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# THE MAXIMAL OPERATOR IN WEIGHTED VARIABLE EXPONENT SPACES ON METRIC SPACES

by

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**Abstract**

We study the boundedness of the maximal operator in weighted variable exponent spaces $L^{p(\cdot)}(X,\varrho)$ on a doubling measure metric space $X$. When $X$ is bounded, the weight belongs to a version of a Muckenhoupt-type class, which is narrower than the expected Muckenhoupt condition for variable exponent, but coincides with the usual Muckenhoupt class $A_p$ in the case of constant $p$. For a bounded $X$ we also consider a class of weights of the form $\varrho(x) = \prod_{k=1}^{n} w_k(d(x,x_k))$, $x_k \in X$, where the functions $w_k(r)$ have finite upper and lower indices $m(w)$ and $M(w)$ satisfying the condition $-\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)} < m(w) \le M(w) < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_k)}$, where $\underline{\mathfrak{dim}}(X)$ is a certain version of the lower dimension of the space $X$. In the case of unbounded $X$ we admit weights of the form $w_0[1+d(x_0,x)]\prod_{k=1}^{m} w_k[d(x_k,x)]$.

Some of the results are new even in the case of constant $p$. We also deal with some new notions of upper and lower local dimensions of measure metric spaces.

*Key Words and Phrases*: maximal functions, weighted Lebesgue spaces, variable exponent spaces, metric space, doubling condition, Zygmund conditions, Bary-Stechkin class.

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# 1 Introduction

Within the last decade a significant progress has been made in the study of maximal, singular and potential operators in variable exponent Lebesgue spaces $L^{p(\cdot)}$ in the Euclidean setting, including weighted estimates. We refer in particular to the surveying articles [7], [21], [42] and papers [3], [4], [5], [6], [8], [24], [26], [27], [28], [29], [31], [36] and references therein.

In the case of constant $p$ these classical operators have been studied in a more general setting of measure metric spaces, see in particular [2], [9], [10], [11], [13], [22]. In the variable exponent setting, investigation of the classical operators of harmonic analysis on measure metric spaces started several years ago. In [23], [25], [30], [31] there were proved results on the boundedness, including certain weighted cases, of maximal, singular and potential operators on an arbitrary Carleson curve, which is a typical example of Ahlfors-regular measure metric space with constant dimension. The non-weighted boundedness of the maximal operator on a bounded doubling measure metric space was proved in [16] and [20]. We refer also to [15], where Sobolev-type theorem for potential operators on bounded metric spaces in $\mathbb{R}^n$ with variable dimension was obtained and to [35], where continuity of Sobolev functions on metric spaces in the limiting case was studied.

We obtain weighted estimates for the maximal operator on a doubling measure metric space $(X, d, \mu)$ with weights in a certain subclass of Muckenhoupt class $A_{p(\cdot)}$. We also specially consider the case of radial-type weights $w(d(x_0, x))$, which are functions of the distance $d(x_0, x)$ in $X$, and show that in this case the condition of weighted boundedness may be written in some natural terms of relations between certain (Matuszewska-Orlicz type) indices of the weight $w(r)$ and the lower dimension $\underline{\mathfrak{dim}}(X)$ of the space $X$.

The paper is organized as follows. In Section 2 we recall some facts for variable exponent Lebesgue spaces on measure metric spaces. In Section 3 we formulate the main results - Theorems A, B and C - on the weighted boundedness of the maximal operator. Theorem A gives a kind of "Muckenhoupt-looking" condition when $X$ is bounded, which we called an "ersatz" of the Muckenhoupt-expected condition. In Theorem B we deal with a bounded space $X$ and radial-type oscillating weights $w[d(x_0, x)]$ and obtain sufficient conditions for the boundedness in the form

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_0)} < m(w) \leq M(w) < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_0)}, \tag{1.1}$$

where

$$\underline{\mathfrak{dim}}(X) = \lim_{t \to 0} \frac{\ln \left( \limsup\limits_{r \to 0} \inf\limits_{x \in X} \frac{\mu B(x, rt)}{\mu B(x, r)} \right)}{\ln t},$$

and $m(w), M(w)$ are Matuszewska-Orlicz indices of the weight $w$, see Section 4 for these indices and Subsection 5.4 for the way in which the dimension $\underline{\mathfrak{dim}}(X)$ appeared as a kind of the Matuszewska-Orlicz index of the measure $\mu B(x, r)$. Finally, in Theorem C we give a version of Theorem B for the case of unbounded spaces $X$.

In Section 4 we give some results on the upper and lower Matuszewska-Orlicz-type indices of weights in the Zygmund-Bary-Stechkin class, which we need to prove Theorem B. In Section 5 we find some Zygmund-type conditions on functions $w$ under which the weights $w[d(x_0, x)]$ are Muckenhoupt weights. These conditions are given in terms of certain integral inequality imposed on the functions $\mu B(x, r)$ and $w$. In Section 6 we prove a weighted version of pointwise Diening's estimate for the maximal function, which in the non-weighted case on measure metric spaces was proved in [16] and [20]. In Sections 7, 8 and 9 we give the proof of Theorems A, B and C. In the case where $X$ is a Carleson curve on the complex plane so that $\underline{\mathfrak{dim}}(X) = \overline{\mathfrak{dim}}(X) = 1$, Theorems A, B and C were obtained in [25].

## 2 Some basics for variable exponent spaces

In the sequel $(X, d, \mu)$ is a space with quasimetric $d$ and a non-negative measure $\mu$; we refer to [13], [14], [17] for the basic notions of function spaces on metric spaces. The quasidistance $d : X \times X \to \mathbb{R}^1$ is assumed to satisfy the standard conditions: $d(x, y) = d(y, x) \geq 0,\ d(x, y) = 0 \iff x = y$, and

$$d(x, y) \leq a[d(x, z) + d(z, y)], \quad a \geq 1, \tag{2.1}$$

where $x, y, z \in X$. In the sequel we assume that the measure satisfies the doubling condition

$$\mu(B(x, 2r)) \le C_\mu \mu B(x, r). \tag{2.2}$$

The variable exponent $p(x)$ defined on $X$ is supposed to satisfy the conditions

$$1 < p_- \le p(x) \le p^+ < \infty, \quad x \in X \tag{2.3}$$

and

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{d(x,y)}}, \quad d(x,y) \le \frac{1}{2}, \quad x,\, y \in X. \tag{2.4}$$

By $L^{p(\cdot)}(X, \varrho)$, where $\varrho(x) \ge 0$, we denote the weighted Banach space of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(X,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int\limits_X \left| \frac{\varrho(x) f(x)}{\lambda} \right|^{p(x)} d\mu(x) \le 1 \right\} < \infty. \tag{2.5}$$

We write $L^{p(\cdot)}(X, 1) = L^{p(\cdot)}(X)$ and $\|f\|_{L^{p(\cdot)}(X)} = \|f\|_{p(\cdot)}$ in the case $\varrho(t) \equiv 1$. The generalized Lebesgue spaces $L^{p(\cdot)}(X)$ with variable exponent on measure metric spaces have been considered in [15], [16], [20], see also references therein. It is known that the Hölder inequality holds in the form

$$\int\limits_X |f(x) g(x)|\, d\mu(x) \le k \big\|f\big\|_{p(\cdot)} \cdot \big\|g\big\|_{p'(\cdot)} \tag{2.6}$$

with $k = \frac{1}{p_-} + \frac{1}{p'_-}$, $p'(x) = \frac{p(x)}{p(x)-1}$, $p'_- = \inf\limits_{x \in X} p'(x)$. The modular

$$I^p(f) = I^p_X(f) = \int\limits_X |f(x)|^{p(x)}\, d\mu(x)$$

and the norm $\|f\|_{p(\cdot)}$ are simultaneously greater than one and simultaneously less than 1.

We note also that the embedding

$$L^{p(\cdot)} \subseteq L^{s(\cdot)}, \quad \big\|f\big\|_{s(\cdot)} \le C \big\|f\big\|_{p(\cdot)}, \tag{2.7}$$

is valid for $1 \le s(x) \le p(x) \le p^+ < \infty$, when $\mu(X) < \infty$.

**Lemma 2.1.** *Let $p$ be any function on $X$ satisfying condition (2.4) and let $w$ be a function on* $[0, \ell]$, $\ell =$ diam $X$.
*i) Let $\ell < \infty$. If there exist exponents $a, b \in \mathbb{R}^1$ and constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 r^a \le w(r) \le c_2 r^b \quad \textit{for} \quad r \in [0, \ell], \tag{2.8}$$

*then*

$$\frac{1}{C} [w(d(x, x_0))]^{p(x_0)} \le [w(d(x, x_0))]^{p(x)} \le C [w(d(x, x_0))]^{p(x_0)}, \tag{2.9}$$

*where $C > 1$ does not depend on $x, x_0 \in X$.*
*ii) Let $\ell = \infty$. Suppose that $p(x)$ additionally satisfies the condition that there exists $\lim_{x\to\infty} p(x) =: p(\infty)$ and $|p(x) - p(\infty)| \leq \frac{A}{\ln(2+d(x,x_0))}$. If $w$ satisfies condition (2.8) for $r \leq 1$ and the condition $c_3 r^\alpha \leq w(r) \leq c_4 r^\beta$ for $r \geq 1$, then (2.9) holds for $d(x, x_0) \leq 1$, while*

$$\frac{1}{C}[w(d(x,x_0))]^{p(\infty)} \leq [w(d(x,x_0))]^{p(x)} \leq C[w(d(x,x_0))]^{p(\infty)} \quad \textit{for} \quad d(x,x_0) \geq 1. \tag{2.10}$$

Proof. Let $l < \infty$. We denote $g(x, x_0) = [w(d(x, x_0))]^{p(x)-p(x_0)}$. To show that $\frac{1}{C} \leq g(x, x_0) \leq C$, that is, $|\ln g(x, x_0)| \leq C_1,\ C_1 = \ln C$, we observe that

$$|\ln g(x,x_0)| = |p(x) - p(x_0)| \cdot |\ln w(d(x,x_0))| \leq A\ell \frac{|\ln w(d(x,x_0))|}{\ln \frac{2\ell}{d(x,x_0)}}$$

which is bounded by the condition on $w$. The case $\ell = \infty$ is similarly treated. □

# 3 Main Results

## 3.1 Background

We use the notation $\mathcal{M}^\varrho$ for

$$\mathcal{M}^\varrho f(x) = \sup_{r>0} \frac{\varrho(x)}{\mu(B(x,r))} \int\limits_{B(x,r)} \frac{|f(y)|}{\varrho(y)}\, d\mu(y) \tag{3.1}$$

where $B(x, r) = B_X(x, r) = \{y \in X : d(x, y) < r\}$, and write $\mathcal{M} = \mathcal{M}^1$ when $\varrho(t) \equiv 1$. In the case of constant $p \in (1, \infty)$ the following result is known, where $A_p = A_p(X)$ is the class of weights satisfying the Muckenhoupt condition

$$\sup_{x\in X, r>0} \left( \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |\varrho(y)|^p d\mu(y) \right) \left( \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \frac{d\mu(y)}{|\varrho(y)|^{p'}} \right)^{p-1} < \infty. \tag{3.2}$$

**Theorem 3.1.** *([2], [33]) Let $(X, d, \mu)$ be a doubling measure metric space and $\varrho \in A_p$. Then the maximal operator $\mathcal{M}$ is bounded in the space $L^p(X, \varrho)$.*

The non-weighted boundedness result obtained in [16] and [20] runs as follows.

**Theorem 3.2.** *Let $X$ be bounded and doubling and the exponent $p(x)$ satisfy assumptions (2.3)-(2.4). Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(X)$.*

The boundedness of the operator $\mathcal{M}^\varrho$ in $L^{p(\cdot)}(X, \varrho)$ was proved in the case of the power weight $\varrho(x) = |x - x_0|^\beta, x_0 \in X$ in [26], [29] for the case where $X = \Omega$ is a bounded domain in $\mathbb{R}^n$ or $X = \Gamma$ is a Carleson curve on the complex plane, and there was shown that the necessary and sufficient condition for such a boundedness is $-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}$ with $n = 1$ in the case of

Carleson curves. The case of radial weights of the form $w(|x-x_0|)$ with $x_0 \in \Omega \subset \mathbb{R}^n$ or $x_0 \in \Gamma$ was treated in [25], [24], [31], with the last condition replaced by

$$-\frac{n}{p(x_0)} < m(w) \le M(w) < \frac{n}{p'(x_0)} \tag{3.3}$$

where $m(w)$ and $M(w)$ are the so called lower and upper indices of the function $w$, see Subsection 4.2 on these index numbers.

## 3.2 Statements of the main results

For a measure metric space $(X, d, \mu)$ we prove three main results given in Theorems A, B and C stated below. The space $(X, d, \mu)$ is assumed to satisfy the conditions: 1) all the balls $B(x, r)$ are measurable, 2) the space $C(X)$ of uniformly continuous functions on $X$ is dense in $L^1(\mu)$. In most of the statements we also suppose that the measure $\mu$ is doubling. Let

$$A_{p(\cdot)}(X) \;=\; \text{Muckenhoupt class}$$

be the class of weights for which the maximal operator is bounded in the space $L^{p(\cdot)}(X, \varrho)$. In Theorem A we make use of the class $\widetilde{A}_{p(\cdot)}(X)$ of weights, which satisfy the condition

$$\sup_{x\in X, r>0} \left(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |\varrho(y)|^{p(y)} d\mu(y)\right) \left(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \frac{d\mu(y)}{|\varrho(y)|^{\frac{p(y)}{p_- -1}}}\right)^{p_- -1} < \infty. \tag{3.4}$$

This class $\widetilde{A}_{p(\cdot)}(X)$ is narrower than $A_{p(\cdot)}$, which may be seen on power weights, see Theorem B, conditions of which cover a wider range of exponents of power weights than that covered by the class $\widetilde{A}_{p(\cdot)}(X)$. However, $\widetilde{A}_{p(\cdot)}(X)$ coincides with the Muckenhoupt class $A_p$ in case $p$ is constant. Theorem A states that $\widetilde{A}_{p(\cdot)}(X) \subset A_{p(\cdot)}(X)$ under batural conditions.

**Theorem A.** *Let $X$ be a bounded doubling measure metric space, let $p(x)$ satisfy conditions (2.3), (2.4) and $\varrho$ fulfill condition (3.4). Then the operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X, \varrho)$.*

In Theorems B and C we deal with a special class of radial type weights in the Zygmund-Bary-Stechkin class of the type of almost monotonic functions, when the final statement may be formulated in terms of numerical inequalities for the Matuszewska-Orlicz indices of the weights. To this end, we arrive at the necessity to relate the properties of the weight to those of the measure $\mu B(x, r)$ as stated in (1.1). Such a result for the Euclidean case was earlier obtained in [24]. Theorem B is proved by means of Theorem A, but it is not contained in Theorem A, being more general in its range of applicability. In Theorem B we consider weights of the form

$$\varrho(x) = \prod_{k=1}^{N} w_k(d(x, x_k)), \quad x_k \in X, \tag{3.5}$$

where $w_k(r)$ may oscillate between two power functions as $r \to 0+$ (radial Zygmund-Bary-Stechkin type weights), and in Theorem C we consider similar weights of the form

$$\varrho(x) = w_0[1 + d(x_0, x)] \prod_{k=1}^{N} w_k[d(x, x_k)], \quad x_k \in X, k = 0, 1, ...N. \tag{3.6}$$

For lower and upper local dimensions at a point $x \in X$, we use an approach different from known in the fractal geometry and used in the variable exponent analysis on measure metric spaces in [15], see also [12], [16]. To this end, we apply the same Matuzewska-Orlicz indices to measures of balls to obtain local lower and upper dimensions in terms of these indices. This idea to introduce local dimensions by the following definition was borrowed from [40].

**Definition 3.3.** The numbers

$$\underline{\mathfrak{dim}}(X;x) = \sup_{r>1} \frac{\ln \left( \underline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)} \right)}{\ln\ r}, \quad \overline{\mathfrak{dim}}(X;x) = \inf_{r>1} \frac{\ln \left( \overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)} \right)}{\ln\ r} \tag{3.7}$$

will be referred to as local lower and upper dimensions of $X$ at a point $x \in X$.

Observe that the dimensions (3.7) are Matuszewska-Orlicz indices of measures $\mu B(x,r)$ of balls, see Subsection 4.2 on these indices.

The dimension $\underline{\mathfrak{dim}}(X;x)$ may be also rewritten in terms of the upper limit as well:

$$\underline{\mathfrak{dim}}(X;x) = \sup_{0<r<1} \frac{\ln \left( \overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)} \right)}{\ln\ r}. \tag{3.8}$$

Since the function $\mu_0(x,r) = \overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}$ is semimultiplicative in $r$, by properties of such functions ([32], Ch.II, Theorem 1.3) we obtain that

$$\underline{\mathfrak{dim}}(X;x) \le \overline{\mathfrak{dim}}(X;x)$$

and we may rewrite these dimensions also in the form

$$\underline{\mathfrak{dim}}(X;x) = \lim_{r\to 0} \frac{\ln \mu_0(x,r)}{\ln\ r}, \quad \overline{\mathfrak{dim}}(X;x) = \lim_{r\to \infty} \frac{\ln \mu_0(x,r)}{\ln\ r}. \tag{3.9}$$

**Lemma 3.4.** *If the measure $\mu$ is doubling, the above local dimensions are finite numbers and*

$$\overline{\mathfrak{dim}}(X;x) \le \log_2 C_\mu, \tag{3.10}$$

*where $C_\mu$ is the constant from the doubling condition* (2.2).

P r o o f. As is known, by the iteration of the doubling condition, the inequality

$$\frac{\mu B(x,R)}{\mu B(x,r)} \le C \left( \frac{R}{r} \right)^{\log_2 C_\mu}, \quad 0 < r \le R < \infty \tag{3.11}$$

may be obtained. Hence $\frac{\mu B(x,tr)}{\mu B(x,r)} \le C t^{\log_2 C_\mu}$. Then from (3.9) we obtain (3.10). □

For lower local dimensions we also introduce their "lower bound"

$$\underline{\mathfrak{dim}}(X) := \lim_{t\to 0} \frac{\ln \left( \limsup_{r\to 0} \inf_{x\in X} \frac{\mu B(x,rt)}{\mu B(x,r)} \right)}{\ln t} \le \inf_{x\in X} \underline{\mathfrak{dim}}(X;x). \tag{3.12}$$

Observe that in many cases the coincidence $\underline{\mathfrak{dim}}(X) = \inf_{x\in X} \underline{\mathfrak{dim}}(X;x)$ holds, for instance, if the meausres of balls have the form $\mu B(x,r) = a(x)[\varphi(r)]^{b(x)}$.

In case where $X$ is unbounded, we will also need similar dimensions connected in a sense with the influence of infinity. Let

$$\mu_\infty(x,r) = \overline{\lim_{h\to\infty}} \frac{\mu B(x,rh)}{\mu B(x,h)}.$$

We introduce the numbers

$$\underline{\mathfrak{dim}}_\infty(X) = \lim_{r\to 0} \frac{\ln \mu_\infty(x,r)}{\ln\, r}, \quad \overline{\mathfrak{dim}}_\infty(X) = \lim_{r\to\infty} \frac{\ln \mu_\infty(x,r)}{\ln\, r}. \tag{3.13}$$

As shown in [40], these limits do not depend on the "starting" point $x$. It is easy to see that they are non-negative. Similarly to Lemma 3.4, it may be shown that $\overline{\mathfrak{dim}}_\infty(X) \le \log_2 C_\mu < \infty$ in the case of doubling measure.

In the sequel, we always assume that the dimensions never degenerate, that is,

$$\underline{\mathfrak{dim}}(X) > 0 \quad \text{and} \quad \underline{\mathfrak{dim}}_\infty(X) > 0.$$

**Remark 3.5.** Observe that for an arbitrarily small $\varepsilon > 0$ we have

$$c_1 r^{\overline{\mathfrak{dim}}(X;x)+\varepsilon} \le \mu B(x,r) \le c_1 r^{\underline{\mathfrak{dim}}(X;x)-\varepsilon}, \quad 0 < r \le R < \infty \tag{3.14}$$

and

$$c_3 r^{\underline{\mathfrak{dim}}_\infty(X)-\varepsilon} \le \mu B(x,r) \le c_4 r^{\overline{\mathfrak{dim}}_\infty(X)+\varepsilon}, \quad 0 < r_0 \le r, \tag{3.15}$$

where $c_i, i = 1,2,3,4$, depend on $\varepsilon > 0$, but do not depend on $r$ and $x$, the bounds in (3.15) needed in the case $X$ is unbounded. Bounds in (3.14)-(3.15) follow from properties (4.15) and (4.21) given in Subsection 4.3; the fact that the $c_i$ do not depend on $x$, was proved in [40].

To formulate the next Theorems B and C, we need the following additional assumptions on the measure $\mu$:

$$\text{the measure} \quad \mu \quad \text{is non-atomic;} \tag{3.16}$$

$$\mu B(x,r) \quad \text{is continuous in} \quad r \quad \text{for every fixed} \quad x \in X; \tag{3.17}$$

$$\inf_{x\in X} \mu B(x,r) > 0 \quad \text{for every} \quad r > 0. \tag{3.18}$$

Condition (3.18) is fulfilled for every doubling measure $\mu$, if $X$ is bounded, since in this case there exists a number $d > 0$ such that $\mu B(x,r) \ge Cr^d$ with $C > 0$ not depending on $x$ and $r$. So we will have to refer to condition (3.18) only in Theorem **C** for unbounded spaces $X$.

The Zygmund-Bary-Stechkin class $\Phi^0_\gamma$ of weights and upper and lower indices of weights used in the theorem below are defined in Section 4. Various examples of functions in Zygmund-Bary-Stechkini-type classes with coinciding such indices may be found in [37], Section II; [38], Section 2.1, and with non-coinciding indices in [39].

**Theorem B**. *Let $X$ be a bounded doubling measure metric space satisfying assumptions (3.16), (3.17), let $p(x)$ fulfill conditions (2.3), (2.4) on $X$. The operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X,\varrho)$ with weight (3.5), if $r^{\frac{\mathfrak{dim}(X)}{p(x_k)}} w_k(r) \in \Phi^0_{\underline{\mathfrak{dim}}(X)}$*, or equivalently $w_k \in \widetilde{W}([0,\ell])$, $\ell =$ diam $X$, and

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)} < m(w) \le M(w) < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_k)}\ , \quad k = 1,2,...,N. \tag{3.19}$$

In the case where $X$ is a bounded open set in $\mathbb{R}^n$, Theorem B was proved in [24] for Zygmund-Bary-Stechkin type weights and in [29] for power weights.

**Theorem C**. *Let*
*i) $X$ be an unbounded measure metric space satisfying the doubling condition and assumptions* (3.16), (3.17) *and* (3.18)*;*
*ii) $p$ satisfy conditions (2.3)-(2.4) and let there exist a ball $B(x_0, R), x_0 \in X$ such that $p(x) \equiv p_\infty = const$ for $x \in X \backslash B(x_0, R)$.*
*Then the maximal operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X, w)$, with weight (3.6), if $w_k \in \widetilde{W}(\mathbb{R}^1_+)$ and*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)} < m(w_k) \leq M(w_k) < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_k)}, \; k = 1, ..., N, \tag{3.20}$$

*and*

$$-\frac{\underline{\mathfrak{dim}}_\infty(X)}{p_\infty} < \sum_{k=0}^{N} m_\infty(w_k) \leq \sum_{k=0}^{N} M_\infty(w_k) < \frac{\underline{\mathfrak{dim}}_\infty(X)}{p'_\infty} - \Delta_{p_\infty}, \tag{3.21}$$

*where* $\Delta_{p_\infty} = \frac{\overline{\mathfrak{dim}}_\infty(X) - \underline{\mathfrak{dim}}_\infty(X)}{p_\infty}$.

In particular, for the power type weight

$$\varrho(x) = (1 + d(x_0, x))^{\beta_0} \prod_{k=1}^{N} [d(x, x_k)]^{\beta_k}, \quad x_k \in X, k = 0, 1, ...N \tag{3.22}$$

conditions (3.20)-(3.21) take the form

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)} < \beta_k < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_k)}, \qquad k = 1, ..., N, \tag{3.23}$$

and

$$-\frac{\underline{\mathfrak{dim}}_\infty(X)}{p_\infty} < \sum_{k=0}^{N} \beta_k < \frac{\underline{\mathfrak{dim}}_\infty(X)}{p'_\infty} - \Delta_{p_\infty}. \tag{3.24}$$

The bounds in (3.21) turn to take a "natural" form $-\frac{\mathfrak{dim}_\infty(X)}{p_\infty}$ and $\frac{\mathfrak{dim}_\infty(X)}{p'_\infty}$ with $\Delta_{p_\infty} = 0$ when the dimensions $\underline{\mathfrak{dim}}_\infty(X)$ and $\overline{\mathfrak{dim}}_\infty(X)$ coincide with each other.

The Euclidean space version of Theorem C for variable exponents and power weights was obtained in [19].

# 4 Preliminaries on Zygmund-Bary-Stechkin classes.

In this section we follow some ideas of papers [1], [18], [37], [39], [41]. Let $0 < \ell \leq \infty$. A non-negative function $\varphi$ on $[0, \ell]$ is said to be almost increasing (a.i.) or almost decreasing (a.d.), if there exists a constant $C \geq 1$ such that $\varphi(x) \leq C\varphi(y)$ for all $x \leq y$ (or $x \geq y$, respectively). Let

$$W = \{w \in C([0, \ell]) : \; w(t) > 0 \; \text{ for } \; t > 0, \; w(t) \; \text{ is a.i.}\} \tag{4.1}$$

and

$$W_0 = \{w \in W : w(0) = 0\}. \tag{4.2}$$

Besides $W$ we also need a wider class

$$\widetilde{W}([0,\ell]) = \{\varphi : \ \exists a = a(\varphi) \in \mathbb{R}^1 \ \text{ such that } \ x^a\varphi(x) \in W([0,\ell])\}. \tag{4.3}$$

## 4.1 The Zygmund-Bary-Stechkin type classes $\Phi^\alpha_\beta = \Phi^\alpha_\beta([0,\ell])$ and $\Psi^\alpha_\beta = \Psi^\alpha_\beta([\ell,\infty]), \ 0 < \ell < \infty$.

The following class $\Phi^\alpha_\beta$ of Zygmund-Bary-Stechkin type in the case $\alpha = 0$ and $\beta = 1, 2, 3, ...$ was introduced in [1] (in [1] functions $w$ were increasing, not almost increasing).

**Definition 4.1.** The Zygmund-Bary-Stechkin type class $\Phi^\alpha_\beta = \Phi^\alpha_\beta([0,\ell]), \ -\infty < \alpha < \beta < \infty$, is defined as $\Phi^\alpha_\beta := \mathcal{Z}^\alpha \cap \mathcal{Z}_\beta$, where $\mathcal{Z}^\alpha$ is the class of functions $w \in \widetilde{W}$ satisfying the condition

$$\int_0^h \frac{w(t)}{t^{1+\alpha}} dt \le c\frac{w(h)}{h^\alpha} \tag{$\mathbb{Z}^\alpha$}$$

and $\mathcal{Z}_\beta$ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(t)}{t^{1+\beta}} d(t) \le c\frac{w(h)}{h^\beta}, \tag{$\mathbb{Z}_\beta$}$$

where $c = c(w) > 0$ does not depend on $h \in (0,\ell]$.

We will also need an analogous class of functions with a similar behaviour at infinity. Let $C_+([\ell,\infty)), \ 0 < \ell < \infty$, be the class of functions $w(t)$ on $[\ell,\infty)$, continuous and positive at every point $t \in [\ell,\infty)$ and having a finite or infinite limit $\lim\limits_{t\to\infty} w(t) =: w(\infty)$. We define

$$W([\ell,\infty)) = \{w \in C_+([\ell,\infty) : \ w(t) \ \text{ is a.i.}\} \tag{4.4}$$

and

$$\widetilde{W}([\ell,\infty)) = \{\varphi : \ \exists a = a(\varphi) \in \mathbb{R}^1 \ \text{ such that } \ x^a\varphi(x) \in W([\ell,\infty)\}. \tag{4.5}$$

**Definition 4.2.** Let $-\infty < \alpha < \beta < \infty$. We put $\Psi^\beta_\alpha := \widehat{\mathcal{Z}}^\beta \cap \widehat{\mathcal{Z}}_\alpha$, where $\widehat{\mathcal{Z}}^\beta$ is the class of functions $w \in \widetilde{W}([\ell,\infty))$ satisfying the condition

$$\int_r^\infty \left(\frac{r}{t}\right)^\beta \frac{w(t)\,dt}{t} \le cw(r), \quad r \to \infty, \tag{4.6}$$

and $\widehat{\mathcal{Z}}_\alpha$ is the class of functions $w \in W([\ell,\infty))$ satisfying the condition

$$\int_\ell^r \left(\frac{r}{t}\right)^\alpha \frac{w(t)\,dt}{t} \le cw(r), \quad r \to \infty \tag{4.7}$$

where $c = c(w) > 0$ does not depend on $r \in [\ell,\infty)$.

Observe that properties of functions in the class $\Psi^\beta_\alpha([\ell,\infty))$ are easily derived from those of functions in $\Phi^\alpha_\beta([0,\ell])$ because of the following equivalence

$$w \in \Psi^\beta_\alpha([\ell,\infty)) \iff w_* \in \Phi^{-\beta}_{-\alpha}([0,\ell^*]), \tag{4.8}$$

where $w_*(t) = w\left(\frac{1}{t}\right)$ and $\ell_* = \frac{1}{\ell}$.

## 4.2 Indices $m(w)$ and $M(w)$ of non-negative a. i. functions

The numbers

$$m(w) = \sup_{t>1} \frac{\ln\left(\liminf\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \sup_{0<t<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \lim_{t\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \tag{4.9}$$

and

$$M(w) = \sup_{t>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \lim_{t\to\infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \tag{4.10}$$

will be referred to as *the lower and upper Matuszewska-Orlicz indices* of the function $w(t)$, see [34], p. 20. We refer to [37], [39] for various properties of these indices. We have $0 \leq m(w) \leq M(w) \leq \infty$ for $w \in W$.

The indices $m(w)$ and $M(w)$ may be also well defined for functions $w(t)$ positive for $t > 0$ which do not necessarily belong to $W$, for example, for $w \in \widetilde{W}$. Observe that

$$m(w_a) = a + m(w), \quad M(m_a) = a + M(w) \quad \text{where} \quad w_a(t) := t^a w(t) \tag{4.11}$$

and

$$m(w^\lambda) = \lambda m(w), \quad M(w^\lambda) = \lambda M(w), \quad ,\lambda \geq 0 \tag{4.12}$$

for every $w \in \widetilde{W}$.

The indices $m_\infty(w)$ and $M_\infty(w)$ responsible for the behavior of functions $w \in \Psi_\alpha^\beta([\ell, \infty))$ at infinity are introduced in the way similar to Definition in (4.9) and (4.10):

$$m_\infty(w) = \sup_{x>1} \frac{\ln\left[\underline{\lim}_{h\to\infty} \frac{w(xh)}{w(h)}\right]}{\ln\ x}, \quad M_\infty(w) = \inf_{x>1} \frac{\ln\left[\overline{\lim}_{h\to\infty} \frac{w(xh)}{w(h)}\right]}{\ln\ x}. \tag{4.13}$$

## 4.3 Properties of functions $w \in \Phi_\beta^\alpha([0, \ell])$ in terms of the indices $m(w)$ and $M(w)$

The following statement is valid, see [37],[39] for $\alpha = 0, \beta = 1$ and [18] for the general case (observe that in [18] it was supposed that $\alpha \geq 0$, the case where $\alpha < 0$ being a consequence of the former in view of relations (4.11)).

**Theorem 4.3.** *Let $w \in \widetilde{W}([0, \ell]),\ 0 < \ell < \infty$. Then $w \in \mathcal{Z}^\alpha$ if and only if $\alpha < m(w) < \infty$, and $w \in \mathcal{Z}_\beta,\ \beta > 0$, if and only if $-\infty < M(w) < \beta$, so that*

$$w \in \Phi_\beta^\alpha \iff \alpha < m(w) \leq M(w) < \beta. \tag{4.14}$$

*Besides this, for $w \in \Phi_\beta^\alpha$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that*

$$c_1 t^{M(w)+\varepsilon} \leq w(t) \leq c_2 t^{m(w)-\varepsilon}, \qquad 0 \leq t \leq \ell. \tag{4.15}$$

*The following properties are also valid*

$$m(w) = \sup\{\mu \in \mathbb{R}^1 : \ t^{-\mu} w(t) \quad \textit{is a.i.} \quad \textit{on} \quad [0, \ell]\}, \tag{4.16}$$

$$M(w) = \inf\{\nu \in \mathbb{R}^1 : \ t^{-\nu} w(t) \quad \textit{is a.d.} \quad \textit{on} \quad [0, \ell]\}. \tag{4.17}$$

**Lemma 4.4.** *Let* $w \in \widetilde{W}$ *and* $-\infty < m(w) \leq M(w) < \infty$. *Then* $\frac{1}{w} \in \widetilde{W}$ *and*

$$m\left(\frac{1}{w}\right) = -M(w), \qquad M\left(\frac{1}{w}\right) = -m(w). \tag{4.18}$$

Proof. By (4.11) and (4.17) the function $\frac{w(t)}{t^\lambda}$ is a.d. under the choice $\lambda > M(w)$. Then $\frac{t^\lambda}{w(t)}$ is a.i. so that $\frac{1}{w} \in \widetilde{W}$. Relation (4.18) follows directly from the definition of the indices in (4.9)-(4.10). □

Note also that

$$m(uv) \geq m(u) + m(v), \qquad M(uv) \leq M(u) + M(v) \tag{4.19}$$

for $u, v \in \widetilde{W}$.

**Remark 4.5.** If $w \in \widetilde{W}$ and $m(w) > 0$, then $w \in W$.

Indeed, let $a \in \mathbb{R}^1$ be such that $w_a(t) = t^a w(t) \in W$. Then according to (4.16) the function $\frac{w_a(t)}{t^{m(w_a)-\varepsilon}}$ is a.i. for every $\varepsilon > 0$. But $m(w_a) = m(w) + a$, so that $\frac{w(t)}{t^{m(w)-\varepsilon}}$ is a.i. for every $\varepsilon > 0$. Since $m(w) > 0$, then the function $w$ itself is a.i., which means that it is in $W$.

In the sequel we extend functions $w \in \widetilde{W}$ for $x > \ell$ as $w(x) \equiv w(\ell)$ whenever necessary.

One can easily reformulate properties of functions of the class $\Phi_\gamma^\beta$ near the origin, given in Theorem 4.3 and Lemma 4.4 for the case of the corresponding behavior at infinity of functions of the class $\Psi_\alpha^\beta$. This reformulation is an easy task because of the relation (4.8). Observe in particular that for $w \in C_+([\ell, \infty))$ one has $w_*(t) := \ w\left(\frac{1}{t}\right) \in C_+\left(\left[0, \frac{1}{\ell}\right]\right)$ and the direct calculation shows that

$$m_\infty(w) = -M(w_*), \quad M_\infty(w) = -m(w_*), \quad w_*(t) := \ w\left(\frac{1}{t}\right) \tag{4.20}$$

and the corresponding analogues of properties (4.15), (4.16) and (4.17) for functions in $\Psi_\alpha^\beta([\ell, \infty))$ take the form

$$c_1 t^{m_\infty(w)-\varepsilon} \leq w(t) \leq c_2 t^{M_\infty(w)+\varepsilon}, \qquad t \geq \ell, \quad w \in \Psi_\alpha^\beta([\ell, \infty)), \tag{4.21}$$

$$m_\infty(w) = \sup\{\mu \in \mathbb{R}^1 : \ t^{-\mu} w(t) \quad \text{is a.i.} \quad \text{on} \quad [\ell, \infty)\}, \tag{4.22}$$

$$M_\infty(w) = \inf\{\nu \in \mathbb{R}^1 : \ t^{-\nu} w(t) \quad \text{is a.d.} \quad \text{on} \quad [\ell, \infty)\}. \tag{4.23}$$

**Remark 4.6.** Observe that from (4.20) it follows that properties (4.18) and (4.19) hold also for the indices $m_\infty(w)$ and $M_\infty(w)$.

Making use of Theorem 4.3 for $\Phi_\beta^\alpha([0, 1])$ and relations (4.20), we easily arrive at the following version of statement (4.14) for the case $\ell = \infty$.

**Lemma 4.7.** *Let* $w \in \widetilde{W}(\mathbb{R}^1_+)$. *Then*

$$w \in \mathcal{Z}^\alpha(\mathbb{R}^1_+) \quad \Longleftrightarrow \quad \alpha < \min\{m(w), m_\infty(w)\} < \infty \tag{4.24}$$

*and*

$$w \in \mathcal{Z}_\beta(\mathbb{R}^1_+) \quad \Longleftrightarrow \quad -\infty < \max\{M(w), M_\infty(w)\} < \beta. \tag{4.25}$$

**Remark 4.8.** Every function $w \in \widetilde{W}([0,\ell]),\ 0 < \ell \le \infty$ has the property

$$w(\lambda r) \le C\lambda^\nu w(r), \quad r > 0, \quad \lambda \ge 1, \tag{4.26}$$

if $M(w) < \infty$ in the case $\ell < \infty$ and both $M(w) < \infty$ and $M_\infty(w) < \infty$ in the case $\ell = \infty$), where $\nu > M(w)$ when $\ell < \infty$, and $\nu > \max(M_\infty(w), M(w))$ when $\ell = \infty$, and $C = C(\nu)$ does not depend on $r$ and $\lambda$, so that $w(r)$ satisfies the doubling condition. Inequality (4.26) follows from the fact that the function $\frac{w(r)}{r^\nu}$ is a.d. according to (4.17) and (4.23).

We are interested in a statement of the type of Theorem 4.3 for parameter depending functions $w(x,r)$ with the parameter $x$ running our measure metric space $X$. The main question in such a generalization is a characterization, in terms of the bounds of the Matuszewska-Orlicz indices, of the Zygmund conditions with a constant $C$ not depending on $x$. Results of such a type were proved in [40]. Theorem 4.10 below is a consequence of parts I and II of Theorems 3.1 and 3.2 in [40]. To formulate Theorem 4.10, we need the following definition which extends the classes introduced in (4.1) and (4.3).

**Definition 4.9.** By $W = W(X \times [0,\ell])$ we denote the class of functions $w$ with the properties 1) $w \in L^\infty(X \times [0,\ell])$; 2) $w(x,r)$ is continuous in $r \in [0,\ell]$ for any fixed $x \in X$; 3) $w(x,0) = 0$, but

$$\operatorname*{ess\,inf}_{x\in X} w(x,r) := d_0(r) > 0 \quad \text{for every} \quad r > 0, \tag{4.27}$$

where $\operatorname*{ess\,inf}_{x\in X} w(x,r)$ is considered with respect to the measure $\mu$ on $X$; 4) for any fixed $x \in X$ the function $w(x,r)$ is a.i. in $r$ with the uniform estimate $w(x,r_1) \le C_w w(x,r_2), \quad 0 \le r_1 \le r_2 \le \ell$. We also put

$$\widetilde{W}(X \times [0,\ell]) = \{w : \exists a = a(w) \in \mathbb{R}^1 \ \text{ such that } \ r^a w(x,r) \in W(X \times [0,\ell])\}. \tag{4.28}$$

For functions $w \in \widetilde{W}(X \times [0,\ell])$ we consider their indices with respect to the variable $r$, uniform with respect to $x$:

$$m(w) = \sup_{r>1} \frac{\ln\left(\liminf\limits_{h\to 0}\, \inf\limits_{x\in X} \frac{w(x,rh)}{w(x,h)}\right)}{\ln\ r}, \quad M(w) = \inf_{r>1} \frac{\ln\left(\limsup\limits_{h\to 0}\, \sup\limits_{x\in X} \frac{w(x,rh)}{w(x,h)}\right)}{\ln\ r}. \tag{4.29}$$

**Theorem 4.10.** *Let* $w \in \widetilde{W}(X \times [0,\ell])$ *and* $\beta, \gamma \in \mathbb{R}^1$. *The Zygmund conditions*

$$\int_0^r \frac{w(x,t)}{t^{1+\beta}} dx \le A\frac{w(x,r)}{r^\beta}, \quad \int_r^\ell \frac{w(x,t)}{t^{1+\gamma}} dt \le A\frac{w(x,r)}{r^\gamma}, \quad 0 < r \le \ell$$

*with a constant $A > 0$ not depending on $x$ and $r$ are equivalent to the numerical inequalities*

$$m(w) > \beta, \quad M(w) < \gamma,$$

*respectively.*

**Remark 4.11.** A similar reformulation of Lemma 4.7 for parameter depending functions $w(x,r)$ is also valid, being derived from Theorem 4.10 by direct arguments.

# 5 Radial-type weights $\varrho(x) = w[d(x_0, x)]$ as Muckenhoupt weights for $L^{p(\cdot}(X, \varrho)$.

## 5.1 Auxiliary lemmas

The role of the indices $m(w_k), M(w_k)$ of weights involved in (3.5) and (3.6) and of similar indices related to the measure $\mu$ (that is, dimensions $\underline{\mathfrak{dim}}_\infty(X)$ and $\overline{\mathfrak{dim}}_\infty(X)$) may be seen from the following lemmas.

**Lemma 5.1.** *Let $X$ be an unbounded doubling measure metric space and let $\alpha > \overline{\mathfrak{dim}}_\infty(X)$. Then for every $0 < \varepsilon < \alpha - \overline{\mathfrak{dim}}_\infty(X)$ there exists a constant $C = C(\varepsilon)$, not depending on $x$ and $r$ such that*

$$\int\limits_{X \setminus B(x,r)} \frac{d\mu(y)}{[d(x,y)]^\alpha} \le C(x) r^{\overline{\mathfrak{dim}}_\infty(X) - \alpha + \varepsilon}, \qquad 0 < r_0 \le r < \infty, \tag{5.1}$$

Proof. Let $r \ge 1$ and $A_k(x) = \{y \in X : 2^k r \le d(x,y) \le 2^{k+1} r\}$. We obtain

$$\int\limits_{X \setminus B(x,r)} \frac{d\mu(y)}{[d(x,y)]^\alpha} = \sum_{k=0}^{\infty} \int\limits_{A_k(x)} \frac{d\mu(y)}{[d(x,y)]^\alpha}$$

$$\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^\alpha} \int\limits_{A_k(x)} d\mu(y) \le r^{-\alpha} \sum_{k=0}^{\infty} \frac{\mu B(x, 2^{k+1} r)}{2^{k\alpha}}.$$

Hence by (3.15) we have $\int\limits_{X \setminus B(x,r)} \frac{d\mu(y)}{[d(x,y)]^\alpha} \le C r^{\overline{\mathfrak{dim}}_\infty(X) - \alpha + \varepsilon} \sum\limits_{k=0}^{\infty} \frac{1}{2^{k[\alpha - \overline{\mathfrak{dim}}_\infty(X) - \varepsilon]}}$ which proves (5.1).

□

**Lemma 5.2.** *Let $X$ be a doubling measure metric space, $\varrho$ a weight of form (3.5) and let $p(\cdot)$ satisfy conditions (2.3), (2.4). Then*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)} < m(w_k), \quad k = 1, 2, ..., N, \quad \Longrightarrow \quad \varrho \in L^{p(\cdot)}_{loc}(X). \tag{5.2}$$

$$M(w_k) < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_k)}, \quad k = 1, 2, ..., N, \quad \Longrightarrow \quad \frac{1}{\varrho} \in L^{p'(\cdot)}_{loc}(X). \tag{5.3}$$

Proof. Let $\Omega$ be an arbitrary bounded open set in $X$ containing the point $x_k$. To check (5.3), it suffices to show that

$$I_k(\Omega) := \int\limits_{\Omega} \frac{d\mu(x)}{\{w[d(x,x_k)]\}^{p'(x_k)}} < \infty, \quad k = 1, 2, ..N,$$

with Lemma 2.1 taken into account. Let $\ell =$ diam $X$. By property (4.15) we have

$$I_k(\Omega) \leq C \int\limits_{d(x,x_k)<\ell} \frac{d\mu(x)}{[d(x,x_k)]^{(M(w_k)+\varepsilon)p'(x_k)}} = \sum_{k=0}^{\infty} \int\limits_{A_k} \frac{d\mu(x)}{[d(x,x_k)]^{(M(w_k)+\varepsilon)p'(x_k)}},$$

where $A_k = \{x \in X : 2^{-k-1}\ell < d(x,x_k) < 2^{-k}\ell\}$. Hence $I_k(\Omega) \leq C \sum\limits_{k=0}^{\infty} \frac{\mu B(x,2^{-k}\ell)}{\left(2^{-k-1}\ell\right)^{(M(w_k)+\varepsilon)p'(x_k)}}$. Making use of (3.14), we obtain $I_k(\Omega) \leq C \sum\limits_{k=0}^{\infty} 2^{-k[(M(w_k)+\varepsilon)p'(x_k)-\underline{\mathfrak{dim}}(X)]}$. Since $M(w_k)p'(x_k) > \underline{\mathfrak{dim}}(X)$, the series is convergent under the choice of sufficiently small $\varepsilon > 0$.

Similarly, statement (5.2) is verified. □

## 5.2 Some joint conditions on weight and measure.

In the sequel we suppose that $w \in \widetilde{W}([0,\ell])$, $\ell =$ diam $X$ and consider the condition that $\frac{\mu B(x,r)}{w(r)} \in \mathcal{Z}_0$ uniformly in $x \in X$, that is,

$$\int\limits_0^r \frac{\mu B(x,t)\,dt}{tw(t)} \leq C\frac{\mu B(x,r)}{w(r)}, \tag{5.4}$$

where $C > 0$ does not depend on $x \in X$ and $0 < r <$ diam $X$. In Lemma 5.9 we will give a sufficient condition for the validity of (5.4) in terms of the numbers $M(w)$ and $\underline{\mathfrak{dim}}(X)$.

Observe that Lemma 5.3 does not use the doubling condition for the measure $\mu$.

**Lemma 5.3.** *Let $w \in \widetilde{W}([0,\ell])$, $\ell =$ diam $X$ and let assumption (5.4) be satisfied. Then*

$$\int\limits_{B(x,r)} \frac{d\mu(y)}{w(d(x,y))} \leq C\frac{\mu B(x,r)}{w(r)} \tag{5.5}$$

*where $C > 0$ does not depend on $x \in X$ and $r \in [0,\ell)$.*

Proof. Let $\nu < m(w)$ when $\ell < \infty$ and $\nu < \min(m(w), m_\infty(w))$ when $\ell = \infty$ and let $w_\nu(x) = \frac{w(x)}{x^\nu}$ so that $w_\nu(x)$ is an a.i. function on $[0,\ell]$ according to (4.16) and (4.22). We proceed as follows:

$$J := \int\limits_{B(x,r)} \frac{d\mu(y)}{w(d(x,y))} = \sum_{k=0}^{\infty} \int\limits_{X_k(x,r)} \frac{[d(x,y)]^{-\nu}d\mu(y)}{w_\nu(d(x,y))} \tag{5.6}$$

where $X_k(x,r) = \{y \in X : 2^{-k-1}r \le d(x,y) < 2^{-k}r\}$. Since the measure is non-negative and the function $w_\nu$ is almost increasing, and $[d(x,y)]^{-\nu} \le C\left(2^{-k}r\right)^{-\nu}$ for $y \in X_k(x,r)$, $C = 2^{\max(\nu,0)}$, we obtain

$$J \le C \sum_{k=0}^{\infty} \frac{\left(2^{-k}r\right)^{-\nu} \mu(X_k(x,r))}{w_\nu(2^{-k-1}r)} \le C \sum_{k=0}^{\infty} \frac{\mu(B(x,2^{-k}r))}{w(2^{-k-1}r)}$$

$$= C\frac{\mu B(x,r)}{w\left(\frac{r}{2}\right)} + C\sum_{k=0}^{\infty} \frac{\mu(B(x,2^{-k-1}r))}{w(2^{-k-2}r)}.$$

According to (4.26) the weight $w \in \widetilde{W}$ satisfies the doubling condition. Therefore, we get

$$J \le C\frac{\mu B(x,r)}{w(r)} + C\sum_{k=0}^{\infty} \frac{\mu(B(x,2^{-k-1}r))}{w(2^{-k}r)}. \tag{5.7}$$

The inequality

$$\frac{\mu B(x,2^{-k-1}r)}{w(2^{-k}r)} \le C \int_{2^{-k-1}}^{2^{-k}} \frac{\mu B(x,tr)\,dt}{tw(tr)} \tag{5.8}$$

with $C > 0$ not dependent on $x$ and $r$ is valid, which is obtained by the following direct estimation:

$$\int_{2^{-k-1}}^{2^{-k}} \frac{\mu B(x,tr)\,dt}{tw(tr)} \ge C\frac{\mu B(x,2^{-k-1}r)}{w_\nu(2^{-k}r)} \int_{2^{-k-1}}^{2^{-k}} (tr)^{-\nu}dt \ge C\frac{\mu B(x,2^{-k-1}r)}{w(2^{-k}r)}.$$

By (5.8) from (5.7) we then get

$$J \le C\frac{\mu B(x,r)}{w(r)} + C\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \frac{\mu B(x,tr)dt}{tw(tr)} = C\frac{\mu B(x,r)}{w(r)} + C\int_0^1 \frac{\mu B(x,tr)dt}{w(tr)}$$

which proves (5.5) in view of (5.4). □

## 5.3 Sufficient conditions for $\varrho(x) := w[d(x,x_0)]$ to belong to $\widetilde{A}_{p(\cdot)}(X)$ in terms of the Zygmund condition.

The following lemma is related to condition (5.4).

**Lemma 5.4.** *Let $w \in \widetilde{W}([0,\ell])$, $\ell =$ diam $X$ and the measure $\mu$ and the weight $w$ satisfy condition (5.4). Then a similar condition holds for the function $\frac{\mu B(x,r)}{r^\nu}$:*

$$\int_0^r \frac{\mu B(x,t)\,dt}{t^{1+\nu}} \le C\frac{\mu B(x,r)}{r^\nu}, \tag{5.9}$$

*for every $\nu < m(w)$ when $\ell < \infty$ and $\nu < \min(m(w), m_\infty(w))$ when $\ell = \infty$.*

Proof. Since the function $\frac{w(r)}{r^\nu}$ is almost increasing, we obtain

$$\int_0^r \frac{\mu B(x,t)\,dt}{t^{1+\nu}} = \int_0^r \frac{\mu B(x,t)}{tw(t)}\frac{w(t)}{t^\nu}\,dt \le C\frac{w(r)}{r^\nu}\int_0^r \frac{\mu B(x,t)}{tw(t)}\,dt \le C\frac{\mu B(x,r)}{r^\nu}$$

in view of (5.4). □

In the next lemma, $a$ is the constant from (2.1).

**Lemma 5.5.** *Let $X$ be a measure metric space with doubling condition and let $w \in \widetilde{W}([0,\ell]), \ell = \operatorname{diam} X$. If the measure $\mu$ and the weight $w$ satisfy condition (5.4), then the inequality*

$$\mathcal{M}_r^w(1) := \frac{w(d(x,x_0))}{\mu B(x,r)} \int_{B(x,r)} \frac{d\mu(y)}{w(d(y,x_0))} \le c \tag{5.10}$$

*holds with $c > 0$ not depending on $0 < r < \ell$ and $x \in X$, in each of the following cases:*
*i) $d(x,x_0) \ge 2ar$,*
*ii) $m(w) > 0$ when $\ell < \infty$, and $\min\{m(w), m_\infty(w)\} > 0$ when $\ell = \infty$.*
*In the case $d(x,x_0) \le 2ar$, the estimate*

$$\frac{w(r)}{\mu B(x,r)} \int_{B(x,r)} \frac{d\mu(y)}{w(d(y,x_0))} \le c \tag{5.11}$$

*holds, which is valid jointly with* (5.10) *when $m(w) > 0$.*

Proof.
**$1^0$. The case** $d(x,x_0) \ge 2abr$. By (2.1) we have

$$d(y,x_0) \ge \frac{1}{a}d(x_0,x) - d(y,x) \ge \frac{1}{a}d(x_0,x) - r \ge \frac{1}{2a}d(x_0,x). \tag{5.12}$$

Let $\nu < m(w)$ when $\ell < \infty$ and $\nu < \min\{m(w), m_\infty(w)\}$ when $\ell = \infty$, and $w_\nu(r) = \frac{w(r)}{r^\nu}$. Since $w_\nu$ is an a.i. function, we have $w_\nu(d(y,x_0|) \ge cw_\nu\left(\frac{1}{2a}d(x_0,x)\right)$. Taking also into account property (4.26), we obtain $w_\nu(d(y,x_0)) \ge cw_\nu(d(x_0,x))$. Then we have

$$\mathcal{M}_r^w(1) \le C\frac{w(d(x,x_0))}{w_\nu(d(x,x_0))\mu B(x,r)} \int_{B(x,r)} \frac{d\mu(y)}{[d(y,x_0)]^\nu} = C\frac{[d(x,x_0)]^\nu}{\mu B(x,r)} \int_{B(x,r)} \frac{d\mu(y)}{[d(y,x_0)]^\nu}.$$

If $\nu \ge 0$, we use (5.12) again and obtain (5.10). If $\nu < 0$, then $\frac{1}{[d(y,x_0)]^\nu} = [d(y,x_0)]^{|\nu|} \le C\left([d(y,x)]^{|\nu|} + [d(x,x_0)]^{|\nu|}\right) \le C\left(r^{|\nu|} + [d(x,x_0)]^{|\nu|}\right) \le C_1[d(x,x_0)]^{|\nu|}$, whence (5.10) again follows.

**$2^0$. The case** $d(x,x_0) \le 2ar$. Observe that in this case $B(x,r) \subset B(x_0,Qr)$ with $Q = a(1+2a)$, since $d(x,y) < r \implies d(x_0,y) \le a[d(x_0,x)+d(x,y)] < a(1+2a)r$. Hence

$$\mathcal{M}_r^w(1) \le \frac{w(d(x,x_0))}{\mu B(x,r)} \int_{B(x_0,Qr)} \frac{d\mu(y)}{w(d(y,x_0))} \le C\,\frac{w(d(x,x_0))}{\mu B(x,Qr)} \int_{B(x_0,Qr)} \frac{d\mu(y)}{w(d(y,x_0))}$$

by the doubling condition for $\mu$. Then by Lemma 5.3 we get

$$\mathcal{M}_r^w(1) \leq C\frac{w(d(x,x_0))}{w(Qr)}, \tag{5.13}$$

which gives (5.11) by (4.26). In the case $m(w) > 0$ the function $w(x)$ is a.i. and then (5.13) yields (5.10). □

Theorem 5.7 below provides the following sufficient conditions

$$\int_0^r \frac{\mu B(x,t)[w(t)]^{p(x_0)}}{t} dt \leq C\mu B(x,r)[w(r)]^{p(x_0)}, \tag{5.14}$$

$$\int_0^r \frac{\mu B(x,t)}{t[w(t)]^{q_0}} dt \leq C\frac{\mu B(x,r)}{[w(r)]^{q_0}} \quad \text{with} \quad q_0 = \frac{p(x_0)}{p_- - 1} \tag{5.15}$$

for a function $w[d(x,x_0)]$ to satisfy condition (3.4), $w \in \widetilde{W}$, where $C > 0$ does not depend on $r > 0$ and $x \in X$. To this end, from Lemma 5.5 we deduce the following corollary.

**Corollary 5.6.** *Let $w \in \widetilde{W}([0,\ell]), \ell = \operatorname{diam} X$, let $p(x)$ satisfy conditions (2.3)-(2.4), let $x_0 \in X$ and the measure $\mu$ be doubling. When $\ell < \infty$, condition (5.14) implies the inequality*

$$\frac{1}{\mu B(x,r)} \int_{B(x,r)} [w(d(x_0,y))]^{p(y)} d\mu(y) \leq C[w(\xi)]^{p(x_0)} \tag{5.16}$$

*and condition (5.15) implies the inequality*

$$\frac{1}{\mu B(x,r)} \int_{B(x,r)} \frac{d\mu(y)}{[w(d(x_0,y))]^{\frac{p(y)}{p_- -1}}} \leq \frac{C}{[w(\xi)]^{q_0}}, \tag{5.17}$$

*where $\xi = max(r, d(x,x_0))$. When $\ell = \infty$, the implications (5.14) $\rightarrow$ (5.16) and (5.15) $\rightarrow$ (5.17) are valid in case $p = const, 1 < p < \infty$.*

Proof. When $\ell < \infty$, by Lemma 2.1, the exponent $p(y)$ on the left-hand side of (5.16) and (5.17) may be replaced by $p(x_0)$ from the very beginning, the assumptions of Lemma 2.1 on $w$ being satisfied by (4.15) and (4.26). Then (5.16)-(5.17) follow directly from (5.10)-(5.11) with $w(r)$ replaced by $[w(r)]^{p(x_0)}$ in case of (5.16) and by $[w(r)]^{-q_0}$ in case of (5.17). When $\ell = \infty$ and $p = const$, we have $p(y) = p(x_0)$ and again (5.16)-(5.17) follow from (5.14)-(5.15). □

**Theorem 5.7.** *Let $w \in \widetilde{W}([0,\ell]), \ell = \operatorname{diam} X$, let $p(x)$ satisfy conditions (2.3)-(2.4), let $x_0 \in X$ and the measure $\mu$ satisfy the doubling condition. If the function $w$ and the measure $\mu$ fulfill conditions (5.14)-(5.15), then the function $\varrho(x) = w(d(x,x_0))$ satisfies condition (3.4), if either $\ell < \infty$, or $\ell = \infty$ and $p = const$.*

Proof. By Corollary 5.6, we have $\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} [w(d(x_0,y))]^{p(y)} d\mu(y) \leq C[w(\xi)]^{p(x_0)}$ and $\left( \frac{1}{\mu B(x,r)} \int\limits_{B(x,r))} \frac{d\mu(y)}{[w(d(x_0,y))]^{\frac{p(y)}{p_- -1}}} \right)^{p_- -1} \leq \frac{C}{[w(\xi)]^{p(x_0)}}$, which yields the validity of (3.4). □

**Corollary 5.8.** *Let $w \in \widetilde{W}([0,\ell]), \ell =$ diam $X$ and the measure $\mu$ satisfy the doubling condition, let $x_0 \in X$ and $p(x) = p = const$. Then under conditions (5.14)-(5.15)*

$$w[d(x,x_0)] \in A_p(X), \quad 1 < p < \infty. \tag{5.18}$$

Note that known examples of weights in $A_p(X)$ on metric spaces even for constant $p$ were powers $[\mu B(x_0, d(x_0,x))]^\alpha$ of the measure, see [13], p. 42. The statement of Corollary 5.8 giving examples of radial functions of the distance seems to be new, see also Corollary 5.12.

## 5.4 Sufficient conditions for $\varrho(x) := w[d(x,x_0)]$ to belong to $\widetilde{A}_{p(\cdot)}(X)$ in terms of indices of the weight and the lower local dimension.

**Lemma 5.9.** *Let $w \in \widetilde{W}([0,\ell]), \ell =$ diam $X$. The conditions*

$$\begin{array}{ll} M(w) < \underline{\mathfrak{dim}}(X), & \text{if } \ell < \infty \\ M(w) < \underline{\mathfrak{dim}}(X) \quad \text{and} \quad M_\infty(w) < \underline{\mathfrak{dim}}_\infty(X), & \text{if } \ell = \infty \end{array} \tag{5.19}$$

*are sufficient for the validity of (5.4).*

Proof. By $\mathcal{F}_x(r) = \frac{\mu B(x,r)}{w(r)}$ we denote for brevity the function involved in (5.4). Let

$$m(\mathcal{F}_x) = \sup_{r>1} \frac{\ln \left( \liminf\limits_{h\to 0} \frac{\mathcal{F}_x(rh)}{\mathcal{F}_x(h)} \right)}{\ln r}, \quad m_\infty(\mathcal{F}_x) = \sup_{r>1} \frac{\ln \left[ \liminf\limits_{h\to\infty} \frac{\mathcal{F}_x(rh)}{\mathcal{F}_x(r)} \right]}{\ln r}$$

be the lower index numbers (4.9) and (4.13) of this function with respect to the variable $r$. By Theorem 4.10 and Remark 4.11 we have

$$(5.4) \quad \Longleftrightarrow \quad \begin{array}{ll} 0 < \inf\limits_{x\in X} m(\mathcal{F}_x) < \infty, & \text{if } \ell < \infty \\ 0 < \min\{\inf\limits_{x\in X} m(\mathcal{F}_x), \inf\limits_{x\in X} m_\infty(\mathcal{F}_x)\} < \infty, & \text{if } \ell = \infty. \end{array} \tag{5.20}$$

Note that Theorem 4.10 is applicable because $F_x(r) \in \widetilde{W}(X \times [0,\ell])$ under assumptions (3.16), (3.17), (3.18). Sufficient conditions for (5.20) may be given in terms of the separated index numbers, namely, the upper indices $M(w), M_\infty(w)$ of the weight and the numbers $\underline{\mathfrak{dim}}(X), \underline{\mathfrak{dim}}_\infty(X)$, although it should be noted that in this way we obtain not an equivalent condition, but a sufficient one. Namely, by (4.18)-(4.19) and Remark 4.6, we obtain that the conditions

$$\underline{\mathfrak{dim}}(X;x) - M(w) > 0 \quad \text{and} \quad \underline{\mathfrak{dim}}_\infty(X) - M_\infty(w) > 0 \tag{5.21}$$

imply $\min\{\inf\limits_{x\in X} m(\mathcal{F}_x), \inf\limits_{x\in X} m_\infty(\mathcal{F}_x)\} > 0$. Then conditions (5.19) imply (5.4).

□

**Corollary 5.10.** *Condition (5.4) for the function $v(r) = r^\beta$ is satisfied, if*

$$\begin{array}{ll} \beta < \underline{\mathfrak{dim}}(X), & when \ \ell < \infty \\ \beta < \min\{\underline{\mathfrak{dim}}(X), \underline{\mathfrak{dim}}_\infty(X)\}, & when \ \ell = \infty. \end{array} \tag{5.22}$$

**Theorem 5.11.** *Let $w \in \widetilde{W}([0,\ell])$, let $p(x)$ satisfy conditions (2.3)-(2.4) and let $x_0 \in X$ .*
*I) The case* $\operatorname{diam} X < \infty$*; if*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_0)} < m(w) \le M(w) < \frac{\underline{\mathfrak{dim}}(X)}{q_0}, \tag{5.23}$$

*where $q_0 = \frac{p(x_0)}{p_- - 1}$, then the function $\varrho(x) = w(d(x,x_0))$ satisfies condition (3.4).*
*II) The case* $\operatorname{diam} X = \infty$*; if $p = const,\ 1 < p < \infty$ and*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p} < m(w) \le M(w) < \frac{\underline{\mathfrak{dim}}(X)}{p'} \tag{5.24}$$

*and*

$$-\frac{\underline{\mathfrak{dim}}_\infty(X)}{p} < m_\infty(w) \le M_\infty(w) < \frac{\underline{\mathfrak{dim}}_\infty(X)}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{5.25}$$

*then $w(d(x,x_0)) \in A_p(X)$.*

Proof. In view of Theorem 5.7, it suffices to verify that condition (5.23) or conditions (5.24)-(5.25) imply conditions (5.14)-(5.15). This is easily done by means of Lemma 5.9 applied to $w_1(r) = [w(r)]^{-p(x_0)}$ and $w_2(r) = [w(r)]^{q_0}$. Indeed, by (4.18) and (4.12) we have

$$M(w_1) = -p(x_0)m(w) \qquad \text{and} \qquad M(w_2) = q_0 M(w),$$

respectively. Then the conditions $M(w_i) < \underline{\mathfrak{dim}}(X), i = 1,2$, of Lemma 5.9 are nothing else but (5.23), and similarly conditions $M_\infty(w_i) < \underline{\mathfrak{dim}}_\infty(X), i = 1,2$, coincide with (5.24)-(5.25). □

**Corollary 5.12.** *Let $p = const, 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1$, and let $x_0 \in X$.*
*I) In the case $\ell < \infty$ the inclusion $[d(x,x_0)]^\gamma \in A_p(X)$ holds, if*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p} < \gamma < \frac{\underline{\mathfrak{dim}}(X)}{p'}; \tag{5.26}$$

*II) in the case $\ell = \infty$ the inclusion $[1 + d(x,x_0)]^\beta [d(x,x_0)]^\gamma \in A_p(X)$ holds, if*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p} < \gamma < \frac{\underline{\mathfrak{dim}}(X)}{p'} \quad and \quad -\frac{\underline{\mathfrak{dim}}_\infty(X)}{p} < \beta + \gamma < \frac{\underline{\mathfrak{dim}}_\infty(X)}{p'} / \tag{5.27}$$

Proof. I. $\ell < \infty$. By (5.26) and Theorem 5.11, condition (3.4) is satisfied for $\varrho(x) = [d(x_0,x)]^\gamma$ with $p = const$. Then $\varrho \in A_p(X)$ according to Theorem 3.1.
II. The case $\ell = \infty$ is similarly treated taking into account that for $w = w_1 w_2$ with $w_1(r) = (1+r)^\beta$ and $w_2(r) = r^\gamma$ one has $m(w) = M(w) = \gamma$ and $m_\infty(w) = M_\infty(w) = \beta + \gamma$. □

Theorem 5.11 and Corollary 5.12 contain statements new even for the case of constant $p$.

# 6 A weighted pointwise estimate of the maximal function.

The following pointwise estimate

$$|\mathcal{M}\psi(x)|^{p_1(x)} \le c\Big(1+\mathcal{M}\big[\psi^{p_1(\cdot)}\big](x)\Big) \tag{6.1}$$

valid for for all $\psi \in L^{p_1(\cdot)}(X)$ with $\|\psi\|_{p_1(\cdot)} \le C$, is due to L.Diening [5] for the Euclidean case; it was extended in [16], [20] for doubling measure metric spaces. We need a similar weighted estimate. For our purposes it will be sufficient to have it for a power weight of the form $\varrho(x) = [d(x_0,x)]^\beta$, $x_0 \in X$. For the weighted means

$$\mathcal{M}_r^\varrho f(x) = \frac{\varrho(x)}{\mu B(x,r)} \int\limits_{B_r(x)} \frac{|f(y)|}{\varrho(y)}\, dy \tag{6.2}$$

we prove the following theorem (obtained earlier for the Euclidean case in [29]).

**Theorem 6.1.** *Let $\mu(X) < \infty$, $p(x)$ satisfy (2.3) and (2.4), $x_0 \in X$ and let $\varrho(x) = [d(x_0,x)]^\beta$. If $0 \le \beta < \frac{\mathfrak{dim}(X)}{p'(x_0)}$, then*

$$\big[\mathcal{M}_r^\varrho f(x)\big]^{p(x)} \le C\left(1+\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)} |f(y)|^{p(y)}\, dy\right) \tag{6.3}$$

*for all $f \in L^{p(\cdot)}(X)$ such that $\|f\|_{p(\cdot)} \le c < \infty$, where $C = C(c,p,\beta) < \infty$ is a constant not depending on $x$, $r$ and $x_0$.*

Proof. It suffices to consider the case $c = 1$. By the condition on $\beta$ and the continuity of $p(x)$ we conclude that there exists a $\delta > 0$ such that

$$\beta p'(x) < \underline{\mathfrak{dim}}(X) \ \text{ for all } x \text{ with } \ d(x_0,x) \le \delta. \tag{6.4}$$

We may assume that $\delta \le 1$. We denote

$$p_r(x) = \min_{d(x,y)\le r} p(y)$$

and $\frac{1}{q_r(x)} = 1 - \frac{1}{p_r(x)}$. From (6.4) and the inequality $\frac{\mu B(x,r)}{2\mu(X)} \le \frac{1}{2}$ it is easily seen that

$$\beta q_r(x) < \underline{\mathfrak{dim}}(X) \ \text{ if } \ d(x_0,x) \le \frac{\delta}{2a} \ \text{ and } \ 0 < r \le \frac{\delta}{4a^2} \tag{6.5}$$

where $a$ is the constant from (2.1), and in the sequel it is convenient to distinguish the cases $0 < r \le \frac{\delta}{4a^2}$ and $\frac{\delta}{4a^2} \le r$.

**$1^0$ The case $\mathbf{d(x_0,x) \le \frac{\delta}{2a}}$ and $\mathbf{0 < r \le \frac{\delta}{4a^2}}$ (the main case).** In this case, applying the Hölder inequality with the exponents $p_r(x)$ and $q_r(x)$ to the integral on the right-hand side of the equality

$$\left|\mathcal{M}_r\Big(\frac{f(y)}{[d(x_0,y)]^\beta}\Big)\right|^{p(x)} = \frac{C}{[\mu B(x,r)]^{p(x)}}\left(\int\limits_{B(x,r)} \frac{|f(y)|}{[d(x_0,y)]^\beta}\, dy\right)^{p(x)}$$

where $\mathcal{M}_r$ stands for $\mathcal{M}_r^\varrho|_{\varrho\equiv 1}$, we get

$$\left|\mathcal{M}_r\left(\frac{f(y)}{[d(x_0,y)]^\beta}\right)\right|^{p(x)} \le$$

$$\le \frac{C}{[\mu B(x,r)]^{p(x)}}\left(\int\limits_{B(x,r)} |f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}} \cdot \left(\int\limits_{B(x,r)} \frac{dy}{[d(x_0,y)]^{\beta q_r(x)}}\right)^{\frac{p(x)}{q_r(x)}}. \tag{6.6}$$

We make use of estimate (5.10) with $w(t) = t^{\beta q_r(x)}$. This estimate is applicable by Lemma 5.5. Indeed, the condition $m(w) = \beta q_r(x) \ge 0$ is fulfilled, the case $\beta = 0$ being trivial. According to Lemma 5.9, the condition $\beta q_r(x) < \underline{\mathfrak{dim}}(X)$ provided by (6.5), is sufficient for the validity of the corresponding condition (5.4) required by Lemma 5.5. We obtain

$$\left|\mathcal{M}_r\left(\frac{f(y)}{[d(x_0,x)]^\beta}\right)\right|^{p(x)} \le C\,\frac{[d(x_0,x)]^{-\beta p(x)}}{[\mu B(x,r)]^{\frac{p(x)}{p_r(x)}}}\left(\int\limits_{B(x,r)} |f(y)|^{p_r(x)}\,dy\right)^{\frac{p(x)}{p_r(x)}}.$$

Here

$$\int\limits_{B(x,r)} |f(y)|^{p_r(x)}\,dy \le \int_{B(x,r)} dy + \int\limits_{\substack{B(x,r)\\ \{y:\ |f(y)|\ge 1\}}} |f(y)|^{p(y)}\,dy,$$

since $p_r(x) \le p(y)$ for $y \in B(x,r)$. Hence

$$\left|\mathcal{M}_r\left(\frac{f(y))}{[d(x_0,x)]^\beta}\right)\right|^{p(x)} \le C_1C_2\,\frac{[d(x_0,x)]^{-\beta p(x)}}{[\mu B(x,r)]^{\frac{p(x)}{p_r(x)}}}\left[\frac{\mu B(x,r)}{2\mu(X)} + \frac{1}{2}\int\limits_{B(x,r)} |f(y)|^{p(y)}\,dy\right]^{\frac{p(x)}{p_r(x)}},$$

where $C_2 = \{\max[2\mu(X),1]\}^{\frac{p^+}{p_-}}$. The expression in the brackets is less than or equal to 1. Since $\frac{p(x)}{p_r(x)} \ge 1$, we obtain

$$|\mathcal{M}_r^\varrho f|^{p(x)} \le \frac{C}{[\mu B(x,r)]^{\frac{p(x)}{p_r(x)}}}\left[\frac{\mu B(x,r)}{2\mu(X)} + \frac{1}{2}\int\limits_{B(x,r)} |f(y)|^{p(y)}\,dy\right] \le$$

$$\le C\,[\mu B(x,r)]^{\frac{p_r(x)-p(x)}{p_r(x)}}\left[1 + \frac{1}{\mu B(x,r)}\int\limits_{B(x,r)} |f(y)|^{p(y)}\,dy\right].$$

From here (6.3) follows, since

$$[B(x,r)]^{\frac{p_r(x)-p(x)}{p_r(x)}} \le C. \tag{6.7}$$

Indeed, $[\mu B(x,r)]^{\frac{p_r(x)-p(x)}{p_r(x)}} = e^{\frac{p(x)-p_r(x)}{p_r(x)}\ln\frac{1}{\mu B(x,r)}}$, where

$$\left|\frac{p(x)-p_r(x)}{p_r(x)}\ln\frac{1}{\mu B(x,r)}\right| \le \left|p(x) - p(\xi_r)\right|\ln\frac{1}{\mu B(x,r)}$$

with $\xi_r \in B(x,r)$, and then by (2.4),

$$\left|\frac{p(x)-p_r(x)}{p_r(x)}\ln\frac{1}{\mu B(x,r)}\right| \le A\frac{\ln\frac{1}{\mu B(x,r)}}{\ln\frac{1}{|x-\xi_r|}} \le A\frac{\ln\frac{1}{\mu B(x,r)}}{\ln\frac{1}{r}},$$

since $|x-\xi_r| \le r$. As is known, when $X$ is bounded and the measure $\mu$ is doubling, then there exist an exponent $Q>0 (Q=\log_2 C_\mu)$ and a constant $c_0>0$ such that

$$\mu B(x,r) \ge c_0 r^Q, \tag{6.8}$$

which follows from (3.11). According to the bounds in (3.14), (3.15), one may also choose $Q$ as a number greater than $\sup\limits_{x\in X}\overline{\mathfrak{dim}}(X;x)$. Then we easily obtain that $\ln\frac{1}{\mu B(x,r)} \le c_1\ln\frac{1}{r}$, $\quad c_1 = Q+\frac{|\ln c_0|}{\ln\frac{4}{\delta_0}}$. Hence $\left|\frac{p(x)-p_r(x)}{p_r}\ln\frac{1}{\mu B(x,r)}\right| \le Ac_1$ which proves (6.7) and yields estimate (6.3).

**$2^0$ The case $\mathbf{d(x_0,x)\ge\frac{\delta}{2a},\ 0<r\le\frac{\delta}{4a^2}}$.** This case is trivial, because $d(x_0,y) \ge \frac{1}{a}d(x_0,x)-d(x,y) \ge \frac{\delta}{2a^2}-\frac{\delta}{4a^2} = \frac{\delta}{4a^2}$. Thus $[d(x_0,y)]^\beta \ge \left(\frac{\delta}{4a^2}\right)^\beta$. Since $[d(x_0,x)]^\beta \le (\operatorname{diam} X)^\beta$, it follows that $\mathcal{M}_r^\varrho f(x) \le c\mathcal{M}_r f(x)$, and one may proceed as above for the case $\beta=0$.

**$3^0$ The case $\mathbf{r\ge\frac{\delta}{4a^2}}$.** This case is also easy. It suffices to show that the left-hand side of (6.3) is bounded. We have

$$\mathcal{M}_r^\varrho f(x) \le \frac{C(\operatorname{diam} X)^\beta}{\mu B\left(x,\frac{\delta}{4a^2}\right)}\left\{\int\limits_{d(x_0,y)\le\frac{\delta}{8a^2}}\frac{|f(y)|}{[d(x_0,y)]^\beta}\,dy+\int\limits_{d(x_0,y)\ge\frac{\delta}{8a^2}}\frac{|f(y)|}{[d(x_0,y)]^\beta}\,dy\right\}.$$

Here $\frac{(\operatorname{diam} X)^\beta}{\mu B\left(x,\frac{\delta}{4a^2}\right)} \le C$ by (6.8). The first integral is bounded by Hölder inequality with the exponents $p_\delta = \min_{d(x_0,y)\le\frac{\delta}{8a^2}} p(y)$ and $q_\delta = p'_\delta$ which is possible because from (6.4) we have $\beta q_\delta < \underline{\mathfrak{dim}}(X)$ which, by Lemmas 5.3 and 5.9, guarantees the convergence of the integral $\int_{d(x_0,y)\le\frac{\delta}{8a^2}}\frac{d\mu(y)}{[d(x_0,y)]^{\beta q_\delta}}$. The estimate of the second integral is trivial since $d(x_0,y)\ge\frac{\delta}{8a^2}$. □

**Corollary 6.2.** *Let* $\varrho(r)=r^\beta$ *and* $0\le\beta<\frac{\underline{\mathfrak{dim}}(X)}{p'(x_0)}$. *Under conditions* (2.3) *and* (2.4) $|\mathcal{M}^\varrho f(x)|^{p(x)} \le C\left(1+\mathcal{M}\left[|f(\cdot)|^{p(\cdot)}\right](x)\right)$ *for all* $f\in L^{p(\cdot)}(X)$ *such that* $\|f\|_{p(\cdot)}\le 1$.

# 7 Proof of Theorem A.

Let $\|f\|_{p(\cdot)}\le 1$. We follow the known trick ([5]) and represent $I^p(\mathcal{M}^\varrho f)$ as

$$I^p(\mathcal{M}^\varrho f) = \int\limits_X\left([\varrho(x)]^{p_1(x)}\left|\mathcal{M}\left(\frac{f(y)}{\varrho(y)}\right)(x)\right|^{p_1(x)}\right)^{p_-}d\mu(x), \tag{7.1}$$

where $p_1(x) = \frac{p(x)}{p_-}$ . We make use of estimate (6.1) valid for for all $\psi \in L^{p_1(\cdot)}(X)$ with $\|\psi\|_{p_1(\cdot)} \le C$. We intend to choose $\psi(y) = \frac{f(y)}{\varrho(y)}$ with $f \in L^{p(\cdot)}(X)$ in (6.1). This is possible because

$$\int\limits_X \left|\frac{f(y)}{\varrho(y)}\right|^{\frac{p(y)}{p_-}} d\mu(y) \le C, \tag{7.2}$$

for all $f \in L^{p(\cdot)}$ with $\|f\|_p \le 1$. Estimate (7.2) is obtained by means of the Hölder inequality with the exponents $p_-$ and $p'_- = \frac{p_-}{p_- - 1}$, taking into account that $\int\limits_X [\varrho(y)]^{-\frac{p(y)}{p_- - 1}} d\mu(y) < \infty$, the latter following from condition (3.4). In view of (7.2), we may apply estimate (6.1). Then (7.1) implies

$$I^p(\mathcal{M}^\varrho f) \le c \int\limits_X [\varrho(x)]^{p(x)} \left[1 + \mathcal{M}\left(\left|\frac{f(y)}{\varrho(y)}\right|^{p_1(y)}\right)\right]^{p_-} d\mu(x).$$

Since $\int\limits_X [\varrho(x)]^{p(x)} d\mu(x) < \infty$ by (3.4), we obtain

$$I^p(\mathcal{M}^\varrho f) \le c + c\int_X \left[\mathcal{M}^{\varrho_1}(|f(\cdot)|^{p_1(\cdot)})(x)\right]^{p_-} d\mu(x) \tag{7.3}$$

under notation (3.1) with $\varrho_1(x) = [\varrho(x)]^{p_1(x)}$. By Theorem 3.1, the weighted $L^p$-boundedness, $p = p_- = const$, of the maximal operator is valid if $\varrho_1(x) = [\varrho(x)]^{p_1(x)} \in A_{p_-}$, that is,

$$\sup_{x\in X, r>0} \left(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |\varrho_1(y)|^{p_-} d\mu(y)\right)^{\frac{1}{p}} \left(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \frac{d\mu(y)}{|\varrho_1(y)|^{p'_-}}\right)^{\frac{1}{p'_-}} < \infty. \tag{7.4}$$

It remains to note that (7.4) is nothing else but condition (3.4).

Therefore, by the boundedness of the weighted operator $\mathcal{M}^{\varrho_1}$ in $L_{p_-}$, from (7.3) we get

$$I^p(\mathcal{M}^\varrho f) \le c + c\int\limits_X |f(y)|^{p_1(y)\cdot p_-} d\mu(y) = c + c\int\limits_X |f(y)|^{p(y)} d\mu(y) \le c < \infty. \tag{7.5}$$

# 8 Proof of Theorem B

It suffices to prove Theorem B for a single weight $w(d(x, x_0))$, $x_0 \in X$, $r^{\frac{1}{p(x_0)}} w(r) \in \Phi^0_1$, the reduction to the case of a single weight being made by standard arguments, we refer for instance to [24], Subsection 5.1, where the Euclidean case was considered.

## 8.1 Proof of Theorem B for the case of power weights $\varrho(x) = [d(x_0, x)]^\beta$.

Let $\varrho(x) = [d(x_0, x)]^\beta$. We assume that $\|f\|_{p(\cdot)} \le 1$ and have to show that $I^p(\mathcal{M}^\varrho f) \le C < \infty$ when

$$-\frac{\mathfrak{dim}(X)}{p(x_0)} < \beta < \frac{\mathfrak{dim}(X)}{p'(x_0)}. \tag{8.1}$$

We start from the representation in (7.1), which yields

$$I^p(\mathcal{M}^\varrho f) \le C \int\limits_X \left( [d(x_0,x)]^{\beta p_1(x_0)} \left| \mathcal{M}\left( \frac{f(y)}{[d(x_0,y)]^\beta} \right)(x) \right|^{p_1(x)} \right)^{p_-} d\mu(x), \tag{8.2}$$

where $p_1(x) = \frac{p(x)}{p_-}$ and distinguish between the cases $\beta \le 0$ and $\beta \ge 0$.

**$1^0$ The case** $-\frac{\mathfrak{dim}(X)}{p(x_0)} < \beta \le 0$. We make use of estimate (6.1) valid for all $\psi \in L^{p_1(\cdot)}(X)$ with $\|\psi\|_{p_1(\cdot)} \le C$, where $C > 0$ is any fixed constant. For $\psi(x) = \frac{f(x)}{[d(x_0,x)]^\beta}$ we have $\|\psi\|_{p_1(\cdot)} \le a_0 \|f\|_{p_1(\cdot)}, \quad a_0 = (\operatorname{diam} X)^{|\beta|}$. From imbedding (2.7) we then have $\|\psi\|_{p_1(\cdot)} \le a_0 C \|f\|_{p(\cdot)} \le a_0 C \ell$. Therefore, by (6.1) we get

$$I^p(\mathcal{M}^\varrho f) \le c \int\limits_X \left( [d(x_0,x)]^{\beta p_1(x_0)} \left[ 1 + \mathcal{M}\left( \left| \frac{f(y)}{[d(x_0,y)]^\beta} \right|^{p_1(y)} \right) \right] \right)^{p_-} d\mu(x)$$

$$\le c \int\limits_X \left\{ [d(x_0,x)]^{\beta p(x_0)} + \left( [d(x_0,x)]^{\beta p_1(x_0)} \mathcal{M}\left( \frac{f(y)|^{p_1(y)}}{[d(x_0,y)]^{\beta p_1(x_0)}} \right) \right)^{p_-} \right\} d\mu(x) \le$$

$$\le c + c \int\limits_X \left( \mathcal{M}^{\varrho_1}\left( |f(\cdot)|^{p_1(\cdot)} \right)(x) \right)^{p_-} d\mu(x),$$

where $\varrho_1(x) = [d(x_0,x)]^\gamma, \ \gamma = \beta p_1(x_0)$. By Theorem 3.1, the weighted maximal operator $\mathcal{M}^{\varrho_1}$ is bounded in $L^{p_-}$ with constant $p_-$ if $\varrho_1 \in A_{p_-}$. By Corollary 5.12 this is the case, if $-\frac{\mathfrak{dim}(X)}{p_-} < \gamma < \frac{\mathfrak{dim}(X)}{p'_-}$, which is satisfied since $-\frac{\mathfrak{dim}(X)}{p(x_0)} < \beta \le 0$. Therefore,

$$I^p(\mathcal{M}^\varrho f) \le c + c \int\limits_X |f(y)|^{p_1(y) \cdot p_-} dy = c + c \int\limits_X |f(y)|^{p(y)} dy < \infty.$$

**$2^0$ The case** $0 \le \beta < \frac{\mathfrak{dim}(X)}{p'(x_0)}$. We represent the functional $I^p(\mathcal{M}^\varrho f)$ in the form

$$I^p(\mathcal{M}^\varrho f) = \int\limits_X \left( \left| \mathcal{M}^\varrho f(x) \right|^{p_1(x)} \right)^\lambda d\mu(x) \tag{8.3}$$

with $p_1(x) = \frac{p(x)}{\lambda} > 1, \ \lambda > 1$, where $\lambda$ will be chosen in the interval $1 < \lambda < p_-$. In (8.3), we wish to use the pointwise weighted estimate

$$|\mathcal{M}^\varrho f(x)|^{p_1(x)} \le c\left[1 + \mathcal{M}(f^{p_1(\cdot)})(x)\right], \tag{8.4}$$

obtained in (6.3). This estimate is applicable according to Theorem 6.1 if $\|f\|_{p_1(\cdot)} \le c$ and $\beta < \frac{\mathfrak{dim}(X)}{[p_1(x_0)]'}$. The condition $\|f\|_{p_1(\cdot)} \le c$ is satisfied since $p_1(x) \le p(x)$, and the condition on $\beta$ is fulfilled if $\lambda < \frac{\mathfrak{dim}(X) - \beta}{\mathfrak{dim}(X)} p(x_0)$. Therefore, under the choice

$$1 < \lambda < \min\left( p_-, \frac{\mathfrak{dim}(X) - \beta}{\mathfrak{dim}(X)} p(x_0) \right)$$

we may apply (8.4) in (8.3). This yields $I^p(\mathcal{M}^\varrho f) \le c + c\int\limits_X \left|\mathcal{M}(|f|^{p_1(\cdot)})(x)\right|^\lambda d\mu(x) \le c + c\int\limits_X \left(|f(x)|^{p_1(x)}\right)^\lambda d\mu(x)$ by the boundedness of the maximal operator $\mathcal{M}$ in $L^\lambda(X)$, $\lambda > 1$. Hence $I^p(\mathcal{M}^\varrho f) \le c + c\int\limits_X |f(x)|^{p(x)}\, d\mu(x) \le c$.

## 8.2 Proof of Theorem B in the general case.

**$1^0$ The case (5.23).** This case is covered by Theorem A, because the weight $w(d(x,x_0))$ satisfies condition (3.4) by Theorem 5.7.

**$2^0$ The remaining case**. The interval in (5.23) coincides with the interval in (3.19), if $q_0 = p'(x_0)$, that is, the maximal value $q_0 = \frac{p(x_0)}{p_- -1} = \max_{x\in X}\frac{p(x_0)}{p(x)-1}$ is taken on at the point $x_0$. Therefore, to get rid of the right-hand side bound in (5.23), we may split integration over $X$ into two parts, one over a small neighborhood $B_\delta = B(x_0,\delta)$ of the point $x_0$, and another over its exterior $X\backslash B_\delta$, and to choose $\delta$ sufficiently small so that the number $\frac{p_-(B_\delta)-1}{p(x_0)}$ is arbitrarily close to $\frac{p(x_0)-1}{p(x_0)} = \frac{1}{p'(x_0)}$. To this end, under notation (3.1) with $\varrho(x) = w[d(x_0,x)]$ we put

$$\mathcal{M}^\varrho = 1_{B_\delta}\mathcal{M}^\varrho 1_{B_\delta} + 1_{B_\delta}\mathcal{M}^\varrho 1_{X\backslash B_\delta} + 1_{X\backslash B_\delta}\mathcal{M}^\varrho 1_{B_\delta} + 1_{X\backslash B_\delta}\mathcal{M}^\varrho 1_{X\backslash B_\delta} \tag{8.5}$$

$$=: \mathcal{M}_1^\varrho + \mathcal{M}_2^\varrho + \mathcal{M}_3^\varrho + \mathcal{M}_4^\varrho.$$

Since the weight is strictly positive beyond any neighborhood of the point $x_0$, we have $\mathcal{M}_4^\varrho f(x) \le C\mathcal{M}f(x)$. For $\mathcal{M}_3^\varrho$ we have

$$\mathcal{M}_3^\varrho f(x) = \sup_{r>0}\frac{1_{X\backslash B_\delta}(x)}{\mu B(x,r)}\int\limits_{B(x,r)\cap B_\delta\cap X}\frac{w(d(x_0,x))}{w(d(x_0,y))}|f(y)|\, d\mu(y).$$

Here $d(x_0,x) > \delta > d(x_0,y)$. Observe that the function $w_\varepsilon(t) = \frac{w(t)}{t^{M(w)+\varepsilon}}$ is a.d. for any $\varepsilon > 0$ according to (4.17). Therefore

$$\frac{w(d(x_0,x))}{w(d(x_0,y))} = \frac{w_\varepsilon(d(x,x_0))}{w_\varepsilon(d(x_0,y))}\cdot\frac{[d(x_0,x)]^{M(w)+\varepsilon}}{[d(x_0,y)]^{M(w)+\varepsilon}} \le C\left[\frac{d(x_0,x)}{d(x_0,y)}\right]^{M(w)+\varepsilon}.$$

Hence

$$\mathcal{M}_3^\varrho f(x) \le C\mathcal{M}_{M(w)+\varepsilon}f(x) \tag{8.6}$$

where by $\mathcal{M}_{M(w)+\varepsilon}f(x)$ we denoted the weighted maximal function with the power weight $[d(x,x_0)]^{M(w)+\varepsilon}$. Similarly $\mathcal{M}_2^w f(x) \le C\mathcal{M}_{m(w)-\varepsilon}f(x)$. Thus from (8.5) we obtain

$$\mathcal{M}^\varrho f(x) \le 1_{B_\delta}\mathcal{M}^\varrho 1_{B_\delta}f(x) + \mathcal{M}f(x) + \mathcal{M}_{M(w)+\varepsilon}f(x) + \mathcal{M}_{m(w)-\varepsilon}f(x). \tag{8.7}$$

The operator $\mathcal{M}$ is bounded by Theorem 3.2, the boundedness of the maximal operators $\mathcal{M}_{M(w)+\varepsilon}$ and $\mathcal{M}_{m(w)-\varepsilon}$ with power weights was proved in Subsection 8.1, the boundedness condition (8.1) being satisfied for $\beta = M(w)+\varepsilon$ and $\beta = m(w)-\varepsilon$ under a choice of $\varepsilon$ sufficiently small.

It remains to prove the boundedness of the first term on the right-hand side of (8.7). This is nothing else but the boundedness of the same operator $\mathcal{M}^{\varrho}$ over a "small" measure metric space $X_\delta = B_\delta = B(x_0, \delta)$ with the measure induced by that on $X$. This measure is also doubling. According to the previous case, the required boundedness on $X_\delta$ holds if

$$-\frac{\mathfrak{dim}(X)}{p(x_0)} < m(w) \le M(w) < \frac{\mathfrak{dim}(X)}{p'_\delta} \tag{8.8}$$

where $p'_\delta = \frac{p_-(X_\delta)-1}{p(x_0)}$ and $p_-(X_\delta) = \min\limits_{x\in X_\delta} p(x)$. Let us show that, given the condition $-\frac{\mathfrak{dim}(X;x)}{p(x_0)} < m(w) \le M(w) < \frac{\mathfrak{dim}(X;x)}{p'(x_0)}$, one can always choose $\delta$ sufficiently small such that (8.8) holds. Given $M(w) < \frac{\mathfrak{dim}(X)}{p'(x_0)}$, we have to choose $\delta$ so that $M(w) < \frac{\mathfrak{dim}(X)}{p'_\delta} \le \frac{\mathfrak{dim}(X)}{p'(x_0)}$. We have

$$\frac{\mathfrak{dim}(X)}{p'_\delta} = \frac{\mathfrak{dim}(X)}{p'(x_0)} - a(\delta), \qquad \text{where} \qquad a(\delta) = \frac{\mathfrak{dim}(X)}{p(x_0)}\left[p(x_0) - p_-(X_\delta)\right].$$

By the continuity of $p(x)$ we can choose $\delta$ so that $a(\delta) < \frac{\mathfrak{dim}(X)}{p'(x_0)} - M(w)$. Then $\frac{\mathfrak{dim}(X)}{p'_\delta} > M(w)$ and (8.8) is fulfilled. Then $\mathcal{M}^w$ is bounded in $L^{p(\cdot)}(B_\delta)$ which completes the proof.

# 9 Proof of Theorem C.

We have to show that $I_X^p(\varrho\mathcal{M}f) \le C < \infty$ provided that $\|\varrho f\|_{p(\cdot)} \le 1$. Let $X_R = X \cap B(x_0, R)$. We may choose $R$ large enough so that all the points $x_k, k = 1, \dots, N$, lie inside the ball $B(x_0, R)$. We split the function $f$ as $f = f\cdot 1_{B_{2R}} + f \cdot 1_{X\setminus B_{2R}} = \varphi + \psi$, so that $I_X^p(\varrho\mathcal{M}f) \le I_X^p(\varrho\mathcal{M}\varphi) + I_X^p(\varrho\mathcal{M}\psi)$. When estimating $I_X^p(\varrho\mathcal{M}\varphi)$, we distinguish the cases $x \in B_{4R}$ and $x \in X\setminus B_{4R}$.

Let first $x \in B_{4R}$. We find it convenient to introduce a notation for the maximal function with respect to the portion $B_{4R}$ of $X$, that is, $\mathcal{M}_{B_{4R}}f(x) = \sup_{r>0}\frac{1}{\mu\{B(x,r)\cap B_{4R}\}}\int_{B(x,r)\cap B_{4R}}|f(y)|d\mu(y)$, $x \in B_{4R}$. For $\mathcal{M}f(x) = \mathcal{M}_X f(x)$ we have

$$\mathcal{M}\varphi(x) \le \sup_{r>0}\frac{1}{\mu\{B(x,r)\cap B_{4R}\}}\int\limits_{B(x,r)\cap B_{4R}}|f(y)|d\mu(y) = \mathcal{M}_{B_{4R}}(x). \tag{9.1}$$

Then by (9.1) and Theorem B,

$$\int\limits_{B_{4R}}[\varrho(x)\mathcal{M}\varphi(x)]^{p(x)}d\mu(x) \le C\int\limits_{B_{4R}}[\varrho(x)\mathcal{M}_{B_{4R}}f(x)]^{p(x)}d\mu(x) \le C \tag{9.2}$$

since $\|\varrho f\|_{L^{p(\cdot)}(B_{4R})} \le \|\varrho f\|_{L^{p(\cdot)}(X)} \le 1$.

Let $x \in X\setminus B_{4R}$. If $r < \frac{2R}{a}$, where $a \ge 1$ is the constant from (2.1), then $B(x,r)\cap B_{2R} = \emptyset$ and $\mathcal{M}_r\varphi(x) = 0$. So we consider $r \ge \frac{2R}{a}$. It can be also easily seen then that whenever the set $B(x,r)\cap B_{2R}$ is non-empty, we have $d(x,x_0) \le a(2R+r) \le a(a+1)r$. Consequently

$$\mathcal{M}_r\varphi(x) = \frac{1}{\mu\{B(x,r)\}}\int\limits_{B(x,r)}|\varphi(y)|d\mu(y) \le \frac{1}{\mu B\left(x, \frac{d(x,x_0)}{a(a+1)}\right)}\int\limits_{B_{2R}}|f(y)|d\mu(y).$$

We use statement (5.3) of Lemma 5.2 and the fact that the measure is doubling and get $\mathcal{M}_r\varphi(x) \le \frac{C}{\mu B(x,d(x,x_0))}\left[1+\|\varrho f\|_{p(\cdot)}\right]$ and then in view of (3.15) we obtain

$$\mathcal{M}\varphi(x) \le \frac{C}{[d(x,x_0)]^{\underline{\mathfrak{dim}}_\infty(X)-\varepsilon}} \qquad \text{for} \qquad x \in X\backslash B_{4R} \tag{9.3}$$

with an arbitrary small $\varepsilon > 0$. Observe that for $x \in X\backslash B_{4R}$

$$\varrho(x) \sim \prod_{k=0}^{N} w_k[d(x,x_0)] \le C[d(x,x_0)]^{\lambda+\varepsilon}, \quad \lambda = \sum_{k=0}^{N} M_\infty(w_k)$$

with an arbitrarily $\varepsilon > 0$, according to (4.21). Therefore,

$$\int\limits_{X\backslash B_{4R}} [\varrho(x)\mathcal{M}\varphi(x)]^{p(x)}\, d\mu(x) \le C \int\limits_{X\backslash X(x_0,4R)} \frac{d\mu(x)}{[d(x,x_0)]^{[\underline{\mathfrak{dim}}_\infty(X)-\lambda-2\varepsilon]p_\infty}} = C_1 < \infty \tag{9.4}$$

By Lemma 5.1 the last integral is convergent if $[\underline{\mathfrak{dim}}_\infty(X)-\lambda]p_\infty > \overline{\mathfrak{dim}}_\infty(X)$, that is, $\lambda < \underline{\mathfrak{dim}}_\infty(X) - \frac{\overline{\mathfrak{dim}}_\infty(X)}{p_\infty}$, which is satisfied by condition (3.21). Combining (9.2) and (9.4), we get

$$I_X^p(\varrho\mathcal{M}\varphi) \le C < \infty. \tag{9.5}$$

Now we pass to the function $\psi$. Let first $x \in B_R$. If $r < R$, then $B(x,r) \cap X\backslash B_{2R} = \emptyset$ and $\mathcal{M}_r\psi(x) = 0, x \in X_R$. Therefore, we have to consider only $r \ge R$ and then

$$\mathcal{M}_r\psi(x) = \frac{1}{\mu\{B(x,r)\}} \int\limits_{B(x,r)\cap B_{2R}} |f(y)| d\mu(y) \le \frac{1}{\mu\{B(x,r)\}} \int\limits_{B(x,r)\cap B_{2R}} (1+|f(y)|^{p_\infty})\, d\mu(y).$$

Hence $\mathcal{M}_r\psi(x) \le C < \infty$. Thus, $\mathcal{M}\psi(x) \le C$ for $x \in X_R$ and then

$$\int\limits_{X_R} [\varrho(x)\mathcal{M}\psi(x)]^{p(x)} d\mu(x) \le C \int\limits_{X_R} [\varrho(x)]^{p(x)} d\mu(x) < \infty \tag{9.6}$$

by statement (5.2) of Lemma 5.2. It remains to estimate

$$\int\limits_{X\backslash X_R} [\varrho(x)\mathcal{M}\psi(x)]^{p(t)} d\mu(x) = \int\limits_{X\backslash X_R} [\varrho(x)\mathcal{M}\psi(x)]^{p_\infty} d\mu(x).$$

To this end, it suffices to make use of the known boundedness of the maximal operator in the Lebesgue space with constant $p_\infty > 1$, see Theorem 3.1. Theorem 3.1 is applicable in this case since our weight $\varrho$ is in $A_{p_\infty}$ according to Part II of Theorem 5.11. (Note that since all the points $x_k, k = 1, ..., N$ and $\infty$ are distinct, it is easily verified that from belonging to $A_{p_\infty}(X)$ of separate weights there follows belonging to $A_{p_\infty}(X)$ of of their products). Therefore,

$$\int\limits_{X\backslash X_R} [\varrho(t)\mathcal{M}\psi(x)]^{p(t)} d\mu(t) \le C \int\limits_{X\backslash X_R} [\varrho(t)\psi(x)]^{p_\infty} d\mu(t) \le C \int\limits_{X} [\varrho(t)f(t)]^{p(t)} d\mu(t) \le C$$

which together with (9.6) yields $I_X^p(\mathcal{M}\psi) \le C < \infty$ and proves the theorem.

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