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TO THE 100th ANNIVERSARY OF BIRTHDAY OF SOLOMON GRIGOR'EVICH MIKHLIN

Dedicated to the memory of Solomon Grigor'evich Mikhlin

Singular Operators and Fourier Multipliers in Weighted Lebesgue Spaces with Variable Index

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Abstract—Mikhlin's ideas and results related to the theory of spaces $L_\rho^{p(\cdot)}$ with nonstandard growth are developed. These spaces are called Lebesgue spaces with variable index; they are used in mechanics, the theory of differential equations, and variational problems. The boundedness of Fourier multipliers and singular operators on the spaces $L_\rho^{p(\cdot)}$ are considered. All theorems are derived from an extrapolation theorem due to Rubio de Francia. The considerations essentially use theorems on the boundedness of operators and maximal Hardy–Littlewood functions on Lebesgue spaces with constant index.

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INTRODUCTION

This paper develops ideas and results of S. G. Mikhlin related, in particular, to the theory of spaces of functions with nonstandard growth, known also as generalized Lebesgue spaces with variable index, which have been extensively studied in recent years. Mathematical problems related to spaces with variable index arise in applications to continuum mechanics (in particular, in the theory of electrorheological fluids). They arise also in the theory of differential equations and variational problems. All this has determined a significant interest in these spaces in recent years. A survey of the development of harmonic analysis on spaces with variable index of integrability is contained in [15, 23, 38] (see also the references therein).

In this paper, we prove theorems on the boundedness of Fourier multipliers and singular integral operators, as well as of majorants of partial sums of trigonometric Fourier series, etc., on weighted Lebesgue spaces $L_\rho^{p(\cdot)}$ with variable index $p(x)$; we also give their vector-valued analogues. The proofs of all of these theorems are based on a version of Rubio de Francia's extrapolation theorem [34] for the spaces $L_\rho^{p(\cdot)}$, which is also proved in this paper. We develop some ideas and approaches of [12, 13]. An important role in the proofs is played by well-known theorems on the boundedness of the operators mentioned above for constant p and Muckenhoupt weights. We also consider the boundedness of the maximal Hardy–Littlewood function on Lebesgue spaces with variable index.

1. DEFINITIONS AND AUXILIARY ASSERTIONS

Let (X, d, μ) be a metric space with metric d and nonnegative finite measure μ. By $B(x, r) = \{y \in X: d(x, y) < r\}$ we denote a ball in X. We assume that the measure μ satisfies the doubling condition

$$\mu B(x, 2r) \le C\mu B(x, r),$$

where $C > 0$ does not depend on $r > 0$ and $x \in X$. For locally μ-integrable functions $f: X \longrightarrow \mathbb{R}^1$, consider the operator $\mathcal{M}$ taking each function f to the maximal Hardy–Littlewood function, which is defined by

$$\mathcal{M}f(x) = \sup_{r>0}\frac{1}{\mu B(x, r)}\int\limits_{B(x, r)}|f(y)|d\mu(y).$$

By $A_s = A_s(X)$, where $1 \le s < \infty$, we denote the class of weights $w: X \longrightarrow \mathbb{R}^1$ being locally almost everywhere positive μ-integrable functions satisfying the Muckenhoupt condition

$$\sup_B\left(\frac{1}{\mu B}\int_B w^s(y)d\mu(y)\right)\left(\frac{1}{\mu B}\int_B w^{-s'}(y)d\mu(y)\right)^{s-1} < \infty$$

for $1 < s < \infty$; for $s = 1$, the weights are required to satisfy the condition

$$\mathcal{M}w(x) \le Cw(x)$$

for almost all $x \in X$, where the constant $C > 0$ does not depend on $x \in X$. Obviously, $A_1 \subset A_s$ for $1 < s < \infty$.

In [10, 30], is was proved that the condition $w \in A_s$ is equivalent to the boundedness of the operator $\mathcal{M}$ on the space $L^s_w(X)$, i.e., to

$$\int_X (\mathcal{M}f(x))^s w^s(x)d\mu(x) \le C\int_X |f(x)|^s w^s(x)d\mu(x);$$

here, $C > 0$ does not depend on f.

Let Ω be an open set in X, and let $\mathcal{P}(\Omega)$ denote the class of measurable functions on Ω for which

$$1 < p_- \le p_+ < \infty, \tag{1.1}$$

where $p_- = p_-(\Omega) = \operatorname{ess\,inf}_{x \in \Omega} p(x)$ and $p_+ = p_+(\Omega) = \operatorname{ess\,sup}_{x \in \Omega} p(x)$.

Consider the weighted Banach space $L^{p(\cdot)}_\rho(\Omega)$ of measurable functions $f: \Omega \longrightarrow \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}_\rho} := \|\rho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 \,\middle|\, \int_\Omega \left|\frac{\rho(x)f(x)}{\lambda}\right|^{p(x)} d\mu(x) \le 1\right\} < \infty. \tag{1.2}$$

Definition 1.1. We say that a function $p \in \mathcal{P}(\Omega)$ belongs to the class $\mathcal{B}_\rho(\Omega)$ if the maximal operator $\mathcal{M}$ is bounded on the space $L^{p(\cdot)}_\rho(\Omega)$.

Definition 1.2. A function $p: \Omega \longrightarrow \mathbb{C}^1$ is called a function of class WL (a weakly Lipschitz function) if

$$|p(x) - p(y)| \le \frac{A}{-\ln d(x, y)} \quad \text{for } x, y \in \Omega \text{ such that } d(x, y) \le \frac{1}{2}, \tag{1.3}$$

where $A > 0$ does not depend on x and y.

Let us introduce the following numbers:

(1) the lower boundary of local "dimensions" in X, namely,

$$m(\mu B) = \sup_{t > 1}\frac{\ln\left(\liminf_{r \to \infty} \operatorname{ess\,inf}_{x \in X} \frac{\mu B(x, rt)}{\mu B(x, r)}\right)}{\ln t}; \tag{1.4}$$

(2) the lower and upper boundaries of "dimensions" (related to the influence of infinity), namely,

$$m_\infty(\mu B) = \sup_{t > 1}\frac{\ln\left(\liminf_{r \to \infty} \operatorname{ess\,inf}_{x \in X} \frac{\mu B(x, rt)}{\mu B(x, r)}\right)}{\ln t} \tag{1.5}$$

and

$$M_\infty(\mu B) = \sup_{t > 1}\frac{\ln\left(\limsup_{r \to \infty} \operatorname{ess\,sup}_{x \in X} \frac{\mu B(x, rt)}{\mu B(x, r)}\right)}{\ln t}. \tag{1.6}$$

It is easy to see that $m(\mu B)$, $m_\infty(\mu B)$, and $M_\infty(\mu B)$ are nonnegative. In what follows, when considering these boundaries of dimensions, we always assume that $m(\mu B), m_\infty(\mu B), M_\infty(\mu B) \in (0, \infty)$.

Note that, in (1.4), $\operatorname*{ess\,inf}_{x \in X}$ can be transposed as

$$m(\mu B) = \operatorname*{ess\,inf}_{x \in X} \sup_{t > 1} \frac{\ln\left(\liminf\limits_{r \to \infty} \frac{\mu N(x, rt)}{\mu B(x, r)}\right)}{\ln t},$$

provided that there exists an $\varepsilon > 0$ such that the function $\mu B(x, r)$ is continuous for $r \in (0, \varepsilon)$ uniformly with respect to $x \in X$. A similar fact is valid for expressions (1.5)–(1.6).

2. CLASSES OF WEIGHT FUNCTIONS

We deal, in particular, with the weights

$$\rho(x) = \prod_{k=1}^{N} [d(x, x_k)]^{\beta_k} (1 + d(x_0, x))^{\beta_\infty}, \quad x_k \in X, \quad k = 0, 1, \ldots, N, \tag{2.1}$$

where $\beta_\infty = 0$ for a bounded metric space X. Let $\Pi = \{x_0, x_1, \ldots, x_N\}$.

Definition 2.1. We say that a weight function of the form (2.1) belongs to the class $V_{p(\cdot)}(X, \Pi)$, where $p(\cdot) \in C(X)$, if

$$-\frac{m(\mu B)}{p(x_k)} < \beta_k < \frac{m(\mu B)}{p'(x_k)} \tag{2.2}$$

and

$$-\frac{m_\infty(\mu B)}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < m_\infty(\mu B) - \frac{M_\infty(\mu B)}{p_\infty}. \tag{2.3}$$

In the case of a bounded metric space X, we consider the more general class of weights

$$\rho(x) = \prod_{k=1}^{N} w_k[d(x, x_k)] \tag{2.4}$$

with "radial" weights, where the functions $w_k(\cdot)$ belong to the Zygmund–Bari–Stechkin class, in which oscillation between power functions with different exponents is allowed.

Let $U = U([0, \ell])$ denote the class of functions $u \in C[0, \ell]$ (where $0 < \ell < \infty$) such that $u(0) = 0$, $u(t) > 0$ for $t > 0$, and u is a nondecreasing function on $[0, \ell]$. By $\tilde{U}$ we denote the class of functions u such that $x^a u(x) \in U$ for some $a \in \mathbb{R}^1$.

Definition 2.2 [1]. We say that v belongs to the Zygmund–Bari–Stechkin class Φ_δ^0 if

$$\int_0^h \frac{v(t)}{t} dt \le c v(h) \quad \text{and} \quad \int_h^\ell \frac{v(t)}{t^{1+\delta}} dt \le c \frac{v(h)}{h^\delta},$$

where $c = c(v) > 0$ does not depend on $h \in (0, \ell]$.

As is known, $v \in \Phi_\delta^0$ if and only if $0 < m(v) \le M(v) < \delta$, where

$$m(w) = \sup_{t > 1} \frac{\ln\left(\liminf\limits_{h \to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \quad \text{and} \quad M(w) = \sup_{t > 1} \frac{\ln\left(\limsup\limits_{h \to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \tag{2.5}$$

(see [35, 36, 19]).

Definition 2.3. We say that a weight function ρ of the form (2.4) belongs to the class $V^{\mathrm{osc}}_{p(\cdot)}(X, \Pi)$, where $p(\cdot) \in C(X)$, if

$$r^{1/p(x_k)} w_k(r) \in \Phi^0_{m(\mu B)} \quad \text{for } x_k \in \Pi \text{ and } k = 1, 2, \ldots, N, \tag{2.6}$$

or, equivalently,

$$w_k(r) \in \tilde{U} \quad \text{and} \quad -\frac{m(\mu B)}{p(x_k)} < m(w_k) \le M(w_k) < \frac{m(\mu B)}{p'(x_k)} \quad \text{for } k = 1, 2, \ldots, N. \tag{2.7}$$

Note that, if the metric space X has constant dimension s in the sense that

$$c_1 r^s \le \mu B(x, r) \le c_2 r^s,$$

where $c_1 > 0$ and $c_2 > 0$ are constants not depending on $x \in X$ and $r > 0$, then inequalities (2.2), (2.3), and (2.7) can be written in the forms

$$-\frac{s}{p(x_k)} < \beta_k < \frac{s}{p'(x_k)}, \qquad -\frac{s}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \frac{s}{p'_\infty} \tag{2.8}$$

and

$$-\frac{s}{p(x_k)} < m(w) \le M(w) < \frac{s}{p'(x_k)}, \tag{2.9}$$

where $k = 1, 2, \ldots, N$, respectively.

3. BOUNDEDNESS OF THE MAXIMAL HARDY–LITTLEWOOD OPERATOR ON WEIGHTED SPACES WITH VARIABLE INDEX

The following assertions are valid.

Theorem 3.1. *Suppose that X is a bounded metric space endowed with a measure satisfying the doubling condition, $p \in WL(X)$, and $\rho \in V^{\mathrm{osc}}_{p(\cdot)}(X, \Pi)$. Then, $\mathcal{M}$ is bounded on $L^{p(\cdot)}_\rho(X)$.*

Theorem 3.2. *Suppose that X is an unbounded metric space endowed with a measure satisfying the doubling condition, $p \in WL(X)$, there exists an $R > 0$ such that $p(x) \equiv p_\infty = \mathrm{const}$ for $x \in X \backslash B(x_0, R)$, and $\rho \in V_{p(\cdot)}(X, \Pi)$. Then, $\mathcal{M}$ is bounded on $L^{p(\cdot)}_\rho(X)$.*

Euclidean versions of Theorems 3.1 and 3.2 were proved in [14] for the nonweight case and in [26, 25] for the weight case.

4. AN EXTRAPOLATION THEOREM

Let $\mathcal{F}$ be a family of ordered pairs (f, g) of nonnegative measurable functions f and g defined on an open subset Ω of X. When saying that the inequality

$$\int_\Omega f^{p_0}(x) w(x) d\mu(x) \le C \int_\Omega g^{p_0}(x) w(x) d\mu(x) \tag{4.1}$$

holds for all pairs $(f, g) \in \mathcal{F}$ and weights $w \in A_q(\Omega)$ (for some q, $1 \le q < \infty$), we always mean that it holds for all pairs such that the left-hand side of this inequality is finite, and the constant C depends only on p_0 and the A_q-constant of the weight w.

Theorem 4.1. *Let X be a metric space with a measure, and let Ω be an open set in X. Suppose that* $1 < p_0 < p_-$, *the weight ρ and the index $p(\cdot) \in \mathcal{P}$ satisfy the condition $(\tilde{p})' \in \mathcal{B}_{\frac{1}{\rho^{p_0}}}$* (where $\tilde{p}(\cdot) = p(\cdot)/p_0$), *and $\mathcal{F}$ is a family such that the inequality*

$$\int_{\Omega} f^{p_0}(x)w(x)d\mu(x) \le C\int_{\Omega} g^{p_0}(x)w(x)d\mu(x) \tag{4.2}$$

holds for all $(f, g) \in \mathcal{F}$ *and all* $w \in A_1(\Omega)$. *Then, for all* $(f, g) \in \mathcal{F}$ *with* $f \in L^{p(\cdot)}_{\rho}(\Omega)$, *the inequality*

$$\|f\|_{L^{p(\cdot)}_{\rho}} \le C\|g\|_{L^{p(\cdot)}_{\rho}} \tag{4.3}$$

with a constant $C > 0$ *not depending on f and g holds*.

Note that Theorem 4.1 does not assume the measure on X to satisfy the doubling condition.

Theorem 4.1 combined with Theorems 3.1 and 3.2 implies the following assertion.

Theorem 4.2. *Let X be a metric space with a measure satisfying the doubling condition, and let* Ω *be an open set in X. Suppose that*

(1) *if the set* Ω *is bounded, then* $p \in WL(\Omega)$ *and* $\rho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$, *and*

(2) *if the set* Ω *is unbounded, then* $p \in WL(\Omega)$, $p(x) \equiv p_\infty = \mathrm{const}$ *for* $x \in \Omega\backslash B(x_0, R)$ *for some* $x_0 \in \Omega$ *and* $R > 0$, *and* $\rho \in V_{p(\cdot)}(\Omega, \Pi)$.

Then, the fulfillment of inequality (4.2) *for all* $(f, g) \in \mathcal{F}$ *from some family* $\mathcal{F}$ *and all* $w \in A_1(\Omega)$ *implies that of inequality* (4.3) *for all pairs* (f, g) *from* $\mathcal{F}$ *for which* $f \in L^{p(\cdot)}_{\rho}(\Omega)$.

Remark 4.3. Since the intervals (2.2), (2.3), and (2.7) are open, it follows that there exists a $p_0 \in (1, p_-)$ for which

$$\rho \in V_{p(\cdot)}(\Omega, \Pi) \Rightarrow \rho^{-p_0} \in V_{(\tilde{p})'(\cdot)}(\Omega, \Pi)$$

and

$$\rho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi) \Rightarrow \rho^{-p_0} \in V^{\mathrm{osc}}_{(\tilde{p})'(\cdot)}(\Omega, \Pi),$$

where $\tilde{p}(x) = \dfrac{p(x)}{p_0}$.

Proof of Theorem 4.1. By virtue of the Riesz theorem, which is valid for spaces with variable index provided that $1 < p_- \le p_+ < \infty$ (see [27, 37]), we have

$$\|f\|^{p_0}_{L^{p(\cdot)}_{\rho}} = \left\|f^{p_0}\rho^{p_0}\right\|_{L^{\tilde{p}(\cdot)}} \le \sup\int_{\Omega} f^{p_0}(x)h(x)d\mu(x),$$

where the supremum is over all nonnegative h such that $\left\|h\rho^{-p_0}\right\|_{L^{(\tilde{p})'(\cdot)}} \le 1$ and f is assumed to be nonnegative. Take any such function h. Let us show that

$$\int_{\Omega} f^{p_0}(x)h(x)d\mu(x) \le C\|g\rho\|^{p_0}_{L^{p(\cdot)}}, \tag{4.4}$$

where the constant $C > 0$ does not depend on h, for an arbitrary pair (f, g) from the given family $\mathcal{F}$. By assumption, p and ρ are such that $(\tilde{p})' \in \mathcal{B}_{1/\rho^{p_0}}$, i.e.,

$$\left\|\rho^{-p_0}\mathcal{M}\varphi\right\|_{L^{\tilde{p}'(\cdot)}} \le C_0\left\|\rho^{-p_0}\varphi\right\|_{L^{\tilde{p}'(\cdot)}}, \tag{4.5}$$

where the constant $C_0 > 0$ does not depend on φ.

Let us apply the construction

$$S\varphi(x) = \sum_{k=0}^{\infty}(2C_0)^{-k}\mathcal{M}^k\varphi(x) \tag{4.6}$$

due to Rubio de Francia [34], where each $\mathcal{M}^k$ is the k-times iterated maximal operator and C_0 is the same constant as in (4.5) ($C_0 \geq 1$). The following assertions are obvious:

(1) $\varphi(x) \leq S\varphi(x)$, where $x \in \Omega$, for any nonnegative function φ;

(2) $\|\rho^{-p_0} S\varphi\|_{L^{(\tilde{p})'}(\cdot)} \leq 2\|\rho^{-p_0}\varphi\|_{L^{(\tilde{p})'}(\cdot)}$; (4.7)

(3) $\mathcal{M}(S\varphi)(x) \leq 2c_0 S\varphi(x)$ for $x \in \Omega$, so that $S\varphi \in A_1(\Omega)$, where the constant A_1 does not depend on φ. Therefore, $S\varphi \in A_{p_0}(\Omega)$.

According to results of Section 1, for $\varphi = h$, we have

$$\int_\Omega f^{p_0}(x)h(x)d\mu(x) \leq \int_\Omega f^{p_0}(x)Sh(x)d\mu(x). \tag{4.8}$$

Hölder's inequality for variable indices, the condition in Section 2, and the assumption that $f \in L_\rho^{p(\cdot)}$ imply

$$\int_\Omega f^{p_0}(x)Sh(x)d\mu(x) = \int_\Omega f^{p_0}(x)\rho^{p_0}\rho^{-p_0}Sh(x)d\mu(x) \leq k\|f^{p_0}\rho^{p_0}\|_{L^{\tilde{p}(\cdot)}} \cdot \|\rho^{-p_0}Sh\|_{L^{(\tilde{p})'(\cdot)}}$$

$$\leq C\|f\rho\|^{p_0}_{L^{p(\cdot)}} \cdot \|h\rho^{p_0}\|_{L^{(\tilde{p})'(\cdot)}} \leq C\|f\rho\|_{L^{p(\cdot)}} < \infty.$$

Therefore, the integral $\int_\Omega f^{p_0}(x)Sh(x)d\mu(x)$ is finite, and we can apply condition (4.1) to the right-hand side of (4.8), which yields

$$\int_\Omega f^{p_0}(x)Sh(x)d\mu(x) \leq C\int_\Omega g^{p_0}(x)Sh(x)d\mu(x).$$

Applying Hölder's inequality to the right-hand side, we obtain

$$\int_\Omega f^{p_0}(x)Sh(x)d\mu(x) \leq C\|\rho g\|_{L^{p(\cdot)}}\|\rho^{-p_0}Sh\|_{L(\tilde{p})'(\cdot)}. \tag{4.9}$$

By virtue of (4.8), to prove (4.4), it suffices to show that $\|\rho^{-p_0}Sh\|_{L(\tilde{p})'(\cdot)}$ is bounded by a constant not depending on h. This follows from (4.7) and the normalization condition $\|h\rho^{-p_0}\|_{L^{(\tilde{p})'(\cdot)}} \leq 1$, which completes the proof of the theorem.

5. APPLICATION TO THE BOUNDEDNESS ON $L_\rho^{p(\cdot)}$ OF CLASSICAL OPERATORS OF HARMONIC ANALYSIS

5.1. Multipliers of the Fourier Transform

We say that a measurable function $\mathbb{R}^n \longrightarrow \mathbb{R}^1$ is a Fourier multiplier in the space $L_\rho^{p(\cdot)}(\mathbb{R}^n)$ if the operator T_m defined on the Schwartz space $S(\mathbb{R}^n)$ by

$$\widehat{T_m f} = m\hat{f}$$

extends to a bounded operator on $L_\rho^{p(\cdot)}(\mathbb{R}^n)$.

Below, we generalize classical Mikhlin's theorem on Fourier multipliers to Lebesgue spaces with variable index.

Theorem 5.1. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^n) \cap p \in WL$, $p(x) = p_\infty = \text{const}$ for $|x| \ge R$ *for some* $R > 0$, *and* ρ *is a weight function of the form* (2.1), *where*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)} \quad \text{for} \quad k = 1, 2, \ldots, N \tag{5.1}$$

and

$$-\frac{n}{p_\infty} < \beta + \sum_{k=1}^{N} \beta_k < \frac{n}{p'_\infty}. \tag{5.2}$$

Suppose also that a function $m(x)$ *is continuous on* $\mathbb{R}^n$ *everywhere except*, *possibly*, *at the origin and has mixed derivative* $\frac{\partial^n m}{\partial x_1 x_2 \ldots x_n}$ *and derivatives* $D^\alpha m = \frac{\partial^{|\alpha|} m}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}$, *where* $\alpha = (\alpha_1, \ldots, \alpha_n)$, *of all orders* $|\alpha| = \alpha_1 + \ldots + \alpha_n \le n - 1$ *continuous outside the origin*. *Finally*, *suppose that*

$$|x|^{|\alpha|}|D^\alpha m(x)| \le C \quad \text{if} \quad |\alpha| \le n - 1,$$

where the constant $C > 0$ *does not depend on* x. *Then*, m *is a Fourier multiplier in* $L^{p(\cdot)}_\rho(\mathbb{R}^n)$.

Theorem 5.1 follows from Theorems 4.2 and 3.2 and the fact that, for any constant s such that $1 < s < \infty$ and any weight $\rho \in A_s$, every function m satisfying the conditions of Theorem 5.1 is a Fourier multiplier in $L^s_\rho(\mathbb{R}^n)$. This fact was proved in [28] (see also [2]).

This theorem can be generalized to weighted spaces with variable index. Namely, the following analogue of the Mikhlin–Hörmander theorem is valid.

Theorem 5.2. *Suppose that* $p(\cdot)$ *and* ρ *satisfy the conditions of Theorem* 5.1 *and* $m: \mathbb{R}^n \longrightarrow \mathbb{R}^1$ *is a function such that*

$$\sup_{R>0}\left(R^{s|\alpha| - n} \int_{R < |x| < 2R} |D^\alpha m(x)|^s dx\right)^{1/s} < \infty$$

for some s $(1 < s \le 2)$ *and all* α *with* $|\alpha| \le \ell$, *where* $(\ell p_-)/n > 1$. *Then*, m *is a Fourier multiplier in* $L^{p(\cdot)}_\rho(\mathbb{R}^n)$.

Theorem 5.2 follows from Theorems 4.2 and 3.2 as well.

In the statement of the following theorem, Δ_j denotes any interval of the form $[2^j, 2^{j+1}]$ or $[-2^{j+1}, -2^j]$, where $j \in \mathbb{Z}$.

Theorem 5.3. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^1) \cap WL(\mathbb{R}^1)$ *and* $p(x)$ *is constant outside some finite interval*. *Suppose also that* ρ *has the form* (2.1) *and conditions* (5.1)–(5.2) *with* $n = 1$ *hold*. *Finally*, *suppose that*, *in each of the intervals* Δ_j, *a function* m *can be represented as*

$$m(\lambda) = \int_{-\infty}^{\lambda} d\mu_{\Delta_j}, \quad \lambda \in \Delta_j,$$

where the μ_{Δ_j} *are finite measures such that* $\sup_j \operatorname{var}\mu_{\Delta_j} < \infty$. *Then*, m *is a Fourier multiplier in* $L^{p(\cdot)}_\rho(\mathbb{R}^1)$.

In the case of constant p, this theorem was proved in [29] for $\rho \equiv 1$ and in [2, 3] for $\rho \in A_p$.

5.2. Multipliers of Trigonometric Fourier Series

Using Theorem 4.1 and known results on constant indices, we can generalize theorems about Marcinkiewicz multipliers and Littlewood–Paley expansions for trigonometric Fourier series to weighted spaces with variable index.

Suppose that $\mathbb{T} = [-\pi, \pi]$, f is a 2π-periodic function, and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{5.3}$$

According to the notation introduced above, the expression $\rho \in V_{p(\cdot)}^{\text{osc}}(\mathbb{T}, \Pi)$ means that ρ is a weight of the form (2.4) satisfying condition (2.6) or (2.7) for $n = 1$.

Theorem 5.4. *Suppose that* $p \in \mathcal{P}(\mathbb{T}) \cap WL(\mathbb{T})$, $\rho \in V_{p(\cdot)}^{\text{osc}}(\mathbb{T}, \Pi)$, *and a sequence of* λ_k *satisfies the conditions*

$$|\lambda_k| \le A \quad \text{and} \quad \sum_{k=2^{j-1}}^{2^j - 1} |\lambda_k - \lambda_{k+1}| \le A,$$

where $A > 0$ *does not depend on k and j. Then, there exists a function* $F(x) \in L_\rho^{p(\cdot)}(\mathbb{T})$ *such that the series* $(\lambda_0 a_0)/2 + \sum_{k=0}^{\infty} \lambda_k (a_k \cos kx + b_k \sin kx)$ *is a Fourier expansion for F, and*

$$\|F\|_{L_\rho^{p(\cdot)}} \le cA \|f\|_{L_\rho^{p(\cdot)}},$$

where $c > 0$ *does not depend on* $f \in L_\rho^{p(\cdot)}(\mathbb{T})$.

Theorem 5.5. *If* $p \in \mathcal{P}(\mathbb{T}) \cap WL(\mathbb{T})$ *and* $\rho \in V_{p(\cdot)}^{\text{osc}}(\mathbb{T}, \Pi)$, *then there exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1 \|f\|_{L_\rho^{p(\cdot)}} \le \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{k=2^{j-1}}^{2^{j-1}} A_k(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^{p(\cdot)}} \le c_2 \|f\|_{L_\rho^{p(\cdot)}} \tag{5.4}$$

for all $f \in L_\rho^{p(\cdot)}(\mathbb{T})$, *where* $A_k(x) = a_k \cos kx + b_k \sin kx$, $A_{2^{-1}} = 0$, *and* $A_0 = a_0/2$.

For constant p and $\rho \in A_p$, this theorem was proved in [28].

5.3. Majorants of Partial Sums of Fourier Series

Suppose that

$$S_*(f) = S_*(f, x) = \sup_{k \ge 0} |S_k(f, x)|,$$

where $S_k(f, x)$ is a partial sum of Fourier series (5.3).

Theorem 5.6. *If* $p \in \mathcal{P}(\mathbb{T}) \cap WL(\mathbb{T})$ *and* $\rho \in V_{p(\cdot)}^{\text{osc}}(\mathbb{T}, \Pi)$, *then*

$$\|S_*(f)\|_{L_\rho^{p(\cdot)}} \le c \|f\|_{L_\rho^{p(\cdot)}} \tag{5.5}$$

for all $f \in L_\rho^{p(\cdot)}(\mathbb{T})$, *where the constant* $c > 0$ *does not depend on f.*

For constant p and $\rho \in A_p$, Theorem 5.6 was proved in [18].

5.4. Singular Cauchy Integral

Consider the singular integral

$$S_\Gamma f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau) d\nu}{\tau - t},$$

where Γ is a simple finite Carleson curve and ν is the natural parameter on Γ.

Theorem 5.7. *If* $p \in \mathcal{P}(\Gamma) \cap WL(\Gamma)$ *and* $\rho \in V^{\text{osc}}_{p(\cdot)}(\Gamma, \Pi)$, *then the operator* S_Γ *is bounded on the space* $L^{p(\cdot)}_\rho(\Gamma)$.

In the case of power weights, Theorem 5.7 was proved in [24], where the case of an infinite Carleson curve was also considered. For constant p and $\rho \in A_p(\Gamma)$, Theorem 5.7 was proved by different methods in [22] and [9].

5.5. Multidimensional Singular Operators

Consider the multidimensional singular operator

$$Tf(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} K(x, y) f(y) dy, \text{ where } x \in \mathbb{R}^n. \tag{5.6}$$

We assume that the singular kernel $K(x, y)$ satisfies the following conditions:

$$|K(x, y)| \le C|x-y|^{-n}; \tag{5.7}$$

$$|K(x', y) - K(x, y)| \le C\frac{|x'-x|^\alpha}{|x-y|^{n+\alpha}} \text{ if } |x'-x| < \frac{1}{2}|x-y| \tag{5.8}$$

and

$$|K(x, y') - K(x, y)| \le C\frac{|y'-y|^\alpha}{|x-y|^{n+\alpha}} \text{ if } |y'-y| < \frac{1}{2}|x-y|, \tag{5.9}$$

where α is an arbitrary positive index;

$$\lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} K(x, y) dy \text{ exists}; \tag{5.10}$$

$$\text{the operator (5.6) is bounded on } L^2(\mathbb{R}^n). \tag{5.11}$$

Theorem 5.8. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$; $p(x) \equiv p_\infty = \text{const}$ *outside some ball* $|x| < R$; ρ *is a weight function of the form* (2.1), *where* $d(x, x_k) = |x - x_k|$; *conditions* (5.1) *and* (5.2) *hold*; *and the kernel* $K(x, y)$ *satisfies conditions* (5.7)–(5.11). *Then, the operator T is bounded on the space* $L^{p(\cdot)}_\rho(\mathbb{R}^n)$.

For constant p and $\rho \in A_p(\mathbb{R}^n)$, this theorem was proved in [11]. For variable $p(\cdot)$, the weightless case of Theorem 5.8 was proved in [16].

5.6. Commutators

Consider the commutators

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

generated by operator (5.6) and a function $b \in \text{BMO}(\mathbb{R}^n)$.

Theorem 5.9. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$; $p(x) \equiv p_\infty = \text{const}$ *outside some ball* $|x| < R$; ρ *is a weight function of the form* (2.1), *where* $d(x, x_k) = |x - x_k|$; *conditions* (5.1) *and* (5.2) *hold*; *and the kernel* $K(x, y)$ *satisfies conditions* (5.7)–(5.11). *Then, the commutator* $[b, T]$ *is bounded on the space* $L^{p(\cdot)}_\rho(\mathbb{R}^n)$.

For constant p and $\rho \in A_p(\mathbb{R}^n)$ with $1 < p < \infty$, Theorem 5.9 was proved earlier (see [32]). The weightless case of Theorem 5.9 for variable $p(\cdot)$ was proved in [20].

5.7. The Fefferman–Stein Function

Suppose that f is a measurable locally integrable function on $\mathbb{R}^n$, B is a ball in $\mathbb{R}^n$, and $f_B = \frac{1}{|B|}\int_B f(x)dx$. Consider the maximal Fefferman–Stein function

$$\mathcal{M}^{\#} f(x) = \sup_{B \in X} \frac{1}{|B|} \int_B |f(x) - f_B| dx.$$

Theorem 5.10. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$; $p(x) \equiv p_\infty = \text{const}$ *outside some ball* $|x| < R$; ρ *is a weight function of the form* (2.1), *where* $d(x, x_k) = |x - x_k|$; *and conditions* (5.1) *and* (5.2) *hold. Then*,

$$\|\mathcal{M} f\|_{L_\rho^{p(\cdot)}(\mathbb{R}^n)} \le C \|\mathcal{M}^{\#} f\|_{L_\rho^{p(\cdot)}(\mathbb{R}^n)}, \tag{5.12}$$

where $C > 0$ *does not depend on* f.

For constant p and $\rho \in A_p$, inequality (5.12) was proved in [17].

5.8. Pseudodifferential Operators

Consider the pseudodifferential operator $\sigma(x, D)$ defined by

$$\sigma(x, D) f(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i (x, \xi)} \hat{f}(\xi) d\xi.$$

Theorem 5.11. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$; $p(x) \equiv p_\infty = \text{const}$ *outside a ball* $|x| < R$; ρ *is a weight function of the form* (2.1), *where* $d(x, x_k) = |x - x_k|$; *conditions* (5.1) *and* (5.2) *hold*; *and*

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)\right| \le c_{\alpha\beta}(1 + |\xi|)^{-|\alpha|}.$$

Then, *the operator* $\sigma(x, D)$ *admits a continuous extension to the space* $L_\rho^{p(\cdot)}(\mathbb{R}^n)$.

For constant p and $\rho \in A_p$, Theorem 5.11 was proved in [31]. For variable $p(\cdot)$, the weightless case of Theorem 5.11 was proved in [33] by a different method.

5.9. Vector-Valued Operators

Consider a sequence $f = (f_1, f_2, \dots, f_k, \dots)$ of locally integrable functions $f_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}^1$.

Theorem 5.12. *Suppose that* $p \in \mathcal{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$; $p(x) \equiv p_\infty = \text{const}$ *outside a ball* $|x| < R$; ρ *is a weight function of the form* (2.1), *where* $d(x, x_k) = |x - x_k|$; *and conditions* (5.1)–(5.2) *hold. Then*, *for any* θ *such that* $0 < \theta < \infty$,

$$\left\| \left(\sum_{j=1}^{\infty} (\mathcal{M} f_j)^\theta \right)^{\frac{1}{\theta}} \right\|_{L_\rho^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^\theta \right)^{\frac{1}{\theta}} \right\|_{L_\rho^{p(\cdot)}(\mathbb{R}^n)},$$

where $c > 0$ *does not depend on* f.

For a constant number p and a function ρ from the class A_p, the weight inequalities for vector-valued functions were proved in [2–4] (see also [8]).

The corresponding assertions for vector-valued operators are also valid for singular integrals, commutators, maximal Fefferman–Stein function, Fourier multipliers, etc.

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