) holds too.

One can check that $\lambda_1(A)$, $\lambda_2(A)$, $\nu(A)$ depend on the finite set of the constants $C_{\alpha,\beta,0}(a)$. This implies that there exist $L \in \mathbb{N}$ and a constant $\varkappa = \varkappa\left(\left\{|a|_{r,t}\right\}_{r\le L, t\le L}\right)$ such that

$$\|A\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)} \le c(n,p,w)\varkappa(\left\{|a|_{r,t}\right\}_{r\le L, t\le L}). \tag{5.2}$$

□

Theorem 5.2. *Let a variable exponent* p *satisfy conditions* (2.2)–(2.4)*. Then* $A = Op(a) \left(\in OPS^m_{1,0}\right)$ *is a bounded operator from* $H^{s,p(\cdot)}(\mathbb{R}^n)$ *to the space* $H^{s-m,p(\cdot)}(\mathbb{R}^n)$*, and*

$$\|A\|_{H^{s,p(\cdot)}(\mathbb{R}^n)\to H^{s-m,p(\cdot)}(\mathbb{R}^n)} \le c(n,p,s,m)\,[\lambda_1(A) + \lambda_2(A) + \nu(A)] \tag{5.3}$$

where the constant $c(n,p,s,m)$ *depends only on* n*, the exponent* p*, the order* m *of the operator, and the order* s *of the space. The constants* $\lambda_1(A), \lambda_2(A), \nu(A)$ *are defined by formulas* (3.2)–(3.4)*.*

Proof. By definition of the space $H^{s,p(\cdot)}(\mathbb{R}^n)$ we have

$$\|A\|_{H^{s,p(\cdot)}(\mathbb{R}^n)\to H^{s-m,p(\cdot)}(\mathbb{R}^n)} = \left\|\langle D\rangle^{s-m} A \langle D\rangle^{-s}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)}.$$

The operator $\langle D\rangle^{s-m} A \langle D\rangle^{-s} \in OPS^0_{1,0}$ and it is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. Hence $A : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is bounded and estimate (5.3) holds. □

6. Fredholmness of pseudodifferential operators in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and $H^{s,p(\cdot)}(\mathbb{R}^n)$.

6.1. Sufficient conditions of Fredholmness in $L^{p(\cdot)}(\mathbb{R}^n)$

Theorem 6.1. *Let the function p satisfy conditions* (2.2)–(2.4)*. Then an operator* $A = Op(a) \in OPSO^0$ *is a Fredholm operator in* $L^{p(\cdot)}(\mathbb{R}^n)$, *if*

$$\lim_{R\to\infty} \inf_{|x|+|\xi|\geq R} |a(x,\xi)| > 0. \tag{6.1}$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $\varphi(x,\xi) = 1$ if $|x| + |\xi| \leq 1$ and $\varphi(x,\xi) = 0$ if $|x|+|\xi| \geq 2$. We set $\varphi_R(x,\xi) = \varphi(x/R, \xi/R)$, $\psi_R = 1-\varphi_R$. Condition (6.1) implies that there exists $R > 0$ such that the symbol $b_R(x,\xi) = \psi_R(x,\xi)a^{-1}(x,\xi) \in SO^0$. Then, applying Proposition 4.6 we obtain that

$$Op(b_R)\, Op(a) = Op(\psi_R + t) = I + Op(\varphi_R + t)$$

where $\varphi_R + t \in SO_0^{-1}$.

It is well known (see, for instance, [35], p. 35–38, [40], Chap. 4) that $Op(r)$ $\left(\in OPSO_0^{-1}\right)$ is a compact operator in $L^2(\mathbb{R}^n)$. Since $L^{p(\cdot)}(\mathbb{R}^n)$ is an intermediate space between $L^2(\mathbb{R}^n)$ and $L^{q(\cdot)}(\mathbb{R}^n)$ and $Op(r)$ is a bounded operator in $L^{q(\cdot)}(\mathbb{R}^n)$ and a compact operator in $L^2(\mathbb{R}^n)$, then by Proposition 2.1 $Op(r)$ is a compact operator in $L^{p(\cdot)}(\mathbb{R}^n)$. Thus $Op(\varphi_R + t)$ is a compact operator in $L^{p(\cdot)}(\mathbb{R}^n)$, and $Op(b_R)$ is a left regularizer of $Op(a)$ in $L^{p(\cdot)}(\mathbb{R}^n)$. In the same way one can prove that $Op(b_R)$ is a right regularizer of $Op(a)$. □

6.2. Necessary conditions of the Fredholmness in $L^{p(\cdot)}(\mathbb{R}^n)$

One can check that the following two conditions:

1) there exists a constant $C > 0$ such that for every point $x \in \mathbb{R}^n$

$$\lim_{R\to\infty} \inf_{|\xi|>R} |a(x,\xi)| > C > 0, \tag{6.2}$$

2)

$$\lim_{R\to\infty} \inf_{|x|>R, \xi\in\mathbb{R}^n} |a(x,\xi)| > 0 \tag{6.3}$$

imply condition (6.1).

We will refer to condition (6.2) as a condition of uniform ellipticity of $Op(a)$, and to condition (6.3) as a condition of ellipticity at infinity.

6.2.1. Uniform ellipticity.

We will prove that the Fredholmness of $Op(a) \in OPSO^0$ implies condition (6.2).

Theorem 6.2. *Let the variable exponent satisfy conditions* (2.2)–(2.4)*, and* $Op(a)$ $\left(\in OPSO^0\right)$ *be a Fredholm operator in* $L^{p(\cdot)}(\mathbb{R}^n)$*. Then condition* (6.2) *holds.*

Proof. Fredholmness of $Op(a)$ implies a priory estimate

$$\|Op(a)\, u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C\, \|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} - \|Tu\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{6.4}$$

where $C > 0$ does not depend on u, and T is a compact operator on $L^{p(\cdot)}(\mathbb{R}^n)$. Let $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$ and $u_m(x) = e^{i(h_m,x)}u(x)$. One can see that $\|u_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$ and the sequence u_m weakly converges to 0 for $h_m \to \infty$.

Indeed, under condition (2.2) the general form of the linear functional on $L^{p(\cdot)}(\mathbb{R}^n)$ is

$$f(u) = \int_{\mathbb{R}^n} \bar{f}(x)u(x)dx,$$

where $f \in L^{p'(\cdot)}(\mathbb{R}^n)$, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, see [25], [43]. Since $S(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$, we can consider f and u in $S(\mathbb{R}^n)$. Hence applying the Parseval equality we obtain

$$f(u_m) = \int_{\mathbb{R}^n} \bar{f}(x)e^{i(h_m,x)}u(x)dx = (2\pi)^n \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{u}(\xi + h_m)d\xi \to 0$$

for $m \to \infty$.

Let $U_h u(x) = e^{i(x,h)}u(x)$. One can see that U_h is an isometric operator in $L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, if $Op(a)$ is a pseudodifferential operator, then

$$U_h^{-1} Op(a) U_h = Op(a(x, \xi + h)).$$

Hence inequality (6.4) implies that

$$\|Op(a(x, \xi + h_m))u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C - \|Tu_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} .$$

Since T is a compact operator, the sequence $\|Tu_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} \to 0$. Hence for every function $u : \|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$ there exists m_0 such that for $m > m_0$

$$\|Op\,(a(x, \xi + h_m))u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq \frac{C}{2} > 0. \tag{6.5}$$

In [35], pages 51–55, the following was proved: if $Op(a) \in OPSO^0$, then

$$\lim_{m\to\infty} \|Op\,(a(x, \xi + h_m) - a(x, h_m)\varphi)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = 0 \tag{6.6}$$

for every function $\varphi \in C_0^\infty(\mathbb{R}^n), 0 \leq \varphi(x) \leq 1$. Note that $\|\varphi I\|_{L^{q(\cdot)}\to L^{q(\cdot)}} \leq 1$, and by Theorem 5.1

$$\|Op\,(a(x, \xi + h_m) - a(x, h_m)\varphi)\|_{L^{q(\cdot)}\to L^{q(\cdot)}} \leq C$$

with $C > 0$ independent of m. Then applying Proposition 2.1 we obtain that

$$\lim_{m\to\infty} \|Op\,(a(x, \xi + h_m) - a(x, h_m)\varphi)\|_{L^{p(\cdot)}\to L^{p(\cdot)}} = 0 \tag{6.7}$$

Hence (6.5) and (6.6) implies that for $u \in C_0^\infty(\mathbb{R}^n)$ $\left(\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1\right)$ there exists m_0 such that for $m > m_0$

$$\|a(x, h_m)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq \frac{C}{4} > 0.$$

Choose a function $u \in C_0^\infty(\mathbb{R}^n) : \|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$ with support in a neighbourhood of the point $x_0 \in \mathbb{R}^n$ such that

$$\sup_{x\in suppu} |a(x, h_m) - a(x_0, h_m)| < \varepsilon$$

uniformly with respect to m. By (2.6) we obtain that for sufficiently large $m > m_0$

$$\|(a(x,h_m) - a(x_0,h_m))\, u\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \varepsilon.$$

Hence for sufficiently large $m > m_0$

$$|a(x_0,h_m)| = \|a(x_0,h_m)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq \|a(x,h_m)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} - \varepsilon = \frac{C}{4} - \varepsilon > 0$$

Hence we proved that if $Op(a)$ is a Fredholm operator in $L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a constant $C_1 > 0$ such that for every $x_0 \in \mathbb{R}^n$ and every sequence $h_m \to \infty$

$$|a(x_0,h_m)| \geq C_1 > 0 \tag{6.8}$$

for enough large m.

Let condition (6.2) does not hold. Then for arbitrary $\varepsilon > 0$ there exists an x_0 and a sequence $h_m \to \infty$ such that $\lim_{m\to\infty} |a(x_0,h_m)| < \varepsilon$. Hence we obtained contradiction with (6.8). □

6.2.2. Ellipticity at infinity. Here we will show that condition (6.3) is necessary for the Fredholmness of pseudodifferential operator acting in $L^{p(\cdot)}(\mathbb{R}^n)$.

We denote by V_h the shift operator on the vector $h \in \mathbb{R}^n$, that is, $V_h u(x) = u(x-h), x \in \mathbb{R}^n, u \in S(\mathbb{R}^n)$.

Proposition 6.3. *Let p satisfy conditions* (2.2)–(2.4)*. Let a sequence* $(\mathbb{R}^n \ni)\, h_m \to \infty$*, and $w_m\, (\in C\,(\mathbb{R}^n))$ be a sequence converging in the* sup*-norm on $\mathbb{R}^n$ to a function $w \in C(\mathbb{R}^n)$. Moreover we suppose that there exists a constant $C > 0$ such that for every $m \in \mathbb{N}$*

$$|w_m\,(x)\,| \leq \frac{C}{\langle x\rangle^n}, \quad |w(x)| \leq \frac{C}{\langle x\rangle^n}. \tag{6.9}$$

Then

$$\lim_{m\to\infty} \|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|w\|_{L^{p(\infty)}(\mathbb{R}^n)}\,. \tag{6.10}$$

Proof. Let

$$F(\lambda,m) = \int_{\mathbb{R}^n} \left|\frac{V_{h_m} w_m(x)}{\lambda}\right|^{p(x)} dx = \int_{\mathbb{R}^n} \left|\frac{w_m(x)}{\lambda}\right|^{p(x+h_m)} dx.$$

and let

$$F(\lambda,\infty) = \int_{\mathbb{R}^n} \left|\frac{w(x)}{\lambda}\right|^{p(\infty)} dx, \lambda > 0,$$

First we will prove that there exists the limit

$$\lim_{m\to\infty} F\,(\lambda,m) := F\,(\lambda,\infty) := \int_{\mathbb{R}^n} \left|\frac{w(x)}{\lambda}\right|^{p(\infty)} dx, \tag{6.11}$$

uniformly in λ on every segment $[a,b]\,, 0 < a < b < \infty$.

Let

$$F_{1,R}(\lambda, m) = \int_{|x|\geq R} \left| \frac{w_m(x)}{\lambda} \right|^{p(x+h_m)} dx,$$

$$F_{2,R}(\lambda, \infty) = \int_{|x|\geq R} \left| \frac{w(x)}{\lambda} \right|^{p(\infty)} dx.$$

Taking into account condition (6.9), by a given $\varepsilon > 0$ we can find $R_0 > 0$ such that

$$F_{1,R_0}(\lambda, m) < \varepsilon \tag{6.12}$$

uniformly in m, and

$$F_{2,R_0}(\lambda, \infty) < \varepsilon. \tag{6.13}$$

Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$

$$M_\varepsilon = \left\{ x \in \bar{B}_{R_0} : \sup_m |w_m(x)| \leq \varepsilon \right\}, M_\varepsilon^{'} = \bar{B}_{R_0} \backslash M_\varepsilon.$$

Then

$$I_1(\lambda, m) = \int_{M_\varepsilon} \left| \frac{w_m(x)}{\lambda} \right|^{p(x+h_m)} dx \leq \frac{\varepsilon^{p_-}}{a} |M_\varepsilon| \leq C\varepsilon, \tag{6.14}$$

$$I_2(\lambda) = \int_{M_\varepsilon} \left| \frac{w(x)}{\lambda} \right|^{p(\infty)} dx \leq \frac{\varepsilon}{a} |M_\varepsilon| = C\varepsilon, \tag{6.15}$$

uniformly in $\lambda \in [a, b]$.

Let

$$I_3(\lambda, m) = \int_{M_\varepsilon'} \left| \frac{w_m(x)}{\lambda} \right|^{p(x+h_m)} dx \tag{6.16}$$

It is evident that we can pass to the limit as $m \to \infty$ under the sign of the integral in (6.16). Then we obtain that uniformly in $\lambda \in [a, b]$

$$\lim_{m\to\infty} I_3(\lambda, m) = \int_{M_\varepsilon'} \left| \frac{w(x)}{\lambda} \right|^{p(\infty)} dx. \tag{6.17}$$

Taking into account (6.12), (6.13), (6.14), (6.15), and 6.17) we obtain (6.11).

Let $\mathbb{N} \cup \infty$ be a compactification of $\mathbb{N}$ by the point ∞. The topology on $\mathbb{N} \cup \infty$ is introduced such that it is discrete on $\mathbb{N}$ and the sets $U_R = \{j \in \mathbb{N} : j > R\}, R > 0$ form the fundamental system of neighborhoods of the point ∞. From (6.11) it follows that $F : \mathbb{R}_+ \times (\mathbb{N} \cup \infty) \to \mathbb{R}_+$ is a continuous function.

Further, by the definition of the norm in $L^{p(\cdot)}(\mathbb{R}^n)$

$$\|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \{\lambda > 0 : F(\lambda, m) \leq 1\}.$$

Moreover there exists a partial derivative $F'_\lambda(\lambda, m) < 0$ for every $\lambda \in (0, \infty)$ and $m \in \mathbb{N} \cup \infty$. Hence $F(\cdot, m)$ is a monotonically decreasing function on $(0, \infty)$ for every fix $m \in \mathbb{N} \cup \infty$. This implies that

$$\|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \{\lambda > 0 : F(\lambda, m) \leq 1\} = \lambda(m)$$

where $\lambda(m)$ is a unique solution of the equation $F(\lambda, m) = 1$. One can see that for $m = \infty$ the equation $F(\lambda, \infty) = 1$ has the unique solution $\lambda(\infty) = \|w\|_{L^{p(\infty)}(\mathbb{R}^n)}$. Moreover

$$F'_\lambda \left(\|w\|_{L^{p(\infty)}(\mathbb{R}^n)}, \infty \right) \neq 0.$$

Hence by the Implicit Function Theorem (see for instance [28], p. 360) we obtain that $\lambda(m)$ is a continuous function on $\mathbb{N} \cup \infty$.

Hence

$$\|w\|_{L^{p(\infty)}(\mathbb{R}^n)} = \lambda(\infty) = \lim_{m\to\infty} \lambda(m) = \lim_{m\to\infty} \|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

and we obtain equality (6.10). □

Proposition 6.4. *Let $A = Op(a) \in OPSO^0$ and a sequence $h_m \to \infty$. Then there exists a subsequence h_{m_k} and a symbol $a_h \in OPS^0_{1,0}$ such that for every function $u \in C_0^\infty(\mathbb{R}^n)$*

$$\lim_{k\to\infty} V_{-h_{m_k}} A V_{h_{m_k}} u = Op(a_h(\xi))u$$

in the topology of $S(\mathbb{R}^n)$.

Proof. Let $A = Op(a) \in OPSO^0$ and a sequence $h_m \to \infty$. Then

$$V_{-h_m} A V_{h_m} = Op(a_m), \tag{6.18}$$

where $a_m(x, \xi) = a(x + h_m, \xi)$. Following [35], p. 52-55, one can prove that for every function $u \in C_0^\infty(\mathbb{R}^n)$

$$\lim_{m\to\infty} Op(a(x + h_m, \xi) - a(h_m, \xi))u = 0$$

in the topology of $S(\mathbb{R}^n)$.

The sequence $a(h_m, \xi)$ is uniformly bounded and equi-continuous. Hence by Arcela-Ascoly Theorem there exists a subsequence $a(h_{m_k}, \xi)$ which converges to a limit function $a_h(\xi)$ uniformly on compact sets in $\mathbb{R}^n$. This implies that

$$Op\left(a(h_{m_k}, \xi)\right) u \to Op\left(a_h(\xi)\right) u$$

in the space $S(\mathbb{R}^n)$. It is easy to check that $a_h \in S^0_{1,0}$. □

Theorem 6.5. *Let $A = Op(a) \in OPSO^0$ and A be a Fredholm operator in $L^{p(\cdot)}(\mathbb{R}^n)$ where p satisfies conditions* (2.2)–(2.4). *Then*

$$\lim_{R\to\infty} \inf_{|x|>R, \xi\in\mathbb{R}^n} |a(x, \xi)| > 0. \tag{6.19}$$

Proof. Let $Op(a) : L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$ be a Fredholm operator. Then the following a priori estimate holds

$$\|Op(a)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C \|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} - \|Tu\|_{L^{p(\cdot)}(\mathbb{R}^n)}), \tag{6.20}$$

where $C > 0$ and T is a compact operator.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi(x) = 1$ for x in a neighborhood of the origin, $\varphi_R(x) = \varphi(x/R), \psi_R = 1 - \varphi_R$. One can see that for every $u \in S(\mathbb{R}^n)$

$$\lim_{R\to\infty} I_{p(\cdot)}(\psi_R u) = 0.$$

By (2.7)–(2.8) this implies that

$$\lim_{R\to\infty} \|\psi_R u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 0.$$

Hence the sequence $\psi_R I$ strongly converges in $L^{p(\cdot)}(\mathbb{R}^n)$ to 0-operator for $R \to \infty$. Since T is a compact operator

$$\lim_{R\to\infty} \|T\psi_R I\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)} = 0. \tag{6.21}$$

Formulas (6.20), (6.21) yield that there exist R_0 such that for $R > R_0$

$$\|Op(a)\psi_R u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C/2\, \|\psi_R u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{6.22}$$

for every function $u \in L^{p(\cdot)}(\mathbb{R}^n)$. Let a sequence $h_m \in \mathbb{R}^n$ tend to infinity, and a function $u \in C_0^\infty(\mathbb{R}^n)$. Then for fixed $R > 0$ there exists $m \geq m_0$ such that $\psi_R V_{h_m} u = V_{h_m} u$. Thus, for $m \geq m_0$

$$\begin{aligned} \|V_{h_m}(V_{-h_m} Op(a) V_{h_m} u)\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \|Op(a)\psi_R V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\geq C/2\, \|V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{6.23}$$

Let h_{m_k} be a subsequence of h_m defined as in Proposition 6.4 and let $w_k = V_{-h_{m_k}} Op(a) V_{h_{m_k}} u = Op\left(a(x + h_{m_k}, \xi)\right) u$. Applying Proposition 6.4 we obtain that $w_k \to w = Op(a_h)u$ in the space $S(\mathbb{R}^n)$. Hence we can use Proposition 6.3 and pass to the limit in the inequality

$$\left\| V_{h_{m_k}} w_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C/2 \left\| V_{h_{m_k}} u \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

and obtain that

$$\|Op(a_h(\xi))u\|_{L^{p(\infty)}(\mathbb{R}^n)} \geq C/2\, \|u\|_{L^{p(\infty)}(\mathbb{R}^n)}. \tag{6.24}$$

Going over to the adjoint operator we obtain that

$$\left\|(Op(a_h(\xi)))^* u\right\|_{L^{q(\infty)}(\mathbb{R}^n)} \geq C/2\, \|u\|_{L^{q(\infty)}(\mathbb{R}^n)}, \tag{6.25}$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Hence $Op(a_h(\xi)) : L^{p(\infty)}(\mathbb{R}^n) \to L^{p(\infty)}(\mathbb{R}^n)$ is an invertible operator. This implies (see for instance [45], [46], [47], [34])) that the invertibility of $Op(a_h(\xi))$ in $L^p(\mathbb{R}^n), p \in (1,\infty)$ implies the invertibility of $Op(a_h(\xi))$ in $L^2(\mathbb{R}^n)$ and hence the condition

$$\inf_\xi |a_h(\xi)| > 0. \tag{6.26}$$

Thus we proved that for every sequence $h_m \to \infty$ there exists a subsequence h_{m_k} and a limit symbol $a_h(\xi) \in S^0_{1,0}$ such that $a(h_{m_k}, \xi)$ converges to the limit function $a_h(\xi)$ for which condition (6.26) holds uniformly with respect to ξ on compact sets in $\mathbb{R}^n$.

Suppose now that condition (6.19) is not satisfied. Then there exists a sequence $(h_m, \xi_m), h_m \to \infty$ such that

$$\lim_{m\to\infty} a(h_m, \xi_m) = 0. \tag{6.27}$$

Note that ξ_m can not tend to infinity because in this case (6.27) contradicts to the above proved condition (6.2). Choose a subsequence (h_{m_k}, ξ_{m_k}) of the sequence (h_m, ξ_m) such that $a(h_{m_k}, \xi)$ converges uniformly with respect to ξ on compact sets in $\mathbb{R}^n$ to the limit function $a_h(\xi)$. Suppose that $\xi_{m_k} \to \xi_0 \in \mathbb{R}^n$. (In the contrary case we can pass to a subsequence again). Then

$$a_h(\xi_0) = \lim_{k\to\infty} a(h_{m_k}, \xi_{m_k}) = 0$$

and we obtain the contradiction with (6.26). □

6.3. Fredholmness of pseudodifferential operators in $H^{s,p(\cdot)}(\mathbb{R}^n)$

The result on Fredholmness of pseudodifferential operators in the spaces $H^{s,p(\cdot)}(\mathbb{R}^n)$ is given by the following theorem.

Theorem 6.6. *Let the variable exponent p satisfy conditions* (2.2)–(2.4). *Let $Op(a) \in OPSO^m$. Then*

$$Op(a) : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$$

is a Fredholm operator if and only if

$$\lim_{R\to\infty} \inf_{|x|+|\xi|\geq R} \left| a(x,\xi) \langle\xi\rangle^{-m} \right| > 0. \tag{6.28}$$

Proof. The operator $A : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is Fredholm if and only if the operator $B = \langle D\rangle^{s-m} Op(a) \langle D\rangle^{-s}$ is Fredholm in $L^{p(\cdot)}(\mathbb{R}^n)$. The operator $B = Op(b) \in OPSO^0$ and we can apply Theorems 6.1, 6.2, and 6.5. From Proposition 4.6 it follows that $b(x,\xi) = a(x,\xi) \langle\xi\rangle^{-m} + t(x,\xi)$, where $t \in SO_0^0$. That is, $\lim_{(x,\xi)\to\infty} t(x,\xi) = 0$. Hence the condition

$$\lim_{R\to\infty} \inf_{|x|+|\xi|\geq R} |b(x,\xi| > 0$$

is equivalent to condition (6.28). □

7. Pseudodifferential operators with analytical symbols in $H^{s,p(\cdot)}(\mathbb{R}^n)$

Let B be an open convex domain in $\mathbb{R}^n$ containing the origin. We denote by $S^m_{1,0}(B)$ a subclass of $S^m_{1,0}(\mathbb{R}^n)$ consisting of symbols $a(x,\xi)$ which have an analytic extension with respect to the variable ξ to the tube domain $\mathbb{R}^n_\xi + iB$, and such that for all $l_1, l_2 \in \mathbb{N}\cup 0$

$$|a|_{l_1,l_2,B} = \sup_{x\in\mathbb{R}^n, \xi\in\mathbf{R}^n_\xi, \eta\in B} \langle\xi\rangle^{-m+|\alpha|} \sum_{|\alpha|\leq l_1, |\beta|\leq l_2} \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi+i\eta)\right| < \infty.$$

As above, with a symbol $a \in S^m_{1,0}\,(B)$ we associate a pseudodifferential operator. The class of such pseudodifferential operators is denoted by $OPS^m_{1,0}\,(B)$.

Definition 7.1. We denote by $\mathcal{R}(B)$ a class of positive weights w such that:

1) $\log w \in C^{\infty}(\mathbb{R}^n)$, and
$$N_l(\log w) = \sup_x \sum_{|\beta|\le l} \left|\partial^{\beta}\nabla\left(\log w(x)\right)\right| < \infty$$
for all l;
2) $\nabla\left(\log w(x)\right) \in B$ for every $x \in \mathbb{R}^n$.

A weight $w(x) \in \mathcal{R}(B)$ is called slowly oscillating if:

3) $\lim_{x\to\infty} \frac{\partial\nabla(\log w(x))}{\partial x_j} = 0, j = 1, \dots, n.$

We denote the class of slowly oscillating weights by $\mathcal{R}_{sl}(B)$.

Let
$$g_w(x, y) = \int_0^1 (\nabla \log w)(x - t(x - y))dt.$$
It is easy to check that for all $l_1, l_2 \in \mathbb{N} \cup 0$
$$\sup_{x,y} \sum_{|\alpha|\le l_1, |\beta|\le l_2} \left|\partial_x^{\alpha}\partial_y^{\beta} g_w(x, y)\right| \le C \sup_{x\in\mathbb{R}^n, 1\le|\beta|\le l_1+l_2} \left|\partial^{\beta}\log w(x)\right| < \infty.$$
Moreover, condition 2) implies that $g_w(x, y) \in B$ for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

The following Proposition is a key result for the study of pseudodifferential operators in exponential weighted spaces.

Proposition 7.2. (see [40], p. 243–247). *Let*
$$A = Op(a(x, \xi)) \in OPS^m_{1,0}(\mathbb{R}^n, B);\ \ w(x) \in \mathcal{R}(B).$$
Then the operator $wOp(p)w^{-1} \in OPS^m_{1,0,0}(\mathbb{R}^n)$, *and*
$$wOp(a)w^{-1} = Op_d(a(x, \xi + ig_w(x, y)).$$

Proposition 7.3. (see [40], p. 243–247). *Let*
$$A = Op(a(x, \xi)) \in OPSO^m(B) = OPSO^m \cap OPS^m_{1,0}(B),$$
and a weight $w \in \mathcal{R}_{sl}(B)$. *Then*
$$wAw^{-1}I = Op(a(x, \xi + i\nabla \log w(x))) + Op(t(x, \xi)), \tag{7.1}$$
where $t(x, \xi) \in SO^{m-1}_0(\mathbb{R}^n)$.

By $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ we denote the weighted space with norm
$$\|u\|_{H^{s,p(\cdot)}_w(\mathbb{R}^n)} = \|wu\|_{H^{s,p(\cdot)}(\mathbb{R}^n)}.$$

pagebreak

Theorem 7.4. *Let the variable exponent satisfy conditions* (2.2)–(2.4)*,* $Op(a) \in OPS^m_{1,0}(B)$*,* $w(x) \in \mathcal{R}(B)$*. Then*

$$Op(a) : H^{s,p(\cdot)}_w(\mathbb{R}^n) \to H^{s-m,p(\cdot)}_w(\mathbb{R}^n)$$

is a bounded operator.

Proof. Proof immediately follows from Proposition 7.2 and Theorem 5.2. □

Theorem 7.5. *Let the variable exponent satisfy conditions* (2.2)–(2.4)*. Let* $Op(a) \in OPSO^m \cap OPS^m_{1,0}(B)$ *and* $w \in \mathcal{R}_{sl}(B)$*. Then*

$$Op(a) : H^{s,p(\cdot)}_w(\mathbb{R}^n) \to H^{s-m,p(\cdot)}_w(\mathbb{R}^n)$$

is a Fredholm operator if and only if

$$\lim_{R\to\infty} \inf_{|x|+|\xi>R|} a(x, \xi + i\nabla \log w(x)) \langle\xi\rangle^{-m} > 0. \tag{7.2}$$

Proof. Proof follows directly from Proposition 7.3, and Theorems 6.1–6.5. □

Theorem 7.5 has the following important corollary, in which $sp_{ess}(A : X \to X)$ stands for the essential spectrum of a bounded operator $A : X \to X$ ($\lambda \in \mathbb{C}$ is said to be a point of the *essential spectrum* of A, if $A - \lambda I$ is not Fredholm operator).

Theorem 7.6. *Let the variable exponent satisfy conditions* (2.2)–(2.4)*. Let* $Op(a) \in OPSO^0 \cap OPS^0_{1,0}(\mathbb{R}^n, B)$ *be a uniformly elliptic pseudodifferential operator at every point* $x \in \mathbb{R}^n$*,* $w \in \mathcal{R}_{sl}(B)$*. Then*

$$sp_{ess}(Op(a) : H^{s,p(\cdot)}_w(\mathbb{R}^n) \to H^{s,p(\cdot)}_w(\mathbb{R}^n)) = \bigcup_{h\in\Omega(a,w)} \overline{\{\lambda \in \mathbb{C} : \lambda = a_h(\xi + iw_h), \xi \in \mathbb{R}^n\}}$$

where $\Omega(a, w)$ *is the set of all sequences* $h_m \to \infty$ *such that the limit*

$$a_h(\xi + iw_h) = \lim_{h_m\to\infty} a(h_m, \xi + i (\nabla \log w) (h_m)) \tag{7.3}$$

is uniform on every compact set in $\mathbb{R}^n$*.*

Theorem 7.6 shows that the essential spectrum of pseudodifferential operator does not depend on s, p, but it essentially depends on the weight w. General speaking, the essential spectrum of $Op(a) \in OPSO^0 \cap OPS^0_{1,0}(\mathbb{R}^n, B)$ acting in $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ is a massive set in the complex plane $\mathbb{C}$, and massivity of this set depends on oscillations of symbol with respect to x, and oscillations of the characteristic $\nabla (\log w)$ of the weight w.

Theorem 7.7 (Phragmen-Lindelöf principle). *Let the variable exponent satisfy conditions* (2.2)–(2.4)*. Let* $Op(a) \in OPSO^m \cap OPS^m_{1,0}(B)$ *be an elliptic pseudodifferential operator at every point* $x \in \mathbb{R}^n$, $w \in \mathcal{R}_{sl}(B)$, $\lim_{x\to\infty} w(x) = \infty$, *and the domain* B *be symmetric with respect to the origin. Let*

$$\lim_{R\to\infty} \inf_{|x|>R, \xi+i\eta\in\mathbb{R}^n+iB} |a(x,\xi+i\eta)| \langle\xi\rangle^{-m} > 0. \tag{7.4}$$

Then

$$u \in H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n), Op(a)u \in H^{s-m,p(\cdot)}_w(\mathbb{R}^n) \Longrightarrow u \in H^{s,p(\cdot)}_w(\mathbb{R}^n).$$

Proof. In view of Proposition 7.3, the operator $w^\theta Op(a) w^{-\theta}, \theta \in [-1,1]$ can be written as

$$w^\theta Op(a) w^{-\theta} I = Op(a(x, \xi + i\theta\nabla \log w(x)) + Op(t_\theta(x,\xi)),$$

where $t_\theta(x,\xi)$ belongs to $SO^{m-1}_0(\mathbb{R}^n)$. By Theorem 7.5 and condition (7.4), the operator $w^\theta Op(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is a Fredholm operator for all $\theta \in [-1,1]$.

We will prove that the index of $w^\theta Op(a) w^{-\theta} I$ does not depend on the parameter θ. Applying Proposition 4.2 we prove that the mapping $[-1,1] \ni \theta \to w^\theta Op(a) w^{-\theta} I : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is continuous. Theorem 5.2 implies that the family $w^\theta Op(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is uniformly bounded. Hence in light of the Proposition 2.1 the family $w^\theta Op(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is continuous. Hence,

$$\mathit{Index}\ (w^\theta Op(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n))$$

does not depend on $\theta \in [-1,1]$. This yields that

$$\begin{aligned} \mathit{Index}(Op(a) \quad &: \quad H^{s,p(\cdot)}_w(\mathbb{R}^n) \to H^{s-m,p(\cdot)}_w(\mathbb{R}^n)) \\ &= \quad \mathit{Index}(Op(a) : H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}_{w^{-1}}(\mathbb{R}^n). \end{aligned}$$

Moreover, the conditions $\lim_{x\to\infty} w(x) = \infty$ imply that $H^{s,p(\cdot)}_w(\mathbb{R}^n) \subset H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n)$, and the last imbedding is dense.

Then (see [11], p. 308)

$$\ker Op(a) : H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}_{w^{-1}}(\mathbb{R}^n)$$

coincides with $\ker Op(a) : H^{s,p(\cdot)}_w(\mathbb{R}^n) \to H^{s-m,p(\cdot)}_w(\mathbb{R}^n)$. Moreover, if the equation $Op(a)u = f$, where $f \in H^{s-m,p(\cdot)}_w(\mathbb{R}^n)$ is solvable in $H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n)$, then $u \in H^{s,p(\cdot)}_w(\mathbb{R}^n)$. □

8. Appendix. Proof of Lemma 3.9

In view of (3.4) we have

$$\begin{aligned}\int\limits_E |\mathbb{A}f(x)|^s dx &= s\int\limits_0^\infty \lambda^{s-1}|\{x\in E : |\mathbb{A}f(x)|>\lambda\}d\lambda \\ &\le s\int\limits_0^\infty \lambda^{s-1}\min\left(|E|, \frac{\nu(\mathbb{A})}{\lambda}\|f\|_1\right)d\lambda \\ &= s\int\limits_0^{\frac{\nu(\mathbb{A})\|f\|_1}{|E|}} \lambda^{s-1}|E|d\lambda + s\int\limits_{\frac{\nu(\mathbb{A})\|f\|_1}{|E|}}^\infty \lambda^{s-2}\nu(\mathbb{A})\|f\|_1 d\lambda \\ &= (\nu(\mathbb{A})\|f\|_1)^s\,|E|^{1-s} + \frac{s}{1-s}(\nu(\mathbb{A})\|f\|_1)^s\,|E|^{1-s} \\ &= \frac{1}{1-s}|E|^{1-s}\,(\nu(\mathbb{A})\|f\|_1)^s\,.\end{aligned}$$

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