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Characterization of the range of one-dimensional fractional integration in the space with variable exponent

Humberto Rafeiro and Stefan Samko

Abstract. Within the frameworks of weighted Lebesgue spaces with variable exponent, we give a characterization of the range of the one-dimensional Riemann-Liouville fractional integral operator in terms of convergence of the corresponding hypersingular integrals. We also show that this range coincides with the weighted Sobolev-type space $L^{\alpha,p(\cdot)}[(a,b),\varrho]$.

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1. Introduction

Recently the spaces $I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ and $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ of Riesz and Bessel potential spaces were studied within the frameworks of variable exponents $p(\cdot)$ in papers [1] and [2] in the case of the whole space $\mathbb{R}^n$. In particular, the following characterization of the space of Bessel potentials was obtained in [2]:

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = L^{p(\cdot)}(\mathbb{R}^n)\bigcap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f \in L^{p(\cdot)}(\mathbb{R}^n) : \mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)\}, \tag{1.1}$$

where $\mathbb{D}^\alpha f$ is the Riesz fractional derivative.

A similar characterization for potentials over a domain in $\mathbb{R}^n$ remains an open question even in the case of constant p. For an analogue of the Riesz derivative adjusted for domains in $\mathbb{R}^n$ we refer to [14].

In this paper we solve such a problem of characterization in the one-dimensional case $n = 1$. We study the range of fractional integrals over the space $L^{p(\cdot)}(\Omega,\varrho)$ with variable exponent $p(\cdot)$ and a power type weight ϱ, where $\Omega = (a,b)$ is a finite or infinite interval. We obtain a characterization of this range in terms of convergence of the corresponding Marchaud derivatives and show that this range may be also obtained as the restriction on Ω of Bessel potentials with densities in $L^{p(\cdot)}(\Omega, \varrho)$. We refer to [19], p. 229-232, for such results in the non-weighted case and constant p.

Note that an increasing interest to the variable exponent Lebesgue spaces $L^{p(\cdot)}$ observed last years was caused by possible applications (elasticity theory, fluid mechanics, differential equations, see for example [15]). We refer to papers [20] and [13] for basics on the Lebesgue spaces with variable exponents and to the surveys [6], [10], [18] on harmonic analysis in such spaces. One of the breakthrough results obtained for variable $p(x)$ was the statement on the boundedness of the Hardy-Littlewood maximal operator in the generalized Lebesgue space $L^{p(\cdot)}$ under certain conditions on $p(x)$, see [4] and the further development in the above survey papers. The importance of the boundedness of the maximal operator is known due to the fact that many convolution operators occurred in applications may be dominated by the maximal operator. This tool is also used in this paper.

Let $0 < \alpha < 1$ and $x \in (a, b)$. We study the ranges $I^\alpha \left[L^{p(\cdot)}[(a,b), \varrho]\right]$ of the Riemann-Liouville fractional integration operators

$$(I^\alpha_{a+}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}}, \quad (I^\alpha_{b-}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)\,dt}{(t-x)^{1-\alpha}} \tag{1.2}$$

over weighted Lebesgue spaces $L^{p(\cdot)}[(a,b), \varrho]$ with variable exponent $p(x)$. We show that the ranges of operators (1.2) coincide (Theorem (4.5)) under natural assumptions and obtain necessary and sufficient conditions for a function f to belong to this range (Theorem (4.4)). Finally we show that this range coincides with the Sobolev type space $L^{\alpha,p(\cdot)}[(a,b), \varrho]$ (Theorem (4.15)). When developing necessary tools for the proof, we also obtain results of independent interest for Hardy-type operators (Theorems (3.4), (3.8)) and for singular operators with fixed singularity (Theorem (3.6)).

A non-weighted result of a type of Theorem (4.15) for variable exponents was obtained in [1] and [2] for the Riesz potential operator in the case of the whole space $\Omega = \mathbb{R}^n$. We deal not with the Riesz potential operator, but with the fractional integration operator I^α_{a+} which has the unilateral nature. However the main novelty in comparison with [1] and [2] is not only in a different nature of the operator or admission of the weight, but in the fact that the case of a domain in $\mathbb{R}^n$, when we may have an essential influence of the boundary, is more difficult. We show how it is possible to characterize this range in the one-dimensional case with $\Omega = (a, b)$, $-\infty < a < b \leq \infty$. In comparison with [1] and [2], the results obtained in this paper require different terms and methods.

Notation

$|\Omega|$ is the Lebesgue measure of a set $\Omega \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$;
ϱ is a weight, *i.e.*, an a.e. finite and positive function;
$\mathcal{P}(\Omega)$ and $\mathcal{P}_1(\Omega)$, see (2.1)-(2.2);

$w\text{-}Lip\,(\Omega)$, see (2.3);
$w\text{-}Lip_{x_0}\,(\Omega)$, see (3.2);
$\mathcal{M}$ is the maximal operator, see (2.7);
$\mathbb{P}_\varrho(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ such that $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega, \varrho)$.

2. Preliminaries

2.1. On spaces $L^{p(\cdot)}$ with variable exponents

Although our main results concern the one-dimensional case $n = 1$, some auxiliary statements below are given for the multidimensional case. We refer to [13], [16] for details on variable Lebesgue spaces over domains in $\mathbb{R}^n$, but give some necessary definitions. For a measurable function $p : \Omega \to [1, \infty)$, where $\Omega \subset \mathbb{R}^n$ is an open set, we put

$$p^+ = p^+(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad p^- = p^-(\Omega) := \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

In the sequel we use the notation

$$\mathcal{P}(\Omega) := \{p \in L^\infty(\Omega) : 1 < p^- \le p(x) \le p^+ < \infty\} \tag{2.1}$$

and

$$\mathcal{P}_1(\Omega) := \{p \in L^\infty(\Omega) : 1 \le p^- \le p(x) \le p^+ < \infty\} \tag{2.2}$$

The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent is introduced as the set of functions f on Ω for which

$$I_p(f) := \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

By $w\text{-}Lip\,(\Omega)$, for bounded Ω, we denote the class of exponents $p \in L^\infty(\Omega)$ satisfying the log-condition

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \quad |x - y| \le \frac{1}{2}, \; x, y \in \Omega. \tag{2.3}$$

By $p'(x)$ we denote the conjugate exponent: $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$.

The weighted Lebesgue space $L^{p(\cdot)}(\Omega, \varrho)$ is defined as the set of all measurable on Ω functions f for which

$$\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} = \|\varrho f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : I_p\left(\frac{\varrho f}{\lambda}\right) \le 1\right\} < \infty.$$

In [11] the following theorem was proved.

Theorem 2.1. *Let $p \in \mathcal{P}_1(\mathbb{R}^n)$. The class $C_0^\infty(\mathbb{R}^n)$ is dense in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ with an a.e. positive weight ϱ if*

$$[\varrho(x)]^{p(x)} \in L^1_{\text{loc}}(\mathbb{R}^n). \tag{2.4}$$

Observe that condition ((2.4)) implies that the indicator function of sets with finite measure belong to $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$.

Lemma 2.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $p \in \mathcal{P}_1(\Omega) \bigcap w\text{-}Lip\,(\Omega)$. *There exists an extension* $\widetilde{p}(x)$ *of* $p(x)$ *to the whole space* $\mathbb{R}^n$ *such that* $\widetilde{p}(x) \equiv p(x)$ *for* $x \in \Omega$, $\widetilde{p} \in \mathcal{P}(\mathbb{R}^n) \bigcap w\text{-}Lip\,(\mathbb{R}^n)$, $\widetilde{p}(x)$ *is constant outside some large fixed ball and*

$$\widetilde{p}^-(\mathbb{R}^n) = p^-(\Omega); \qquad \widetilde{p}^+(\mathbb{R}^n) = p^+(\Omega). \tag{2.5}$$

Proof. It is known in general that any continuous function defined on an arbitrary closed set in $\mathbb{R}^n$ may be extended to the whole space $\mathbb{R}^n$ with preservation of its continuity modulus, see [21], Ch.6, section 2. This extension $\widetilde{p}$ may be realized in such a way that (2.5) is valid, see for example, [5], Theorem 4.2. To get an extension constant outside some ball, it suffices to arrange a new extension in the form

$$\widetilde{p}_*(x) = \eta\left(\frac{x}{R}\right)\widetilde{p}(x) + \left[1 - \eta\left(\frac{x}{R}\right)\right] C$$

where C is any constant such that $p^-(\Omega) \le C \le p^+(\Omega)$ and $\eta(x)$ is any C_0^∞-function with support in the ball $|x| < 2$ and equal identically to 1 in the ball $|x| < 1$, and R is sufficiently large so that $\Omega \subseteq \{x \in \mathbb{R}^n : |x| \le R\}$. (Then $\widetilde{p}_*(x) \equiv p(x)$ for $x \in \Omega$ and $\widetilde{p}_*(x) \equiv C$ for $|x| \ge 2R$).

□

Everywhere in the sequel, when Ω is unbounded, we assume that there exists the limit $p(\infty) : \lim\limits_{x\to\infty} p(x)$. In the case $p(x) \equiv$ const beyond some big ball, we use the notation $p(x) \equiv p_\infty (= p(\infty)),\ |x| > R$.

In case of unbounded domains we will also use the decay condition

$$|p(x) - p(\infty)| \le \frac{C}{\ln(1 + |x|)}, \quad x \in \Omega. \tag{2.6}$$

2.2. On maximal and convolution operators in $L^{p(\cdot)}$

Let

$$(\mathcal{M}\varphi)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\bigcap\Omega} |\varphi(y)|dy \tag{2.7}$$

be the Hardy-Littlewood maximal operator. The following theorem for the weight

$$\varrho(x) = (1 + |x|)^\gamma \prod_{k=1}^m |x - x_k|^{\beta_k}, \qquad x_k \in \Omega, \quad k = 1, 2, \ldots, m \tag{2.8}$$

was in particular proved in [12] when Ω is bounded and in [9], when Ω is unbounded.

Theorem 2.3. *Let* $p \in \mathcal{P}(\Omega) \bigcap w\text{-}Lip\,(\Omega)$ *and* ϱ *be weight of form* (2.8).
I) When Ω *is bounded, the maximal operator is bounded in* $L^{p(\cdot)}(\Omega, \varrho)$ *if and only if*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \quad k = 1, 2, ..., m. \tag{2.9}$$

II) Let Ω *be unbounded and* p *be constant outside some ball of large radius* $R > 0$*:* $p(x) \equiv p_\infty,\ |x| > R$. *The maximal operator is bounded in* $L^{p(\cdot)}(\Omega, \varrho)$ *if and only* *if condition* (2.9) *and the condition*

$$-\frac{n}{p_\infty} < \gamma + \sum_{k=1}^{m} \beta_k < \frac{n}{p'_\infty} \tag{2.10}$$

are satisfied.

By $\mathbb{P}_\varrho(\Omega)$ we denote the set os exponents $p \in \mathcal{P}(\Omega)$ such that $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega, \varrho)$.

Let $\Omega = \mathbb{R}^n$. For dilatations

$$\mathbb{K}_\varepsilon f(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) f(y)dy$$

the following weighted statement is valid.

Theorem 2.4. *Let ϱ be a weight, $\varrho^{-1} \in L^{p'(\cdot)}_{\text{loc}}$, $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $k(x)$ be an integrable function on $\mathbb{R}^n$ with $A := \int_{\mathbb{R}^n} \sup_{|y| \geq |x|} |k(y)|dx < \infty$. Then*

$$i) \qquad \left|\sup_{\varepsilon>0} \mathbb{K}_\varepsilon f(x)\right| \leq A(\mathcal{M}f)(x) \qquad \text{for all} \qquad f \in L^{p(\cdot)}(\mathbb{R}^n, \varrho),$$

so that

$$ii) \qquad \left\|\sup_{\varepsilon>0} \mathbb{K}_\varepsilon f(x)\right\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leq C_1 \|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)}$$

in the case $p(\cdot) \in \mathbb{P}_\varrho(\mathbb{R}^n)$. If in addition $\int_{\mathbb{R}^n} k(y)dy = 1$ and $\varrho(x)$ satisfies condition (2.4)*, then also*

$$iii) \qquad \mathbb{K}_\varepsilon f(x) \to f$$

as $\varepsilon \to 0$ in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ and almost everywhere.

Proof. For the non-weighted case the statement of the theorem is known, see [4]. Statement *i)* can be proved exactly as in [8] since the step functions are dense in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$; statement *ii)* is an immediate consequence of *i)*.

To prove *iii)*, observe that $C_0^\infty(\mathbb{R}^n)$ is dense in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ by Theorem (2.1). So splitting $f = f_1 + f_\varepsilon$, where $f_1 \in C_0^\infty(\mathbb{R}^n)$ and $\|f_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} < \varepsilon$, we have

$$\begin{aligned}\|\mathbb{K}_\varepsilon f - f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} &\leq \|\mathbb{K}_\varepsilon f_1 - f_1\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} + \|\mathbb{K}_\varepsilon f_\varepsilon - f_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \\ &= I_{1,\varepsilon} + I_{2,\varepsilon}.\end{aligned}$$

For $I_{2,\varepsilon}$ we obtain

$$I_{2,\varepsilon} \leq \left\|\mathbb{K}_\varepsilon f_\varepsilon\right\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} + \|f_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leq C\|f_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leq C\varepsilon. \tag{2.11}$$

The a.e. convergence $\mathbb{K}_\varepsilon f_1 \xrightarrow{a.e.} f_1$ as $\varepsilon \to 0$ with $f_1 \in C_0^\infty(\mathbb{R}^n)$ is obvious, see [21]. The boundedness of the maximal operator implies that $I_p[\varrho(\mathcal{M}f)] < \infty$. Using *i)* and Lebesgue dominated convergence theorem we have that $\lim_{\varepsilon\to 0} I_p[\varrho(\mathbb{K}_\varepsilon f_1 - f_1)] = 0$, thus showing that $I_{1,\varepsilon} \xrightarrow{\varepsilon\to 0} 0$. Thus we have convergence in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$-norm and a.e. convergence. □

By Theorem (2.4), the boundedness in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ of the maximal operator guarantees the boundedness of convolution operators

$$Af(x) = \int\limits_{\mathbb{R}^n} k(y) f(x-y)dy$$

whose kernels $k(x)$ have decreasing integrable dominants. However, the boundedness of the maximal operator requires in general the local log-condition (2.3). Meanwhile, for rather "nice" kernels $k(x)$ this condition may be avoided. Namely, in [7] the following result was obtained.

Theorem 2.5. *Let $k(y)$ satisfy the estimate $|k(y)| \leq \frac{C}{(1+|y|)^\nu}$, $y \in \mathbb{R}^n$ for some $\nu > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$. Then the convolution operator A is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n)$ to the space $L^{q(\cdot)}(\mathbb{R}^n)$ under the only assumption that the exponents $p, q \in \mathcal{P}_1(\mathbb{R}^n)$ satisfy decay condition (2.6) and $q(\infty) \geq p(\infty)$.*

2.3. Boundedness of potential and singular operators in weighted $L^{p(\cdot)}$-spaces

The following result is known, see [12], where Theorem (2.6) was stated for the single power weight; its validity for a finite product of power weights is reduced to the case of a single weight by the standard introduction of the unity partition. For the completeness of presentation we give details of such a reduction in Appendix, see Section (5).

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $\alpha(x) \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_\Omega \alpha(x) > 0$, let $p \in \mathcal{P}(\Omega) \bigcap w$-$Lip(\Omega)$ and let ϱ be weight of form (2.8) with $x_k \in \overline{\Omega}$. Under condition (2.9) the operator*

$$I_\varrho^{\alpha(\cdot)} f(x) = \varrho(x) \int_\Omega \frac{f(y)}{\varrho(y)|x-y|^{n-\alpha(x)}} dy$$

is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following theorem on the boundedness of the singular operator

$$S\varphi(t) = \frac{1}{\pi} \int\limits_a^b \frac{\varphi(t)\, dt}{t-x}, \qquad x \in (a,b)$$

was proved in [11].

Theorem 2.7. *Let* $-\infty < a < b < \infty$ *and let* $p \in \mathcal{P}(a,b) \bigcap w\text{-}Lip\,(a,b)$. *The operator* S *is bounded in the space* $L^{p(\cdot)}[(a,b),\varrho]$, *where* ϱ *is weight* (2.8) *with* $x_k \in [a,b], \quad k = 1,2,\dots,m$, *if and only if*

$$-\frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}, \qquad k = 1,2,\dots,m.$$

3. Hardy-type inequalities in variable exponent setting

3.1. Definition and assumptions

Let now $n = 1$ and $\Omega = [a,b]$, where $-\infty < a < b \le \infty$, and consider the space $L^{p(\cdot)}[(a,b),\varrho]$ with the weight

$$\varrho(x) = \begin{cases} |x-a|^{\mu(x)}|b-x|^{\nu(x)} & \text{when } b < \infty \\ |x-a|^{\mu(x)}(1+|x|)^{\nu(x)} & \text{when } b = \infty \end{cases}, \tag{3.1}$$

where the exponents $\mu(x), \nu(x)$ are bounded functions which have finite limits $\mu(a) = \lim\limits_{x\to a} \mu(x), \nu(b) = \lim\limits_{x\to b} \nu(x)$. We need the following notation for the class of exponents.

Definition 3.1. Let $\Omega = (a,b)$, where $-\infty \le a < b \le \infty$ and let $x_0 \in [a,b]$. By $w\text{-}Lip_{x_0}(\Omega)$ we denote the class

$$w\text{-}Lip_{x_0}(\Omega) = \left\{ \mu \in L^\infty(\Omega) : |\mu(x) - \mu(x_0)| \le \frac{A}{\ln \frac{1}{|x-x_0|}}, \quad |x - x_0| \le \frac{1}{2} \right\}, \tag{3.2}$$

in case $x_0 \ne \infty$, and

$$w\text{-}Lip_\infty(\Omega) = \left\{ \mu \in L^\infty(\Omega) : |\mu(x) - \mu(\infty)| \le \frac{A}{\ln(2+|x|)} \right\}. \tag{3.3}$$

For $\mu \in w\text{-}Lip_a(a,b) \bigcap w\text{-}Lip_b(a,b)$ with $-\infty < a < b < \infty$ one has

$$|x-a|^{\mu(x)}|b-x|^{\nu(x)} \approx |x-a|^{\mu(a)}|b-x|^{\nu(b)}. \tag{3.4}$$

Similarly, for $\mu \in w\text{-}Lip_a(\mathbb{R}^1) \bigcap w\text{-}Lip_b(\mathbb{R}^1) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$

$$|x-a|^{\mu(x)}|b-x|^{\nu(x)} \approx |x-a|^{\mu(a)}|b-x|^{\nu(b)}(1+|x|)^{\mu(\infty)+\nu(\infty)-\mu(a)-\nu(b)}. \tag{3.5}$$

Remark 3.2. From Theorem (2.1) it is easy to derive that the class $C_0^\infty((a,b))$ of infinitely differentiable functions with support in (a,b), $-\infty < a < b < \infty$ is dense in the space $L^{p(\cdot)}[(a,b),\varrho]$ with the weight (3.1), if $p \in \mathcal{P}_1(a,b)$ and $\mu(a)p(a) > -1, \nu(b)p(b) > -1$.

Everywhere in the sequel we assume that

$$p(x) \equiv p_\infty = const \quad \text{for large} \quad |x| \ge R \qquad \text{in the case} \quad b = \infty. \tag{3.6}$$

3.2. Hardy inequalities

The following proposition was proved in [7] (see Theorem 3.3 there).

Proposition 3.3. *Let* $p, r \in \mathcal{P}_1(\mathbb{R}^1_+) \bigcap w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$,

$$p(0) = r(0) \quad \text{and} \quad p(\infty) = r(\infty). \tag{3.7}$$

and $\alpha, \beta \in w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$, *The Hardy operators*

$$\mathcal{H}^{\alpha(\cdot)} f(x) = x^{\alpha(x)-1} \int_0^x \frac{f(y)}{y^{\alpha(y)}} dy \quad \text{and} \quad \mathcal{H}_{\beta(\cdot)} f(x) = x^{\beta(x)} \int_x^\infty \frac{f(y)}{y^{\beta(y)+1}} dy \tag{3.8}$$

are bounded from the space $L^{p(\cdot)}(\mathbb{R}^1_+)$ *into* $L^{r(\cdot)}(\mathbb{R}^1_+)$, *if*

$$\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)} \qquad \text{and} \qquad \beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) < -\frac{1}{p(\infty)}.$$

We need the following weighted statement derived from Proposition (3.3).

Theorem 3.4. *Let* $p, r \in \mathcal{P}_1(\mathbb{R}^1_+) \bigcap w\text{-}Lip_0(\mathbb{R}^1) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$ *and condition* (3.7) *be satisfied, let* $0 < b < \infty$ *and* $\varrho(x) = x^{\mu(x)}|x-b|^{\nu(x)}$, $x > 0$, *where* $\mu \in w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$, $\nu \in w\text{-}Lip_b(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$ *and*

$$-\frac{1}{r(b)} < \nu(b) < \frac{1}{p'(b)}. \tag{3.9}$$

Let also $\alpha, \beta \in w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$. *Then the Hardy-type inequalities*

$$\left\| x^{\alpha(x)-1} \int\limits_0^x \frac{f(y)}{y^{\alpha(y)}} dy \right\|_{L^{r(\cdot)}(\mathbb{R}^1_+, \varrho)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+, \varrho)} \tag{3.10}$$

and

$$\left\| x^{\beta(x)} \int\limits_x^\infty \frac{f(y)}{y^{\beta(y)+1}} dy \right\|_{L^{r(\cdot)}(\mathbb{R}^1_+, \varrho)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+, \varrho)} \tag{3.11}$$

are valid if

$$\alpha(0) + \mu(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) + \mu(\infty) + \nu(\infty) < \frac{1}{p'(\infty)} \tag{3.12}$$

and

$$-\frac{1}{p(0)} < \beta(0) + \mu(0), \quad -\frac{1}{p(\infty)} < \beta(\infty) + \mu(\infty) + \nu(\infty), \tag{3.13}$$

respectively.

Proof. For (3.10) we have to show that the operator

$$Bf(x) = x^{\alpha(x)+\mu(x)-1}|x-b|^{\nu(x)} \int_0^x \frac{f(y)\, dy}{y^{\alpha(y)+\mu(y)}|y-b|^{\nu(y)}}$$

is bounded from $L^{p(\cdot)}(\mathbb{R}^1_+)$ to $L^{r(\cdot)}(\mathbb{R}^1_+)$. We have

$$\|Bf\|_{L^{r(\cdot)}(\mathbb{R}^1_+)} \leq \|Bf\|_{L^{r(\cdot)}\left(0, \frac{b}{2}\right)} + \|Bf\|_{L^{r(\cdot)}\left(\frac{b}{2}, 2b\right)} + \|Bf\|_{L^{r(\cdot)}(2b, +\infty)}.$$

For $0 < x < \frac{b}{2}$, we have $|Bf(x)| \le Cx^{\alpha(0)+\mu(0)-1} \int_0^x \frac{f(y)}{y^{\alpha(0)+\mu(0)}} dy$ so that $\|Bf\|_{L^{r(\cdot)}(0,\frac{b}{2})}$ is covered by Proposition (3.3). For $\frac{b}{2} \le x \le 2b$ we have

$$|Bf(x)| \le C|x-b|^{\nu(b)} \left(\int_0^{\frac{b}{2}} \frac{|f(y)|}{y^{\alpha(0)+\mu(0)}} dy + \int_{\frac{b}{2}}^{2b} \frac{|f(y)|}{|y-b|^{\nu(b)}} dy \right),$$

where both the integrals are finite by the Hölder inequality, and $|x-b|^{\nu(b)} \in L^{r(\cdot)}\left(\left[\frac{b}{2}, 2b\right]\right)$. Finally, when $x > 2b$, we get

$$|Bf(x)| \le Cx^{\alpha(\infty)+\mu(\infty)+\nu(\infty)-1} \left(\int_0^{\frac{b}{2}} \frac{|f(y)|\, dy}{y^{\alpha(0)+\mu(0)}} + \int_{\frac{b}{2}}^{2b} \frac{|f(y)|\, dy}{|y-b|^{\nu(b)}} + \int_{2b}^{x} \frac{|f(y)|\, dy}{y^{\alpha(\infty)+\mu(\infty)+\nu(\infty)}} \right)$$

where the first two integrals are finite by the Hölder inequality and $x^{\alpha(\infty)+\mu(\infty)+\nu(\infty)-1} \in L^{r(\cdot)}(2b,\infty)$, while the last term is dominated by $x^{\alpha(\infty)+\mu(\infty)+\nu(\infty)-1} \int_0^x \frac{|f(y)|\, dy}{y^{\alpha(\infty)+\mu(\infty)+\nu(\infty)}}$, which is covered by Proposition (3.3).

Similarly one can prove inequality (3.11). □

Remark 3.5. Theorem (3.4) is also valid for the weight $\varrho(x) = x^{\mu(x)} \prod_{k=1}^{m} |x-b_k|^{\nu_k(x)}$, where $0 < b_1 < b_2 < \cdots < b_m < \infty$, under natural modification. Namely, besides assumptions (3.12)-(3.13), the following conditions should be imposed

$$-\frac{1}{r(b_k)} < \nu_k(b_k) < \frac{1}{p'(b_k)}, \qquad k = 1, \dots, m;$$

$$\alpha(\infty)+\mu(\infty)+\sum_{k=1}^{m} \nu_k(\infty) < \frac{1}{p'(\infty)} \quad \text{and} \quad -\frac{1}{p(\infty)} < \beta(\infty)+\mu(\infty)+\sum_{k=1}^{m} \nu_k(\infty).$$

3.3. On singular operators with fixed singularity

Theorem 3.6. *Let $p, r \in \mathcal{P}_1(\mathbb{R}^1_+) \bigcap w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$ and condition (3.7) be satisfied, let $0 < b < \infty$ and $\varrho(x) = x^{\mu(x)}|x-b|^{\nu(x)}$, $x > 0$, where $\mu \in w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$ and $\nu \in w\text{-}Lip_b(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$. Let also $\beta \in w\text{-}Lip_0(\mathbb{R}^1_+) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$. Then the operator*

$$H^{\beta(\cdot)}\varphi(x) := x^{\beta(x)} \int_0^\infty \frac{\varphi(t)}{t^{\beta(t)}(x+t)} dt \tag{3.14}$$

is bounded from the space $L^{p(\cdot)}(\mathbb{R}^1_+, \varrho)$ into $L^{r(\cdot)}(\mathbb{R}^1_+, \varrho)$, if

$$-\frac{1}{r(b)} < \nu(b) < \frac{1}{p'(b)}, \quad -\frac{1}{p(0)} < \beta(0)+\mu(0) < \frac{1}{p'(0)} \tag{3.15}$$

and

$$-\frac{1}{p(\infty)} < \beta(\infty)+\mu(\infty)+\nu(\infty) < \frac{1}{p'(\infty)}. \tag{3.16}$$

Proof. Since

$$H^\beta f(x) \le \mathcal{H}^\beta f(x) + \mathcal{H}_\beta f(x) \tag{3.17}$$

where $\mathcal{H}^\beta$ and $\mathcal{H}_\beta$ are the Hardy operators (3.8), Theorem (3.6) immediately follows from Theorem (3.4). □

Remark 3.7. In the case of a similar operator $H^\beta\varphi(x) =: \int_0^\ell \left(\frac{x}{t}\right)^\beta \frac{\varphi(t)}{x+t} dt, 0 < x < \ell < \infty$ on a finite interval, Theorem (3.6) is valid without condition (3.16). Note that a weaker version of Theorem (3.6) for a finite interval was proved in [12].

3.4. On a Hardy-Littlewood inequality

The following extension of the Hardy-Littlewood inequality to the case of variable exponents is valid. In the case where the exponent p is constant, this inequality is well known, being due to Hardy and Littlewood, see for instance [19], p. 104-106 (we take this opportunity to note that there are misprints on p. 104 in formulas (5.45)-(5.46): there should be $x^{-\alpha p}$ instead of $x^{\alpha p}$). In the case of variable p, an inequality of Hardy type for the multidimensional Riesz-type potentials over bounded domains in $\mathbb{R}^n$ with the weight $|x - x_0|^\beta$ was proved in [17] in the case $0 < \alpha < n,\ \alpha - \frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}$. We admit infinite intervals (a, b) and thanks to the unilateral structure of the Riemann-Liouville integral we can consider an arbitrary $\alpha > 0$ and the weight exponents in the interval $\left(-\frac{1}{p(x_0)}, \frac{1}{p'(x_0)}\right)$.

Theorem 3.8. *Let* $\alpha > 0,\ -\infty < a < b < \infty$*,* $p \in \mathcal{P}(a, b) \bigcap w\text{-}Lip\,(a, b)$ *and* ϱ *be weight of form* (3.1) *with* $\mu \in w\text{-}Lip_a(a, b), \nu \in w\text{-}Lip_b(a, b)$*. Then*

$$\left\| \frac{1}{(x-a)^\alpha} \int\limits_a^x \frac{\varphi(t)\, dt}{(x-t)^{1-\alpha}} \right\|_{L^{p(\cdot)}[(a,b),\varrho]} \le C \, \|\varphi\|_{L^{p(\cdot)}[(a,b),\varrho]} \tag{3.18}$$

under the conditions:

$$-\frac{1}{p(a)} < \mu(a) < \frac{1}{p'(a)}, \quad -\frac{1}{p(b)} < \nu(b) < \frac{1}{p'(b)}. \tag{3.19}$$

Inequality (3.18) *is also valid in the case* $b = \infty$*, if additionally* $p(x)$ *satisfies assumption* (3.6)*,* $\mu, \nu \in w\text{-}Lip_\infty(a, \infty)$ *and the second condition in* (3.19) *is replaced by*

$$-\frac{1}{p_\infty} < \nu(\infty) + \mu(\infty) < \frac{1}{p'_\infty}. \tag{3.20}$$

Proof. The proof follows the principal idea in [17], but uses the unilateral nature of the one-dimensional integration. Let $a = 0$ for simplicity. We continue $\varphi(t)$ as zero beyond the interval $(0, b)$ and have

$$\frac{1}{x^\alpha} \int\limits_0^x \frac{\varphi(t)\, dt}{(x-t)^{1-\alpha}} = \frac{1}{x} \int\limits_{\mathbb{R}^1} \mathcal{L}\left(\frac{x-t}{x}\right) \varphi(t)\, dt, \quad x > 0, \tag{3.21}$$

where $\mathcal{L}(\xi) = \begin{cases} \xi^{\alpha-1}, & \xi \in [0,1] \\ 0, & \xi \notin [0,1] \end{cases}$. The right-hand side in (3.21) may be extended for all $x \in \mathbb{R}^1$, if in the denominator we replace x by $|x|$. Then we have

$$\frac{1}{|x|} \int\limits_{\mathbb{R}^1} \mathcal{L}\left(\frac{x-t}{|x|}\right) \varphi(t)\, dt \le C\mathcal{M}\varphi(x), \tag{3.22}$$

where the domination by the maximal operator is possible by the pointwise inequality *i)* of Theorem (2.4). Then from (3.21) and (3.22) we obtain

$$\left\| \frac{1}{x^\alpha} \int\limits_0^x \frac{\varphi(t)\, dt}{(x-t)^{1-\alpha}} \right\|_{L^{p(\cdot)}[(0,b),\varrho]} \le C\|\mathcal{M}\varphi\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\varrho_*)}$$

where $\widetilde{p}(x)$ is an extension of $p(x)$ from (a,b) to $\mathbb{R}^1$ provided by Lemma (2.2). An extension $\varrho_*(x)$ of the weight may be taken, according to (3.4)-(3.5), as

$$\varrho_*(x) = \begin{cases} |x-a|^{\mu(a)}|b-x|^{\nu(b)}, & b < \infty, \\ |x-a|^{\mu(a)}(1+|x|)^{\mu(\infty)-\mu(a)+\nu(\infty)}, & b = \infty \end{cases}, \tag{3.23}$$

With this extension, the maximal operator is bounded in the space $L^{\widetilde{p}(\cdot)}(\mathbb{R}^1, \varrho_*)$ by Theorem (2.3) and we arrive at (3.18). □

4. On fractional integrals and derivatives in $L^{p(\cdot)}[(a,b),\varrho]$

4.1. On Marchaud derivative

The Marchaud fractional derivative ([19], p. 200)

$$\mathbb{D}^\alpha_{a+} f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}} dt, \tag{4.1}$$

of order $0 < \alpha < 1$, for "not so nice" functions $f(x)$ is understood as

$$\lim_{\varepsilon\to 0} \mathbb{D}^\alpha_{a+,\varepsilon} f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon\to 0} \int_a^{x-\varepsilon} \frac{f(x)-f(t)}{(x-t)^{1+\alpha}} dt, \quad \varepsilon > 0,$$

where $f(x)$ is assumed to be continued as zero beyond the interval $[a,b]$. It is known ([19], p. 200) that

$$\mathbb{D}^\alpha_{a+,\varepsilon} f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \mathbb{A}_\varepsilon f(x), \tag{4.2}$$

where

$$\mathbb{A}_\varepsilon f(x) = \int_a^{x-\varepsilon} \frac{f(x)-f(t)}{(x-t)^{1+\alpha}} dt \quad \text{for} \quad a+\varepsilon \le x \le b, \tag{4.3}$$

$$\mathbb{A}_\varepsilon f(x) = \frac{f(x)}{\alpha} \left[\frac{1}{\varepsilon^\alpha} - \frac{1}{(x-a)^\alpha}\right] \quad \text{for} \quad a \le x \le a+\varepsilon. \tag{4.4}$$

Lemma 4.1. *Let* $-\infty < a < b < \infty$*,* $\alpha > 0$*, let* $p \in \mathcal{P}(a,b)$ *and* ϱ *be weight of form* (3.1) *with* $\mu \in w\text{-}Lip_a(a,b)$*,* $\nu \in w\text{-}Lip_b(a,b)$*. The truncated fractional differentiation operator* $\mathbb{D}^\alpha_{a+,\varepsilon} f$ *is bounded in* $L^{p(\cdot)}[(a,b),\varrho]$ *for any fixed* $\varepsilon > 0$ *under conditions* (3.19)*.*

This is also valid for $b = \infty$*, if we additionally assume that* $\mu, \nu \in w\text{-}Lip_\infty(a,\infty)$ *and* $p(\cdot) \in \mathbb{P}_\varrho[(a,\infty)]$*; for the latter inclusion, the following conditions are sufficient:* $p \in \mathcal{P}(a,\infty) \bigcap w\text{-}Lip\,(a,\infty)$*, and* (3.6)*,* (3.20) *and the first condition in* (3.19) *hold.*

Proof. After easy calculations we obtain

$$\mathbb{D}^\alpha_{a+,\varepsilon} f(x) = \frac{f(x)}{\Gamma(1-\alpha)\varepsilon^\alpha} - \frac{\alpha \chi_{[a+\varepsilon,b]}(x)}{\Gamma(1-\alpha)} \int\limits_\varepsilon^{x-a} \frac{f(x-t)\,dt}{t^{1+\alpha}}, \qquad x \in [a,b],$$

where the second term is bounded in $L^{p(\cdot)}[(a,b),\varrho]$. Indeed, for $x > a+\varepsilon$ by the Hölder inequality we have $\int\limits_\varepsilon^{x-a} \frac{f(x-t)\,dt}{t^{1+\alpha}} \leq \frac{1}{\varepsilon^{1+\alpha}} \|f\|_{L^{p(\cdot)}[(a,b),\varrho]} \cdot \left\|\varrho^{-1}\right\|_{L^{p'(\cdot)}(a,b)}$ where the last factor is finite under the conditions $\mu(a)p'(a) < 1$, $\nu(b)p'(b) < 1$, in the case of finite b.

In the case $b = \infty$ we have

$$\int_\varepsilon^{x-a} \frac{f(x-t)}{t^{1+\alpha}} dt \leq \int_{\mathbb{R}} k(t) f(x-t) dt \tag{4.5}$$

with $k(t) = t^{-1-\alpha} \cdot \chi_{(\varepsilon,\infty)}(t)$ and then the boundeness follows by Theorem (2.4) when $p(\cdot) \in \mathbb{P}_\varrho[(a,\infty)]$. The sufficiency of the conditions for the latter inclusion, mentioned in the theorem, follows from (4.5) and Theorem (2.3). □

Remark 4.2. The statement of Lemma (4.1) for $b = \infty$ in the case $\mu(a) = \mu(\infty) + \nu(\infty) = 0$ is valid for an arbitrary $p \in \mathcal{P}(a,\infty)$ satisfying condition (2.6). This follows from (4.5) and Theorem (2.5)

4.2. The left-hand side inverse operator to the Riemann-Liouville operator I^α_{a+}

When considering the operator left inverse to I^α_{a+}, we may not follow the same lines as in the known proof for the case of constant p, see [19], Section 13, since the proof there uses the p-mean continuity of the L^p space, which is no more valid in the case of variable p, see [13]. Thus we have to modify the arguments from [19] and make use of the maximal operator.

Theorem 4.3. *Let* $-\infty < a < b < \infty$*,* $0 < \alpha < 1$ *and*

$$f = I^\alpha_{a+}\varphi, \quad \varphi \in L^{p(\cdot)}[(a,b),\varrho],$$

where $p \in \mathcal{P}(a,b) \bigcap w\text{-}Lip\,(a,b)$ *and* ϱ *is weight* (3.1) *with* $\mu \in w\text{-}Lip_a(a,b)$*,* $\nu \in w\text{-}Lip_b(a,b)$*. Then*

$$\mathbb{D}^\alpha_{a+} f = \varphi,$$

where $\mathbb{D}^\alpha_{a+}f = \lim\limits_{\varepsilon\to 0}\mathbb{D}^\alpha_{a+,\varepsilon}f$ *with the limit in the norm of the space* $L^{p(\cdot)}[(a,b),\varrho]$, *under conditions* (3.19).

This is also valid in the case $b=\infty$, *if additionally* $\mu,\nu \in w\text{-}Lip_\infty(a,b)$ *and* (3.6) *and* (3.20) *hold.*

Proof. Without loss of generality we take $a=0$. We need to show that

$$\lim_{\varepsilon\to 0} \| \mathbb{D}^\alpha_{0+,\varepsilon}f-\varphi \|_{L^{p(\cdot)}[(0,b),\varrho]} = 0. \tag{4.6}$$

In [19], p. 227-228, there was proved the following representation

$$\mathbb{D}^\alpha_{0+,\varepsilon}I^\alpha_{0+}\varphi = \begin{cases} \displaystyle\int\limits_0^{\frac{x}{\varepsilon}} \mathcal{K}(t)\varphi(x-\varepsilon t)dt =: A_\varepsilon\varphi(x), & \varepsilon\le x\le b \\ \displaystyle\frac{\sin\alpha\pi}{\pi\varepsilon^\alpha}\int\limits_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt =: B_\varepsilon\varphi(x), & 0\le x\le\varepsilon, \end{cases} \tag{4.7}$$

with

$$\mathcal{K}(t) = \frac{\sin\alpha\pi}{\pi}\frac{t^\alpha_+-(t-1)^\alpha_+}{t}, \qquad t^\alpha_+ = \begin{cases} t^\alpha, & t>0 \\ 0, & t<0 \end{cases}, \tag{4.8}$$

valid for $\varphi\in L^p$, where $1\le p<\infty$, and therefore valid for "nice" functions.

In the sequel the function $\varphi(t)$ is assumed to be continued as zero beyond $[a,b]$ whenever necessary, so that $A_\varepsilon(x)$ and $B_\varepsilon(x)$ are well defined on the whole line $\mathbb{R}^1$.

By Remark (3.2), "nice" functions are dense in $L^{p(\cdot)}[(a,b),\varrho]$, so that to verify (4.7) on $L^{p(\cdot)}[(a,b),\varrho]$, we only need to check the boundedness of all the operators involved in (4.7). The operators $\mathbb{D}^\alpha_{0+,\varepsilon}$ and I^α_{0+} are bounded by Lemma (4.1) and Theorem (2.6), respectively. The operator A_ε is bounded by Theorem (2.4) (after the corresponding extension of p and ϱ to the whole line $\mathbb{R}^1$). In the case $b=\infty$, it suffices to have the boundedness on any $(a,N), N<\infty$, since all the operators are of Volterra type.

Note that the kernel $\mathcal{K}(t)$ has a radial integrable decreasing majorant, so that by Theorem (2.4)

$$|A_\varepsilon\varphi(x)| \le C\mathcal{M}\varphi(x). \tag{4.9}$$

Representation (4.7) may be rewritten as

$$\mathbb{D}^\alpha_{0+,\varepsilon}f(x) = \chi_{[\varepsilon,b]}(x)A_\varepsilon\varphi(x) + \chi_{[0,\varepsilon]}(x)B_\varepsilon\varphi(x), \qquad x\in[0,b] \tag{4.10}$$

and then

$$\mathbb{D}^\alpha_{0+,\varepsilon}f(x)-\varphi(x) = A_\varepsilon\varphi(x)-\varphi(x)+\chi_{[0,\varepsilon]}(x)[B_\varepsilon\varphi(x)-A_\varepsilon\varphi(x)], \quad x\in[0,b]. \tag{4.11}$$

By Part *iii)* of Theorem (2.4)

$$\|A_\varepsilon\varphi-\varphi\|_{L^{p(\cdot)}[(0,b),\varrho]} \le \left\| \frac{1}{\varepsilon}\int_{\mathbb{R}^1}\mathcal{K}\left(\frac{x-\xi}{\varepsilon}\right)\varphi(\xi)d\xi - \varphi(x)\right\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^1,\varrho_*)} \to 0 \tag{4.12}$$

where the extension $\widetilde{p}(x)$ of $p(x)$ has been chosen according to Lemma (2.2) and the extension $\varrho_*(x)$ of the weight is defined in (3.23). The condition $\widetilde{p} \in \mathbb{P}_\varrho(\mathbb{R}^1)$ of Theorem (2.4) is satisfied according to Theorem (2.3). The term $\chi_{[0,\varepsilon]}(x)[B_\varepsilon\varphi(x) - A_\varepsilon\varphi(x)]$ is estimated uniformly in ε by the maximal function:

$$\chi_{[0,\varepsilon]}(x)|B_\varepsilon\varphi(x) - A_\varepsilon\varphi(x)| \leq C\chi_{[0,\varepsilon]}(x)\mathcal{M}\varphi(x). \tag{4.13}$$

Indeed, taking into account the inequality $\frac{1}{\varepsilon^\alpha} \leq \frac{1}{x^\alpha}$ for $0 < x < \varepsilon$ and the estimate obtained in (3.21)-(3.22) and estimate (4.9), we get

$$|B_\varepsilon\varphi(x) - A_\varepsilon\varphi(x)| \leq \frac{\sin\alpha\pi}{\pi x^\alpha}\int_0^x \frac{\varphi(y)\,dy}{(x-y)^{1-\alpha}} + C\mathcal{M}\varphi(x) \leq C_1\mathcal{M}\varphi(x).$$

From (4.13) we have

$$\|\chi_{[0,\varepsilon]}(x)[B_\varepsilon\varphi(x) - A_\varepsilon\varphi(x)]\|_{L^{p(\cdot)}[(0,b),\varrho]} \leq C\|\mathcal{M}\varphi\|_{L^{p(\cdot)}[(0,\varepsilon),\varrho]} \tag{4.14}$$

which tends to zero as $\varepsilon \to 0$ by the boundedness of the maximal operator.

It remains to conclude that from (4.11) there follows (4.6) by (4.12) and (4.14). □

4.3. The range $I^\alpha_{a+}\left[L^{p(\cdot)}[(a,b),\varrho]\right]$ of the fractional integration operator

In the next theorem, we derive necessary and sufficient conditions for the representability of a function $f(x)$ by the fractional integral of a function in $L^{p(\cdot)}[(a,b),\varrho]$.

Theorem 4.4. *Let $-\infty < a < b < \infty$ and $p \in \mathcal{P}(a,b) \bigcap w\text{-}Lip\,(a,b)$, let ϱ be weight of form (3.1) with $\mu \in w\text{-}Lip_a(a,b)$, $\nu \in w\text{-}Lip_b(a,b)$ and let conditions (3.19) be satisfied.*

In order that a function $f(x)$ be representable as $f = I^\alpha_{a+}\varphi$ with $\varphi \in L^{p(\cdot)}[(a,b),\varrho]$, it is necessary and sufficient that $f \in L^{p(\cdot)}[(a,b),\varrho]$ and there exists $\lim\limits_{\varepsilon\to 0}\mathbb{A}_\varepsilon f(x)$ in $L^{p(\cdot)}[(a,b),\varrho]$ where $\mathbb{A}_\varepsilon f(x)$ is the function defined in (4.3)-(4.4).

This statement remains valid in the case $b = \infty$, if the condition $f \in L^{p(\cdot)}[(a,b),\varrho]$ is replaced by

$$\frac{f(x)}{(x-a)^\alpha} \in L^{p(\cdot)}[(a,\infty),\varrho] \tag{4.15}$$

and we additionally assume that $\mu,\nu \in w\text{-}Lip_\infty(a,\infty)$ and conditions (3.6) and (3.20) hold.

Proof. Necessity part is a consequence of Theorems (4.3), (3.8) and (2.6) because by (4.2) we have

$$\mathbb{A}_\varepsilon f = \mathbb{D}^\alpha_{a+,\varepsilon}f - \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha}.$$

The necessity of condition (4.15) follows from (3.18).

Sufficiency part. Let $a = 0$ for simplicity. Given $f \in L^{p(\cdot)}[(0,b),\varrho]$, we introduce the functions

$$\mathbb{B}_\varepsilon f(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\mathbb{A}_\varepsilon f \tag{4.16}$$

where the limit $\lim_{\varepsilon\to 0} \mathbb{A}_\varepsilon f(x)$ exists in $L^{p(\cdot)}[(0,b),\varrho]$ by assumption. Observe that $\frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} \in L^{p(\cdot)}[(a,b),\varrho]$, which follows from the fact that $\mathbb{A}_\varepsilon f \in L^{p(\cdot)}[(a,b),\varrho]$, see (4.4). After some transformations we arrive at the representation

$$I^\alpha_{0+}\mathbb{B}_\varepsilon f(x) = \begin{cases} \int\limits_0^{\frac{x}{\varepsilon}} \mathcal{K}(y) f(x-\varepsilon y)dy, & \varepsilon \le x \le b; \\ \frac{\sin\alpha\pi}{\pi\varepsilon^\alpha} \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, & 0 \le x \le \varepsilon. \end{cases} \tag{4.17}$$

similar to (4.7), see details of those transformations in [19], p. 229-230. Observe that in [19] this representation was justified for $f \in L^p$ with constant p; therefore (4.17) is valid for "nice" functions dense in $L^{p(\cdot)}[(a,b),\varrho]$ and consequently for all $f \in L^{p(\cdot)}[(a,b),\varrho]$ thanks to the boundedness of all the operators involved in (4.17) (with fixed $\varepsilon > 0$, see the arguments in the proof of Theorem (4.3) after (4.8)) .

Since $\{\mathbb{A}_\varepsilon f\}$ is convergent in $L^{p(\cdot)}[(a,b),\varrho]$ as $\varepsilon \to 0$, then $\mathbb{B}_\varepsilon f(x)$ converges in $L^{p(\cdot)}[(a,b),\varrho]$ to $\varphi(x) \in L^{p(\cdot)}[(a,b),\varrho]$, where

$$\varphi(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon\to 0} \mathbb{A}_\varepsilon \varphi(x).$$

We need to show that $f = I^\alpha_{0+}\varphi$. Since the operator I^α_{a+} is continuous in $L^{p(\cdot)}[(a,b),\varrho]$ by Theorem (2.6), it is sufficient to prove that $f = \lim_{\varepsilon\to 0} I^\alpha_{0+}\mathbb{B}_\varepsilon f$. To this end, we have to show that the right-hand side of (4.17) tends to f as $\varepsilon \to 0$ in the norm of the space $L^{p(\cdot)}[(a,b),\varrho]$, which is done exactly as in the proof of Theorem (4.3) thanks to the coincidence of the right-hand sides of (4.17) and (4.7). □

4.4. On the interpretation of the range $I^\alpha_{a+}\left[L^{p(\cdot)}[(a,b),\varrho]\right]$ as fractional Sobolev type

In this subsection we show that the range $I^\alpha_{a+}\left[L^{p(\cdot)}[(a,b),\varrho]\right]$ of the fractional integration operator coincides with the fractional Sobolev space on $L^{\alpha,p(\cdot)}[(a,b),\varrho]$ defined as the space of restrictions of Bessel potentials onto $[a,b]$, see Theorem (4.15).

First we observe that the ranges of the left-hand sided and right-hand sided fractional integrals coincide under the appropriate assumptions. Namely, the following theorem is valid.

Theorem 4.5. *Let* $0 < \alpha < 1$, $p \in \mathcal{P}(a,b) \bigcap w\text{-}Lip\,(a,b)$, $-\infty < a < b < \infty$, *and let* ϱ *be weight of form* (3.1) *with* $\mu \in w\text{-}Lip_a(a,b)$, $\nu \in w\text{-}Lip_b(a,b)$. *Then*

$$I^\alpha_{a+}[L^{p(\cdot)}[(a,b),\varrho]] = I^\alpha_{b-}[L^{p(\cdot)}[(a,b),\varrho]], \tag{4.18}$$

under the conditions

$$\alpha - \frac{1}{p(a)} < \mu(a) < \frac{1}{p'(a)}, \quad \alpha - \frac{1}{p(b)} < \nu(b) < \frac{1}{p'(b)}. \tag{4.19}$$

Proof. The coincidence of the ranges stated in (4.18) under conditions (4.19) follows from the known ([19], p. 206) formulas

$$I^\alpha_{b-}\varphi = I^\alpha_{a+}\left(\cos a\pi\varphi + \sin\alpha\pi r_a^{-\alpha} S r_a^\alpha \varphi\right), \tag{4.20}$$

$$I^\alpha_{a+}\varphi = I^\alpha_{b-}\left(\cos a\pi\varphi - \sin\alpha\pi r_b^{-\alpha} S r_b^\alpha \varphi\right) \tag{4.21}$$

where $r_a^{\pm\alpha}(x) = (x-a)^{\pm\alpha}$ and $r_b^{\pm\alpha}(x) = (b-x)^{\pm\alpha}$. Formulas (4.20)-(4.21) being valid for $\varphi \in C_0^\infty$, are extended to the whole space $L^{p(\cdot)}[(a,b),\varrho]$ by continuity of the operators involved. Indeed, the weighted singular operators $r_a^{-\alpha}Sr_a^\alpha,\ r_b^{-\alpha}Sr_b^\alpha$ are bounded in $L^{p(\cdot)}[(a,b),\varrho]$ by Theorem (2.7) when $-1/p(a) < \mu(a)-\alpha < 1/p'(a)$ and $-1/p(b) < \nu(b) - \alpha < 1/p'(b)$, and the fractional integrals I^α_{a+} and I^α_{b-} are bounded in $L^{p(\cdot)}[(a,b),\varrho]$ by Theorem (2.6). □

Corollary 4.6. *Let* $p \in \mathcal{P}(a,b)\bigcap w\text{-}Lip\,(a,b)$*. In the non-weighted case, the coincidence*

$$I^\alpha_{a+}[L^{p(\cdot)}(a,b)] = I^\alpha_{b-}[L^{p(\cdot)}(a,b)], \tag{4.22}$$

holds if $0 < \alpha < \min\{\frac{1}{p(a)}, \frac{1}{p(b)}\}$.

Similarly to Theorem (4.5), the following statement is proved for the whole line $\mathbb{R}^1$ with the help of the known relations ([19], p. 202) between the left-hand-sided and right-hand-sided fractional integrals via the singular operator.

Theorem 4.7. *Let* $0 < \alpha < 1$*, let* $p \in \mathcal{P}(\mathbb{R}^1)\bigcap w\text{-}Lip\,(\mathbb{R}^1)$ *satisfy condition* (3.6) *and let* $\varrho(x) = |x-a|^{\mu(x)}|b-x|^{\nu(x)}$ *with* $\mu \in w\text{-}Lip_a(\mathbb{R}^1)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$ *and* $\nu \in w\text{-}Lip_b(\mathbb{R}^1)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$*. Then under conditions* (4.19) *and the condition*

$$\alpha - \frac{1}{p_\infty} < \mu(\infty) + \nu(\infty) < \frac{1}{p'_\infty}$$

the following coincidence of the ranges holds

$$I^\alpha_+[L^{p(\cdot)}(\mathbb{R}^1,\varrho)] = I^\alpha_-[L^{p(\cdot)}(\mathbb{R}^1,\varrho)] = I^\alpha[L^{p(\cdot)}(\mathbb{R}^1,\varrho)], \tag{4.23}$$

where I^α *is the one-dimensional Riesz operator.*

The *space of Bessel potentials* is known as the range of the Bessel operator:

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f:\ f = \mathcal{B}^\alpha\varphi, \quad \varphi \in L^{p(\cdot)}(\mathbb{R}^n)\}, \quad \alpha \geq 0,$$

where $\mathcal{B}^\alpha$ is the Bessel potential operator which reduces to multiplication by $\left(1+|\xi|^2\right)^{-\frac{\alpha}{2}}$ in Fourier transforms; we refer to [2] for the study of these fractional type spaces with variable exponent, including the characterization of $\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ in terms of convergence of some hypersingular integrals.

Definition 4.8. For a domain $\Omega \subset \mathbb{R}^n$ we define the fractional Sobolev type space $L^{\alpha,p(\cdot)}(\Omega)$ as the space of restrictions onto Ω of functions $f \in \mathcal{B}^\alpha[L^{\widetilde{p}(\cdot)}(\mathbb{R}^n)]$ with some extension $\widetilde{p}(\cdot)$ of $p(\cdot)$ from Ω to $\mathbb{R}^n$:

$$L^{\alpha,p(\cdot)}(\Omega) = \mathcal{B}^\alpha[L^{\widetilde{p}(\cdot)}(\mathbb{R}^n)]\Big|_\Omega \tag{4.24}$$

and for $f = \mathcal{B}\varphi\Big|_\Omega$ define the norm by

$$\|f\|_{L^{\alpha,p(\cdot)}(\Omega)} = \inf \ \|\varphi\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R}^n)}$$

where the infimum is taken with respect to all possible φ in the representation $f = \mathcal{B}\varphi\Big|_\Omega$ and all the extensions $\widetilde{p}$.

We need a similar weighted space. To avoid complications with extension of arbitrary weights from Ω to $\mathbb{R}^n$, we restrict ourselves to the one-dimensional case and power-type weights and use their extensions in the form

$$\widetilde{\varrho}(x) = \begin{cases} |x-a|^{\widetilde{\mu}(x)}|b-x|^{\widetilde{\nu}(x)} & \text{when } \ b < \infty \\ |x-a|^{\widetilde{\mu}(x)}(1+|x|)^{\widetilde{\nu}(x)} & \text{when } \ b = \infty \end{cases}, \tag{4.25}$$

Definition 4.9. Let $-\infty < a < b \leq \infty$. We define the fractional Sobolev type space $L^{\alpha,p(\cdot)}[(a,b),\varrho]$ with weight (3.1) as

$$L^{\alpha,p(\cdot)}[(a,b),\varrho]) = \mathcal{B}^\alpha[L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})]\Big|_{(a,b)} \tag{4.26}$$

where $\widetilde{\varrho}$ is an extension of form (4.25) with $\widetilde{\mu} \in w\text{-}Lip_a(\mathbb{R}^1)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$, $\widetilde{\nu} \in w\text{-}Lip_b(\mathbb{R}^1)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$, and for $f = \mathcal{B}\varphi\Big|_{(a,b)}$ define the norm by

$$\|f\|_{L^{\alpha,p(\cdot)}((a,b),\varrho)} = \inf \ \|\varphi\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})}$$

where the infimum is taken with respect to all possible φ in the representation $f = \mathcal{B}\varphi\Big|_{(a,b)}$ and all the extensions $\widetilde{p}, \widetilde{\mu}$ and $\widetilde{\nu}$.

We refer to [3] for the notion of Banach function spaces and to [19], Section 26, for the notion of the Riesz fractional differentiation $\mathbb{D}^\alpha$ and its truncation $\mathbb{D}^\alpha_\varepsilon$, used in the following result for the Riesz and Bessel potentials.

Theorem 4.10. *Let* $\mathrm{X} = \mathrm{X}(\mathbb{R}^n)$ *be a Banach function space, satisfying the assumptions*

i) C_0^∞ *is dense in* X*;*

ii) the maximal operator $\mathcal{M}$ *is bounded in* X*;*

iii) $I^\alpha f(x)$ *converges absolutely for almost all* x *for every* $f \in \mathrm{X}$ *and* $(1+|x|)^{-n-\alpha}I^\alpha f(x) \in L^1(\mathbb{R}^n)$.

Then

$$\mathcal{B}^\alpha(\mathrm{X}) = \mathrm{X}\bigcap I^\alpha(\mathrm{X}) = \{f \in \mathrm{X} : \mathbb{D}^\alpha f = \lim_{\substack{\varepsilon\to 0\\(\mathrm{X})}} \mathbb{D}^\alpha_\varepsilon f \in \mathrm{X}\}. \tag{4.27}$$

Proof. Theorem (4.10) was proved in [2] for the case of $\mathrm{X} = L^{p(\cdot)}(\mathbb{R}^n)$. The analysis of the proof given in [2] shows that conditions i)-iii) are sufficient for that proof to hold within the frameworks of abstract Banach function spaces. □

As a corollary to Theorem (4.10), we obtain the following result for the case $X = L^{p(\cdot)}(\mathbb{R}^n, \varrho)$, important for the sequel.

Theorem 4.11. *Let* $\Omega = \mathbb{R}^n$, *let* $p \in \mathcal{P}(\mathbb{R}^n)\bigcap w\text{-}Lip\,(\mathbb{R}^n)$ *satisfy condition* (3.6) *in* $\mathbb{R}^n$, *let* ϱ *be weight of form* (2.8) *with the exponents satisfying condition* (2.9) *and the condition*

$$\alpha - \frac{n}{p_\infty} < \gamma + \sum_{k=1}^m \beta_k < \frac{n}{p'_\infty}. \tag{4.28}$$

Then

$$\begin{aligned} L^{\alpha,p(\cdot)}(\mathbb{R}^n,\varrho) &= L^{p(\cdot)}(\mathbb{R}^n,\varrho)\bigcap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n,\varrho)] \\ &= \{f \in L^{p(\cdot)}(\mathbb{R}^n,\varrho) : \mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n,\varrho)\}. \end{aligned} \tag{4.29}$$

Proof. Theorem (4.11) follows from Theorem (4.10). Indeed, condition *i)* of Theorem (4.10) for the space $X = L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ is fulfilled by Theorem (2.1), condition *ii)* is satisfied by Theorem (2.3). Condition *iii)* is checked directly due to the known pointwise estimate

$$|I^\alpha \varphi(x)| \le C\mathcal{M}\varphi(x) + C \int\limits_{|x-y|>1} \frac{|\varphi(y)|\,dy}{|x-y|^{n-\alpha}},$$

where the second term is easily estimated by direct application of the Hölder inequality. □

The next auxiliary Theorems (4.12) and (4.14) provide preliminary facts necessary for the main result of this subsection given in Theorem (4.15).

Theorem 4.12. *Let* $\varphi \in L^{p(\cdot)}(\mathbb{R}^1, \varrho)$, *where* $p \in \mathcal{P}_1(\mathbb{R}^1_+)\bigcap w\text{-}Lip_0(\mathbb{R}^1)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$, $\varrho(x) = |x|^{\mu(x)}|x-b|^{\nu(x)}, 0 < b < \infty$, *where* $\mu \in w\text{-}Lip_0(\mathbb{R}^1_+)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+), \nu \in w\text{-}Lip_b(\mathbb{R}^1_+)\bigcap w\text{-}Lip_\infty(\mathbb{R}^1_+)$. *Then*

$$\chi_{[0,\infty]} I^\alpha_+ \varphi = I^\alpha_+ \psi \tag{4.30}$$

where

$$\psi(x) = \begin{cases} \varphi(x) + \frac{\sin\alpha\pi}{\pi} \displaystyle\int_0^\infty \left(\frac{t}{x}\right)^\alpha \frac{\varphi(-t)}{x+t} dt, & x > 0 \\ 0, & x < 0 \end{cases} \in L^{p(\cdot)}(\mathbb{R}^1, \varrho), \tag{4.31}$$

if the following conditions are satisfied:

$$-\frac{1}{p(b)} < \nu(b) < \frac{1}{p'(b)}, \quad \alpha - \frac{1}{p(0)} < \mu(0) < \frac{1}{p'(0)} \tag{4.32}$$

and

$$\alpha - \frac{1}{p(\infty)} < \mu(\infty) + \nu(\infty) < \frac{1}{p'(\infty)}. \tag{4.33}$$

Proof. Representation (4.30)-(4.31) is known in case p is constant and $\mu \equiv \nu \equiv 0$, see [19], p. 211. By Theorem (2.1), the set $C_0^\infty(\mathbb{R}^1)$ is dense in the space $L^{p(\cdot)}(\mathbb{R}^1, \varrho)$. Therefore, we need to prove that the operator

$$H^{-\alpha}\varphi(x) = \int_0^\infty \left(\frac{t}{x}\right)^\alpha \frac{\varphi(t)}{x+t} dt$$

involved in (4.31) and studied in Theorem (3.6), is bounded from the space $L^{\overline{p}(\cdot)}(\mathbb{R}^1_+, \overline{\varrho})$ into the space $L^{p(\cdot)}(\mathbb{R}^1_+, \varrho)$, where $\overline{p}(x) = p(-x), \overline{\varrho}(x) = \varrho(-x)$. This boundedness does not follow formally from Theorem (3.6), because $\overline{\varrho} \neq \varrho$. However, we observe that $|x-b| \sim |x+b|$ for $x > 2b$ and $0 < x < \frac{b}{2}$. So the estimation of $\int\limits_0^\infty |\varrho(x) H^{-\alpha}\varphi(x)|^{p(x)}\, dx$ is reduced to Theorem (3.6) when integrating over $(0, \frac{b}{2})$ and $(2b, \infty)$, while for $x \in (\frac{b}{2}, 2b)$ the estimation is trivial by the Hölder inequality.

Having proved that $\psi \in L^{p(\cdot)}(\mathbb{R}^1, \varrho)$, we can now proceed exactly as in [19],Theorem 11.6. The main point is the interchange of integrals as in [19], possible by Fubini's theorem, because the double integral $I^\alpha_{0+}(H^{-\alpha}\varphi)$ is absolutely convergent, which is a matter of direct verification.

□

Corollary 4.13. *Let* $\varphi \in L^{p(\cdot)}(\mathbb{R}^1, \varrho)$, *where* $\varrho(x) = |x-a|^{\mu(x)}|b-x|^{\nu(x)}$ *with* $\mu \in w\text{-}Lip_a(\mathbb{R}^1) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$ *and* $\nu \in w\text{-}Lip_b(\mathbb{R}^1) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$. *Under the conditions*

$$\alpha - \frac{1}{p(a)} < \mu(a) < \frac{1}{p'(a)}, \quad \alpha - \frac{1}{p(b)} < \nu(b) < \frac{1}{p'(b)} \tag{4.34}$$

and condition (4.33), *the relation*

$$\chi_{(a,b)} I^\alpha_+ \varphi = I^\alpha_+ \psi$$

holds, where $\psi \in L^{p(\cdot)}(\mathbb{R}^1, \varrho)$.

Let f^* be the zero extension of a function f defined on (a,b) to $\mathbb{R}^1$. The following theorem provides sufficient conditions for f^* to be representable by fractional integral on $\mathbb{R}^1$ if f has such a property on (a,b). To this end, we need to deal with some extension of the exponent $p(x)$ and the weight $\varrho(x)$ to the whole line $\mathbb{R}^1$. In Theorem (4.14) we use the extension $\widetilde{p}(x)$ satisfying the following conditions

$$\widetilde{p} \in \mathcal{P}_1(\mathbb{R}^1) \quad \text{and} \quad \widetilde{p}(\infty) = \widetilde{p}(b) = p(b), \tag{4.35}$$

$$\widetilde{p} \in w{-}Lip_b\,(\mathbb{R}^1) \bigcap w{-}Lip_\infty\,(\mathbb{R}^1), \tag{4.36}$$

and the extension (4.25) of the weight satisfying the conditions

$$\widetilde{\mu} \in w\text{--}Lip_a(\mathbb{R}^1)\bigcap w\text{--}Lip_\infty(\mathbb{R}^1) \quad \text{and} \quad \widetilde{\nu} \in w\text{--}Lip_b(\mathbb{R}^1)\bigcap w\text{--}Lip_\infty(\mathbb{R}^1), \tag{4.37}$$

$$\widetilde{\mu}(\infty) + \widetilde{\nu}(\infty) - \alpha < \frac{1}{\widetilde{p}'(\infty)}. \tag{4.38}$$

Theorem 4.14. *Let* $-\infty < a < b < \infty$ *and let* $f(x) = I^\alpha_{a+}\varphi$, $x \in (a,b)$, *with* $\varphi \in L^{p(\cdot)}((a,b),\varrho)$, *where* $p \in \mathcal{P}_1(a,b)\bigcap w\text{-}Lip_b(a,b)$, $\varrho(x)$ *is weight* (3.1) *with* $\mu \in w\text{-}Lip_a(a,b), \nu \in w\text{-}Lip_b(a,b)$ *and*

$$\mu(a) < \frac{1}{p'(a)}, \quad -\frac{1}{p(b)} < \nu(b) - \alpha < \frac{1}{p'(b)}. \tag{4.39}$$

Then

$$f^*(x) = (I^\alpha_+\varphi_1)(x), \quad x \in \mathbb{R}^1 \tag{4.40}$$

where $\varphi_1(x) \in L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})$ *is given by*

$$\varphi_1(x) = \begin{cases} 0, & x < a \\ \varphi(x), & a < x < b; \\ -\frac{\alpha}{\Gamma(1-\alpha)}\int_a^b \frac{f(t)dt}{(x-t)^{1+\alpha}} = : g(x), & x > b. \end{cases} \tag{4.41}$$

and $\widetilde{p}(x)$ *and* $\widetilde{\varrho}(x)$ *are arbitrary extensions satisfying conditions* (4.35)-(4.38).

Proof. The representation itself (4.40)-(4.41) is known, in the case of constant p, see [19], p. 236. Thus it is valid for C_0^∞-functions. We only have to show that $\varphi_1 \in L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})$. It suffices to show that $g(x) \in L^{\widetilde{p}(\cdot)}[(b,\infty),\widetilde{\varrho}]$. It is known (see (13.33) in [19]) that $g(x)$ has the form

$$g(x) = -\frac{\sin\alpha\pi}{\pi}\int_a^b \left(\frac{b-\tau}{x-b}\right)^\alpha \frac{\varphi(\tau)d\tau}{x-\tau} =: A_1^\alpha\varphi. \tag{4.42}$$

To check that the operator A_1^α is bounded from $L^{p(\cdot)}[(a,b),\varrho]$ to $L^{\widetilde{p}(\cdot)}[(b,\infty),\widetilde{\varrho}]$, it suffices to show that

$$\int\limits_b^\infty |\mathfrak{A}\psi(x)|^{\widetilde{p}(x)}\,dx \le C < \infty$$

for all ψ with $\|\psi\|_{L^{p(\cdot)}(a,b)} \le 1$, where

$$\mathfrak{A}\psi(x) = (x-a)^{\widetilde{\mu}(x)}(x-b)^{\widetilde{\nu}(x)}\int_a^b \left(\frac{b-\tau}{x-b}\right)^\alpha \frac{\psi(\tau)d\tau}{(x-\tau)(\tau-a)^{\mu(a)}(b-\tau)^{\nu(b)}}, \quad x > b.$$

We have

$$\int\limits_b^\infty |\mathfrak{A}\psi(x)|^{\widetilde{p}(x)}\,dx = \int\limits_b^{2b} |\mathfrak{A}\psi(x)|^{\widetilde{p}(x)}\,dx + \int\limits_{2b}^\infty |\mathfrak{A}\psi(x)|^{\widetilde{p}(x)}\,dx =: U_1 + U_2,$$

For U_1 we obtain

$$U_1 \leq C \int\limits_b^{2b} (x-b)^{[\nu(b)-\alpha]p(b)} dx \left| \int\limits_a^c \frac{|\psi(\tau)|\, d\tau}{(\tau-a)^{\mu(a)}} \right|^{\widetilde{p}(x)}$$

$$+C \int\limits_b^{2b} (x-b)^{[\nu(b)-\alpha]p(b)} dx \left| \int\limits_c^b \frac{|\psi(\tau)|\, d\tau}{(b-\tau)^{\nu(b)-\alpha}(x-\tau)} \right|^{\widetilde{p}(x)}$$

where $a < c < b$. The first term here is easily estimated via $\|\psi\|_{L^{p(\cdot)}(a,b)}$ by the Hölder inequality under conditions (4.39). The second term has the form

$$C \int\limits_0^b x^{[\nu(b)-\alpha]p(b)} dx \left| \int\limits_0^{b-c} \frac{|\psi(b-t)|\, dt}{t^{\nu(b)-\alpha}(t+x)} \right|^{\widetilde{p}(x+b)}$$

which is nothing else but the $\widetilde{p}(\cdot + b)$-modular for the operator of the type (3.14)), so it is easily treated by means of Theorem (3.6) with Remark (3.7) taken into account.

Finally, for U_2 we have

$$U_2 \leq C \int\limits_{2b}^{\infty} \frac{dx}{(1+|x|)^{[1+\alpha-\widetilde{\mu}(\infty)-\widetilde{\nu}(\infty)]p(\infty)}} \left| \int\limits_a^b \frac{|\psi(\tau)|\, d\tau}{(\tau-a)^{\mu(a)}(b-\tau)^{\nu(b)-\alpha}} \right|^{\widetilde{p}(x)}$$

where it remains to make use of the Hölder inequality.

□

Theorem 4.15. *Let* $-\infty < a < b < \infty$, $p \in \mathcal{P}(a,b) \bigcap w\text{-}Lip\,(a,b)$ *and* $\varrho(x) = (x-a)^{\mu(x)}(b-x)^{\nu(x)}$, *where* $\mu \in w\text{-}Lip_a(a,b)$, $\nu \in w\text{-}Lip_b(a,b)$. *Then*

$$I^{\alpha}_{a+}\left[L^{p(\cdot)}[(a,b),\varrho]\right] = L^{\alpha,p(\cdot)}[(a,b),\varrho] \tag{4.43}$$

under conditions (4.19).

Proof. Let $f \in L^{\alpha,p(\cdot)}[(a,b),\varrho]$. Then by the definition in (4.26) there exists an extension $\widetilde{p}$ of p and extensions $\widetilde{\mu}$, $\widetilde{\nu}$ of the exponents of the weight to $\mathbb{R}^1$ and a function $g \in L^{\alpha,\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})$ such that

$$f(x) = g(x) \quad \text{for} \quad a \leq x \leq b.$$

By Theorem (4.11), $g \in I^{\alpha}[L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})]$ and consequently $g \in I^{\alpha}_{+}[L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})]$ by Theorem (4.7), which implies that $f \in I^{\alpha}_{a+}[L^{p(\cdot)}(a,b),\varrho]$ by Corollary (4.13).

Conversely, let $f \in I^{\alpha}_{a+}\left[L^{p(\cdot)}[(a,b),\varrho]\right]$. Let f^* be the continuation of this function by zero beyond the interval $[a,b]$ and let $\widetilde{p}$ be the continuation of p to $\mathbb{R}^1$ satisfying conditions *ii)-iii)* of Theorem (4.14). By Theorem (4.14) we have that $f^* \in I^{\alpha}[L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\varrho)]$ and because $f^* \in L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\varrho)$, by Theorem (4.11) we have that $f^* \in L^{\alpha,\widetilde{p}(\cdot)}(\mathbb{R}^1,\varrho)$. Hence $f \in L^{\alpha,p(\cdot)}\left[(a,b),\varrho\right]$.

□

5. Appendix

For simplicity we prove the following technical lemma for the case of a bounded set Ω in $\mathbb{R}^n$, the case of unbounded sets needs some technical modifications. We deal with the weights satisfying the condition

$$w_k \in L^{p(\cdot)}(\Omega), \quad \frac{1}{w_k} \in L^{p'(\cdot)}(\Omega), \quad k = 1, 2, ..., m. \tag{5.1}$$

We denote

$$\Omega_k = \{x \in \Omega : w_k(x) = 0\} \cup \{x \in \Omega : w_k(x) = \infty\}, \quad k = 1, ..., m.$$

From (5.1) it follows that $|\Omega_k| = 0$. We suppose that

$$\overline{\Omega}_k \bigcap \overline{\Omega}_j = \emptyset \quad \text{for all} \quad j \neq k. \tag{5.2}$$

Lemma 5.1. *Let Ω be an open bounded set in $\mathbb{R}^n$, $w_k(x)$ be weights satisfying the assumptions* (5.1) *and* (5.2) *and let a linear operator A fulfill the conditions:*
i) it is bounded in the spaces $L^{p(\cdot)}(\Omega, w_k)$, $k = 1, \ldots, m$,
ii) the operator $\chi_{E_1} A \chi_{E_2}$ is bounded from the spaces $L^{p(\cdot)}(\Omega, w_k), k = 1, ..., m$, into $L^\infty(\Omega)$ for all disjoint sets $E_1, E_2 \subset \Omega$ such that $\overline{E}_1 \bigcap \overline{E}_2 = \emptyset$.
Then the operator A is bounded in the space $L^{p(\cdot)}(\Omega, w)$ with $w(x) = \prod\limits_{k=1}^{m} w_k(x)$.

Proof. We have to prove the boundedness of the operator $wA\frac{1}{w}$ in the space $L^{p(\cdot)}(\Omega)$. We will make use of a corresponding partition of unity $1 = \sum\limits_{k=1}^{m} a_k(x)$. To this end, we consider some neighborhoods E_k and F_k of the sets Ω_k, that is, some open sets E_k and F_k such that

$$\overline{\Omega}_k \subset E_k \subset \overline{E}_k \subset F_k \subset \Omega \quad \text{and} \quad \overline{F}_k \bigcap \overline{F}_j = \emptyset \quad \text{for all} \quad k \neq j. \tag{5.3}$$

Such neighborhoods exist by assumption (5.2). We choose functions $a_k(x)$ such that $a_k(x) \equiv 1$ on E_k and $a_k(x) \equiv 0$ on $\Omega \backslash F_k$, so that $a_k(x)[w_j(x)]^{\pm 1} \equiv 0$ on $\Omega \backslash F_k$ if $k \neq j$. Then

$$\frac{w(x)}{w(y)} = \sum_{k=1}^{m} w_k(x) b_k(x) \sum_{j=1}^{m} \frac{c_j(y)}{w_j(y)}$$

where $b_k(x)$ and $c_j(y)$ are bounded functions supported in the same neighborhoods where the functions $a_k(x)$ and $a_j(y)$ were. Then

$$wA\frac{f}{w} = \sum_{k=1}^{m} b_k w_k A \frac{c_k f}{w_k} + \sum_{\substack{k,j=1 \\ k \neq j}}^{m} b_k w_k A \frac{c_j f}{w_j}.$$

The first sum contains operators bounded in $L^{p(\cdot)}(\Omega)$ by assumption *i)*. It remains to obtain the boundedness of the operators

$$A_{jk} f = b_k w_k A \frac{c_j f}{w_j}, \quad j \neq k.$$

We have

$$\Omega = E_k \cup E_j \cup E_{jk} \quad \text{with} \quad E_{kj} = \Omega \backslash (E_k \cup E_j).$$

We denote for brevity $\chi_k = \chi_{E_k},\ \chi_j = \chi_{E_j},\ \chi_{kj} = \chi_{E_{kj}}$. It is easily seen that

$$c_j \chi_k \equiv 0 \quad \text{and} \quad b_k \chi_j \equiv 0.$$

Taking this into account, we represent the operator $A_{jk} = (\chi_k + \chi_j + \chi_{kj}) A_{jk} (\chi_k + \chi_j + \chi_{kj})$ in the form $A_{jk} = \sum\limits_{i=1}^{4} B_i$ where

$$B_1 = \chi_{kj} w_k b_k A c_j \frac{\chi_{kj}}{w_j}, \quad B_2 = \chi_k w_k b_k A c_j \frac{\chi_{kj}}{w_j},$$

$$B_3 = \chi_{kj} w_k b_k A c_j \frac{\chi_j}{w_j}, \quad B_4 = \chi_k w_k b_k A c_j \frac{\chi_j}{w_j}.$$

The operators B_1, B_2 and B_3, containing the factor χ_{jk} are bounded in $L^{p(\cdot)}(\Omega)$. This follows from condition *i)* because $\chi_{kj}(x) w_k(x)$ and $\frac{\chi_{kj}(x)}{w_j(x)}$ may be represented, whenever necessary, as

$$\chi_{kj}(x) w_k(x) = u(x) w_j(x) \quad \text{and} \quad \frac{\chi_{jk}(x)}{w_j(x)} = \frac{v(x)}{w_k(x)},$$

where u and v are bounded functions.

Finally, for the operator B_4 we observe that from condition *ii)* it follows that the function $\chi_k b_k A c_j \frac{\chi_j}{w_j} f$ is bounded for $f \in L^{p(\cdot)}(\Omega)$ and then it suffices to refer to the fact that $w_k \in L^{p(\cdot)}(\Omega)$.

□

Remark 5.2. An integral operator $Af(x) = \int\limits_{\Omega} K(x,y) f(y)\, dy$ satisfies condition *ii)* of Lemma (5.1) if $\sup\limits_{x,y:|x-y|\geq\varepsilon} |K(x,y)| < \infty$ for any $\varepsilon > 0$, and the weights satisfy the second assumption in (5.1).

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Humberto Rafeiro
University of Algarve
e-mail: hrafeiro@ualg.pt

Stefan Samko
Universidade do Algarve
e-mail: ssamko@ualg.pt