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Vekua's Generalized Singular Integral on Carleson Curves in Weighted Variable Lebesgue Spaces

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Abstract. For a Carleson curve Γ we establish the boundedness, in weighted Lebesgue spaces $L^{p(\cdot)}(\Gamma, \varrho)$ with variable exponent $p(\cdot)$, of the generalized singular integral operator which arises in the theory of I.N.Vekua generalized analytic functions. The obtained result is an extension of the known results even in the case of constant p. We also show that Vekua's generalized singular integral exists a.e. for $f \in L^1(\Gamma)$ on an arbitrary rectifiable curve.

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1. Introduction

We prove the weighted boundedness of the I.N. Vekua generalized singular operator known in the theory of generalized analytic functions, within the frameworks of Lebesgue spaces with variable exponent $p(t)$. We also show that the I.N. Vekua generalized singular integral exists a.e. for every integrable function f even in the case of an arbitrary rectifiable curve, thus proving that the existence properties of the I.N. Vekua generalized singular operator are the same as of the usual singular operator. The obtained results are new even in the case of constant p.

The paper is organized as follows. In Section 2 we give a certain background related to the problem and introduce necessary definitions and auxiliary statements. In Section 3 we prove the main result of the paper.

2. Preliminaries

Let G be a simply connected domain bounded by a simple finite rectifiable curve $\Gamma = \{t \in \mathbb{C} : t = t(s), 0 \leq s \leq \ell < \infty\}$ with arc-length measure $\nu(t) = s$. In the sequel we use the notation

$$\Gamma(t,r) = \Gamma \cap B(t,r), \qquad B(t,r) = \{\tau \in \mathbb{C} : |\tau - t| < \varepsilon\}, \quad t \in \Gamma, \quad r > 0.$$

Recall that a curve Γ is called Carleson if

$$\nu[\Gamma(t,r)] \leq C_0 r$$

with $C_0 > 0$ not depending on t and r.

2.1. Vekua's generalized singular operator

As is known, the theory of generalized analytic functions was developed by L. Bers and I.N. Vekua, we refer to their books [2], [26]–[27]. Generalized analytic functions of the class $U_{r,2}(A,B;G)$, $r > 2$, in the sense of I.N. Vekua, are regular solutions of the equation

$$\partial_{\bar{z}}\Phi(z) + A(z)\Phi(z) + B(z)\overline{\Phi}(z) = 0 \tag{2.1}$$

where $\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), A(z), B(z) \in L^r(G), r > 2.$

Let $f \in L^1(\Gamma)$. It is known that the integral

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \Omega_1(z,\tau)f(\tau)\,d\tau - \Omega_2(z,\tau)\overline{f}(\tau)\,d\overline{\tau}, \tag{2.2}$$

where Ω_1 and Ω_2 are the so-called basic normalized kernels of the class $U_{r,2}(A,B;G)$, is a regular solution of (2.1), see details in [26], [27]. The integral in (2.2) is called the generalized Cauchy type integral. The corresponding generalized singular integral is introduced as

$$\widetilde{S}_\Gamma f(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} \Omega_1(t,\tau)f(\tau)\,d\tau - \Omega_2(t,\tau)\overline{f}(\tau)\,d\overline{\tau}. \tag{2.3}$$

In [17] there was proved the following conventional statement:

Proposition 2.1. *If the singular integral $\widetilde{S}_\Gamma f(t)$ exists for almost all $t \in \Gamma$, then the function $\Phi(z)$ admits angular boundary values almost everywhere when $z \to t$ non-tangentially and for these boundary values the formula holds*

$$\Phi^{\pm}(t) = \widetilde{S}_\Gamma f(t) \pm \frac{1}{2}f(t) \tag{2.4}$$

almost everywhere. Conversely, the almost everywhere existence of the boundary values $\Phi^{\pm}(t)$ yields that of the singular integral (2.3).

In [18] for the case of constant p the following statement was proved.

Proposition 2.2. *Let Γ be a Carleson curve. The operator $\widetilde{S}_\Gamma$ is bounded in the space $L^p(\Gamma)$ if*

$$p > \frac{r}{r-2}. \tag{2.5}$$

We prove a more general result for variable exponents $p(t)$, which is even in the case of constant p is stronger than the existing result of Proposition 2.2, because we admit all the range $1 < p < \infty$, avoiding the restriction in (2.5). Moreover, based on our earlier results on classical integral operators in weighted variable spaces, we are also able to prove the boundedness of the operator $\widetilde{S}_\Gamma$ in weighted spaces $L^{p(\cdot)}(\Gamma, \varrho)$ with a certain class of general weights, see (2.10), including power type weights with a natural range for their exponents. All the results are obtained under the natural assumption on the variable exponent $p(t)$, see (2.6), (2.7) below.

2.1.1. On singular integrals. It is known that the Cauchy singular integral

$$S_\Gamma f(t) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau$$

converges almost everywhere for any $f \in L^1(\Gamma)$ on every rectifiable curve, see for instance, [3], p.137, Theorem 14.4). G.David [4] proved that the singular operator S_Γ is bounded in the space $L^p(\Gamma)$ with constant $p, 1 < p < \infty$, if and only if Γ is a Carleson curve. An extension of David's result for the case of variable $p(t)$ was made in [7], [12], [13].

2.1.2. On convergence of $\widetilde{S}_\Gamma f(t)$. In Theorem 3.1 we show that the the integral $\widetilde{S}_\Gamma f$ converges almost everywhere in case of an arbitrary rectifiable curve and $f \in L^1(\Gamma)$. Then in view of Proposition 2.1 we arrive at the following conclusion.

Conclusion. *Relation* (2.4) *holds almost everywhere for an arbitrary $f \in L^1(\Gamma)$ and a rectifiable curve Γ.*

2.2. On variable exponent Lebesgue spaces.

We refer to [5], [15], [23], [25] for details on Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, but recall some basic definitions with respect to curves on the complex plane. Let a function $p(t)$ be defined on Γ and satisfy the conditions

$$1 < p_* \le p(t) \le p^* < \infty, \quad t \in \Gamma \tag{2.6}$$

and

$$|p(t) - p(\tau)| \le \frac{A}{\ln \frac{1}{|t-\tau|}}, \quad |t-\tau| \le \frac{1}{2}, \quad t, \tau \in \Gamma. \tag{2.7}$$

By $L^{p(\cdot)}(\Gamma, \varrho)$, where $\varrho(t) \ge 0$, we denote the weighted Banach space of measurable functions $f : \Gamma \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\Gamma,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_\Gamma \left|\frac{\varrho(t) f(t)}{\lambda}\right|^{p(t)} d\nu(t) \le 1\right\} < \infty. \tag{2.8}$$

We write

$$L^{p(\cdot)}(\Gamma) := L^{p(\cdot)}(\Gamma, 1).$$

The Hölder inequality holds

$$\left|\int_\Gamma f(t)g(t)d\nu(t)\right| \leq k\|f\|_{L^{p(\cdot)}(\Gamma)}\|f\|_{L^{p'(\cdot)}(\Gamma)}, \qquad \frac{1}{p(t)} + \frac{1}{p'(t)} \equiv 1, \tag{2.9}$$

where the constant $k = \frac{1}{p_*} + \frac{1}{(p')_*} < 2$ does not depend on f and g.

We deal with weights of the form

$$\varrho(t) = \prod_{k=1}^{m} w_k(|t - t_k|), t_k \in \Gamma, \tag{2.10}$$

where $w_k(r)$ may oscillate as $r \to 0+$ between two power functions (radial Zygmund-Bary-Stechkin type weights). The Zygmund-Bary-Stechkin class of admissible weights is defined in Subsection 2.3. In particular, the power weights

$$\varrho(t) = \prod_{k=1}^{m} |t - t_k|^{\beta_k}, \ t_k \in \Gamma, \tag{2.11}$$

with the condition

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \tag{2.12}$$

are admitted.

2.3. On Zygmund-Bary-Stechkin-type weights

We use the abbreviation a.i = almost increasing, a.d. = almost decreasing. Let

$$W = \{w \in C([0,\ell]) : w(0) = 0, \ w(x) > 0 \ \text{ for } \ x > 0, \ \ w(x) \ \text{ is a.i.}\}. \tag{2.13}$$

The numbers

$$m_w = \sup_{x>1} \frac{\ln\left(\liminf\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x} \quad \text{and} \quad M_w = \sup_{x>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(hx)}{w(h)}\right)}{\ln x}$$

(see [20], [22], [21]), are known as as *the lower and upper indices* of the function $w(x)$ (compare these indices with the Matuszewska-Orlicz indices, see [16], p. 20). We have $0 \leq m_w \leq M_w \leq \infty$ for $w \in W$.

Definition 2.3. ([1]) The Zygmund-Bary-Stechkin type class Φ^0_δ, $0 < \delta < \infty$, is defined as $\Phi^0_\delta := \mathcal{Z}^0 \cap \mathcal{Z}_\delta$, where $\mathcal{Z}^0$ is the class of functions $w \in W$ satisfying the condition

$$\int_0^h \frac{w(x)}{x}dx \leq cw(h) \tag{$\mathbb{Z}^0$}$$

and $\mathcal{Z}_\delta$ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(x)}{x^{1+\delta}}dx \leq c\frac{w(h)}{h^\delta}, \tag{$\mathbb{Z}_\delta$}$$

where $c = c(w) > 0$ does not depend on $h \in (0, \ell]$.

The following statement is valid, see [20],[22] for $\delta = 1$ and [6] for an arbitrary $\delta > 0$.

Theorem 2.4. *Let $w \in W$. Then $w \in \mathcal{Z}^0$ if and only if $m_w > 0$, and $w \in \mathcal{Z}_\delta,\ \delta > 0$, if and only if $M_w < \delta$, so that*

$$w \in \Phi_\delta^0 \Longleftrightarrow 0 < m_w \le M_w < \delta. \tag{2.14}$$

Besides this, for $w \in \Phi_\delta^0$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$c_1 x^{M_w+\varepsilon} \le w(x) \le c_2 x^{m_w-\varepsilon}, \qquad 0 \le x \le \ell. \tag{2.15}$$

The following properties are also valid

$$m_w = \sup\{\mu \in \mathbb{R}^1 :\ x^{-\mu} w(x) \quad \text{is a.i.}\}, \tag{2.16}$$

$$M_w = \inf\{\nu \in \mathbb{R}^1 :\ x^{-\nu} w(x) \quad \text{is a.d.}\}. \tag{2.17}$$

Note that the indices m_ω and M_ω may be also well defined for functions $w(x)$ positive for $x > 0$ which do not necessarily belong to W, for example, for functions in the class

$$\widetilde{W} = \{w : \exists a \in \mathbb{R}^1 \text{ such that } w_a(x) := x^a w(x) \in W\}.$$

Obviously,

$$m_{w_a} = a + m_w, \qquad M_{w_a} = a + M_w.$$

Observe that various non-trivial examples of functions in Zygmund-Bary-Stechkin type classes with coinciding indices may be found in [20], Section II; [19], Section 2.1, and with non-coinciding indices in [22].

In the sequel we shall also need the following technical lemma (its Euclidean version was proved in [9], for the Carleson context the proof is the same).

Lemma 2.5. *Let Γ be a bounded Carleson curve, the exponent p satisfy condition (2.7) and let w be any function such that there exist exponents $a, b \in \mathbb{R}^1$ and the constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 r^a \le w(r) \le c_2 r^{-b},\ 0 \le r \le \ell = \operatorname{diam}(\Gamma)$. Then*

$$\frac{1}{C}[w(|t-t_0|)]^{p(t_0)} \le [w(|t-t_0|)]^{p(t)} \le C[w(|t-t_0|)]^{p(t_0)}, \tag{2.18}$$

where $C > 1$ does not depend on $t, t_0 \in \overline{\Gamma}$.

3. The main result

We prove the following theorem.

Theorem 3.1. *I. Let Γ be a closed simple rectifiable curve. The singular integral $\widetilde{S}_\Gamma f$ exists for almost all $t \in \Gamma$ for any $f \in L^1(\Gamma)$.*
II. Let Γ be a Carleson curve and let $p(t)$ satisfy conditions (2.6) and (2.7). Then the operator $\widetilde{S}_\Gamma$ is bounded in the space $L^{p(\cdot)}(\Gamma, \varrho)$ with weight (2.10), where $w_k(r)$ are such functions that

$$r^{\frac{1}{p(t_k)}} w_k(r) \in \Phi_1^0 \tag{3.1}$$

or equivalently $w_k \in \widetilde{W}$ *and*

$$-\frac{1}{p(t_k)} < m_{w_k} \le M_{w_k} < \frac{1}{p'(t_k)} \ , \quad k = 1, 2, \dots, m. \tag{3.2}$$

In particular, the weights (2.11)–(2.12) *are admitted.*

3.1. The required tools

Everywhere in the sequel Γ is a bounded Carleson curve. Let

$$\mathcal{M}f(t) = \sup_{r>0} \frac{1}{\nu[\Gamma(r,t)]} \int\limits_{\Gamma(r,t)} |f(\tau)|\, d\nu(\tau) \tag{3.3}$$

be the maximal operator along Γ. In [8] the following statement was proved (see also [9] for a similar statement for bounded domains in $\mathbb{R}^n$; observe also that Theorem 3.2 for Carleson curves in the case of power weights was proved in [13], see also [10]).

Theorem 3.2. *Let* $p(t)$ *satisfy conditions* (2.6), (2.7)*. The operator* $\mathcal{M}$ *is bounded in* $L^{p(\cdot)}(\Gamma, \varrho)$ *with the weight* (2.10)*, where* $w_k(r)$ *satisfy* (3.1)–(3.2)*.*

The following theorem was proved in [13], [10] in the case of power weights and is similarly extended to the case of Zygmund-Bary-Stechkin-type weights, taking into account Theorem 3.2.

Theorem 3.3. *Let* $p(t)$ *satisfy conditions* (2.6), (2.7) *on* Γ*. The singular operator* S_Γ *is bounded in* $L^{p(\cdot)}(\Gamma, \varrho)$ *with the weight* (2.10)*, where* $w_k(r)$ *satisfy* (3.1)–(3.2)*.*

3.2. Potentials over rectifiable curves

Let

$$I^\alpha f(t) = \int\limits_\Gamma \frac{f(\tau)}{|\tau - t|^\alpha}\, d\nu(\tau), \qquad 0 < \alpha < 1. \tag{3.4}$$

Observe that the behavior of the potential $I_\alpha f(t)$ along an arbitrary rectifiable curve Γ is not quite trivial. Thus the potential $I_\alpha f(t)$ of a bounded function may prove to be an unbounded function. W.E. Sewell [24] in his investigations on approximation in the complex plane specially singled out the class V_α of rectifiable curves Γ along which the integral in (3.4) converges uniformly in t, so that

$$\sup_{t\in\Gamma} \int\limits_\Gamma \frac{d\nu(\tau)}{|\tau - t|^\alpha} < \infty; \tag{3.5}$$

for $\Gamma \in V_\alpha$. Observe that the condition

$$\sup_{t\in\Gamma} \nu[\Gamma(t,r)] \le Cr^\beta \quad \text{with} \quad \beta > \alpha \tag{3.6}$$

is sufficient for (3.5) to be satisfied. Thus (3.5) is in particular valid on Carleson curves.

Lemma 3.4. *Let $0 < \alpha < 1$ and let Γ be a bounded rectifiable curve and $f \in L^1(\Gamma)$. The integral in* (3.4) *exists for almost all $t \in \Gamma$. When Γ is a Carleson curve, the operator I_α is bounded in the space $L^1(\Gamma)$ on every Carleson curve, and on any rectifiable curve with property* (3.5).

Proof. First we prove the almost everywhere convergence. Let s be an arc length of a point t, $t = t(s), \tau = t(\sigma)$. We have

$$\int_\Gamma \frac{f(\tau)d\nu(\tau)}{|t-\tau|^\alpha} = \int_0^\ell \left|\frac{s-\sigma}{t(s)-t(\sigma)}\right|^\alpha \frac{f[\tau(\sigma)]d\sigma}{|s-\sigma|^\alpha}, \tag{3.7}$$

where ℓ is the length of the curve. Let now $\Pi(\Gamma)$ be the set of all points $t = t(s) \in \Gamma$ such that for $t \in \Pi(\Gamma)$ simultaneously two conditions are fulfilled:

$$\int_0^\ell \frac{f[\tau(\sigma)]d\sigma}{|s-\sigma|^\alpha} \quad \text{exists and} \quad |t'(s)| = 1.$$

As is known, in case of a rectifiable curve Γ, almost all points of the curve belong to $\Pi(\Gamma)$. For a fixed $t = t(s) \in \Pi(\Gamma)$ from the condition $|t'(s)| = 1$ and the fact that Γ is assumed to have no intersections it follows that

$$\left|\frac{s-\sigma}{\tau(s)-\tau(\sigma)}\right| \le C = C(s). \tag{3.8}$$

Then from (3.7) we derive the almost everywhere convergence of the integral on the left-hand side.

The L^1-boundedness in the case of Carleson curve is a matter of direct verification. We have

$$\int_\Gamma |I^\alpha f(t)| d\nu(t) \le C \int_\Gamma |f(\tau)| d\nu(\tau) \int_\Gamma \frac{d\nu(t)}{|\tau-t|^\alpha}$$

and by the help of the standard binary decomposition of $\Gamma(t,1)$ into the portions $\Gamma_k(t) = \{\tau \in \Gamma : 2^{-k-1} < |t-\tau| < 2^{-k}\}$ it is easy to check that condition (3.5) holds for an arbitrary Carleson curve. □

Theorem 3.5. *Let $p(t)$ satisfy conditions* (2.6), (2.7) *on Γ. The operator*

$$I^\alpha f(t) = \int_\Gamma \frac{f(\tau)\, d\nu(\tau)}{|\tau-t|^\alpha}$$

is bounded in the space $L^{p(\cdot)}(\Gamma,\varrho)$ with any weight $\varrho(t)$ for which the following conditions are fulfilled:

1) $\int_\Gamma \frac{d\nu(\tau)}{[\varrho(\tau)]^{p'(\tau)}} < \infty$, *and*
2) *the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(\Gamma,\varrho)$, in particular for weights* (2.10) *with $w_k(r)$ satisfying conditions* (3.1)–(3.2).

Proof. The proof follows the standard lines; we refer in particular to [10], where a weighted Sobolev-type theorem was proved for I_α on Carleson curves in the case of power weights, see for instance the domination of $I_\alpha f(t)$ by the maximal operator in formulas (8.6) and (9.33) in paper [10], however we give the proof for the completeness of presentation.

Without loss of generality we may assume that *diam* $\Gamma > 1$. By Hölder inequality (2.9) we obtain

$$\int\limits_{\Gamma\backslash\Gamma(t,1)} \frac{f(\tau)\,d\nu(\tau)}{|\tau - t|^\alpha} \le k\|f\|_{L^{p(\cdot)}(\Gamma,\varrho)} \left\| \frac{[\varrho(\cdot)]^{-1}}{|\cdot - t|^\alpha} \right\|_{L^{p'(\cdot)}(\Gamma\backslash\Gamma(t,1))}$$

$$\le k\|f\|_{L^{p(\cdot)}(\Gamma,\varrho)} \left\|\varrho^{-1}\right\|_{L^{p'(\cdot)}(\Gamma)} \le C\|f\|_{L^{p(\cdot)}(\Gamma,\varrho)} \tag{3.9}$$

since

$$\left\|\varrho^{-1}\right\|_{L^{p'(\cdot)}(\Gamma)} < \infty \quad \Longleftrightarrow \quad \int\limits_\Gamma \frac{d\nu(\tau)}{[\varrho(\tau)]^{p'(\tau)}} < \infty;$$

the existence of the last integral in the case of weights (2.10) with $w_k(r)$ satisfying conditions (3.1)–(3.2), follows from Lemma 2.5, inequalities (2.15) and condition (3.2).

As regards the integral over $\Gamma(t,1)$, it may be dominated by the maximal function in the usual way. Indeed, let $\gamma_k(t) = \Gamma(t, 2^{-k})\backslash\Gamma(t, 2^{-k-1})$. Then

$$\int\limits_{\Gamma(t,1)} \frac{|f(\tau)|\,d\nu(\tau)}{|\tau - t|^\alpha} = \sum_{k=0}^\infty \int\limits_{\gamma_k(t)} \frac{|f(\tau)|\,d\nu(\tau)}{|t-\tau|^\alpha} \le \sum_{k=0}^\infty 2^{\alpha(k+1)} \int\limits_{\Gamma(t,2^{-k})} |f(\tau)|\,d\nu(\tau)$$

$$\le C\sum_{k=0}^\infty 2^{-(1-\alpha)k} Mf(t) = C_1 Mf(t).$$

Then the application of Theorem 3.2 completes the proof of Theorem 3.5. □

Note that in [14] it was proved that in the case of an infinite curve and constant p, the Sobolev theorem for the potential type operator holds if and only if Γ is a Carleson curve. For the extension of the Sobolev theorem on Carleson curves to the case of weighted spaces with variable $p(t)$ we refer to papers [11], [10], [12].

3.3. Proof of Theorem 3.1: reduction to the classical singular operator

The proof is based on the observation that the functions $\Omega_1(z,t)$ and $\Omega_2(z,t)$ have the following structure

$$\Omega_1(z,t) = \frac{1}{t-z} + \frac{m_1(z,t)}{|t-z|^\alpha} \quad \text{and} \quad \Omega_2(z,t) = \frac{m_2(z,t)}{|t-z|^\alpha} \tag{3.10}$$

where $\alpha = \frac{2}{r} \in (0,1)$ and the functions $m_1(z,t)$ and $m_2(z,t)$ are continuous and bounded when t runs Γ and z runs a bounded domain, see [26], [27]. Consequently, for the generalized singular integral $\widetilde{S}_\Gamma f(t)$ we have

$$\widetilde{S}_\Gamma f(t) = S_\Gamma f(t) + I_\alpha f(t), \tag{3.11}$$

where

$$I_\alpha f(t) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{m_1(t,\tau)f(\tau)}{|\tau - t|^\alpha}\, d\tau - \frac{1}{2\pi i}\int\limits_\Gamma \frac{m_2(t,\tau)\overline{f}(\tau)}{|\tau - t|^\alpha}\, d\overline{\tau}. \tag{3.12}$$

Therefore, the question of convergence or boundedness of $\widetilde{S}_\Gamma f(t)$ is reduced to that of the singular integral $S_\Gamma f(t)$ and the integral with a weak singularity. The almost everywhere convergence of the first term on the right-hand side of (3.11) is known, see Subsection 2.1.1, while that of the second term was proved in Theorem 3.5.

Similarly the boundedness of these terms in the space $L^{p(\cdot)}(\Gamma, \varrho)$ under the assumptions of Theorem 3.1 follows from Theorems 3.3 and 3.5, which completes the proof of Theorem 3.1.

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