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MAXIMAL AND POTENTIAL OPERATORS IN VARIABLE EXPONENT MORREY SPACES

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Abstract. We prove the boundedness of the Hardy–Littlewood maximal operator on variable Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ over a bounded open set $\Omega \subset \mathbb{R}^n$ and a Sobolev type $L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}$-theorem for potential operators $I^{\alpha(\cdot)}$, also of variable order. In the case of constant α, the limiting case is also studied when the potential operator I^{α} acts into BMO space.

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1. Introduction

We introduce the variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ over an open set $\Omega \subset \mathbb{R}^n$, well known in the case where p and λ are constant, see for instance [10], [17]. Last decade there was a real boom in the investigation of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the corresponding Sobolev spaces $W^{m,p(\cdot)}(\Omega)$. We refer to surveys [7], [12], [21] on the progress in this field, including topics of Harmonic Analysis and Operator Theory.

In this paper, within the framework of variable Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ over bounded sets $\Omega \subset \mathbb{R}^n$, we consider the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy$$

and potential type operators

$$I^{\alpha(\cdot)}f(x) = \int\limits_{\Omega} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}$$

of variable order $\alpha(x)$.

We prove the boundedness of the maximal operator in Morrey spaces under the log-condition on $p(\cdot)$ and $\lambda(\cdot)$, in the case of Lebesgue spaces ($\lambda(x) \equiv 0$) this result being due to L. Diening [5]. For potential operators, under the same log-conditions and the assumptions $\inf\limits_{x\in\Omega} \alpha(x) > 0$, $\sup\limits_{x\in\Omega} [\lambda(x) + \alpha(x)p(x)] < n$, we prove a Sobolev type $L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}$-theorem. In the case $\lambda(x) \equiv 0$ the Sobolev Theorem for variable Lebesgue spaces is known to be obtained – via the Hedberg approach – from the boundedness of the maximal operator, see [20].

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As a corollary of the Sobolev Theorem, we derive the corresponding boundedness of the fractional maximal operator

$$M^{\alpha(\cdot)}f(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}}\int\limits_{\widetilde{B}(x,r)}|f(y)|dy.$$

In the case of constant α, we also prove a boundedness theorem in the limiting case $p(x)=\frac{n-\lambda(x)}{\alpha(x)}$, when the potential operator I^α acts from $L^{p(\cdot),\lambda(\cdot)}$ into BMO.

Notation:
$\mathbb{R}^n$ is the n-dimensional Euclidean space;
Ω is a non-empty open set in $\mathbb{R}^n$;
d_Ω denotes the diameter of Ω;
χ_E is a characteristic function of a measurable set $E\subset\mathbb{R}^n$;
$|E|$ is the Lebesgue measure of E;
$B(x,r)=\{y\in\mathbb{R}^n : |x-y|<r\}$, $\widetilde{B}(x,r)=B(x,r)\cap\Omega$;
by c and C we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on Variable Exponent Lebesgue Spaces

Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. We assume that

$$1\le p_-\le p(x)\le p_+<\infty, \tag{1}$$

where we use the standard notation

$$p_-:=\operatorname*{ess\,inf}_{x\in\Omega}p(x)\quad\text{ and }\quad p_+:=\operatorname*{ess\,sup}_{x\in\Omega}p(x).$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions f on Ω such that

$$I_{p(\cdot)}(f)=\int\limits_\Omega|f(x)|^{p(x)}dx<\infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)}=\inf\left\{\eta>0:\ I_{p(\cdot)}\left(\frac{f}{\eta}\right)\le1\right\},$$

this is a Banach function space. As is known, the following inequalities hold

$$\|f\|_{p(\cdot)}^{p_+}\le I_{p(\cdot)}(f)\le\|f\|_{p(\cdot)}^{p_-}\quad\text{if}\quad\|f\|_{p(\cdot)}\le1, \tag{2}$$

$$\|f\|_{p(\cdot)}^{p_-}\le I_{p(\cdot)}(f)\le\|f\|_{p(\cdot)}^{p_+}\quad\text{if}\quad\|f\|_{p(\cdot)}\ge1, \tag{3}$$

from which it follows that

$$c_1\le\|f\|_{p(\cdot)}\le c_2\quad\Longrightarrow c_3\le I_{p(\cdot)}(f)\le c_4 \tag{4}$$

and

$$C_1\le I_{p(\cdot)}(f)\le C_2\quad\Longrightarrow C_3\le\|f\|_{p(\cdot)}\le C_4, \tag{5}$$

with $c_3 = \min(c_1^{p_-}, c_1^{p_+})$, $c_4 = \max(c_2^{p_-}, c_2^{p_+})$, $C_3 = \min\left(C_1^{1/p_-}, C_1^{1/p_+}\right)$ and $C_4 =$ $\max\left(C_2^{1/p_-}, C_2^{1/p_+}\right)$.

As usual, we denote by $p'(\cdot)$ the conjugate exponent given by $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$. The Hölder inequality is valid in the form

$$\int\limits_\Omega |f(x)g(x)|\,dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}. \tag{6}$$

If Ω is bounded and $p(x) \leq q(x)$, there holds the embedding

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega). \tag{7}$$

For the basics of variable exponent Lebesgue spaces we refer to [22], [16].

The $L^{p(\cdot)}$-boundedness of the Hardy–Littlewood maximal operator was proved by L. Diening [5] under the conditions

$$1 < p_- \leq p(x) \leq p_+ < \infty \tag{8}$$

and

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \Omega, \tag{9}$$

where $A > 0$ does not depend on x, y.

The proof of the boundedness of the maximal operator was based on the following pointwise estimate.

Lemma 1. ([5]) *Let Ω be bounded and let $p(\cdot)$ satisfy conditions* (8), (9). *Then there exists a constant $C > 0$ such that, for all $\|f\|_{p(\cdot)} \leq 1$,*

$$(Mf(x))^{\frac{p(x)}{p_-}} \leq C\left[M\left(|f(\cdot)|^{\frac{p(\cdot)}{p_-}}\right)(x) + 1\right]. \tag{10}$$

We will also make use of the following statement proved in [20], Theorem 1.17.

Theorem 1. *Let $p(\cdot)$ satisfy assumptions* (1) *and* (9) *and $\beta(\cdot)$ satisfy the conditions*

$$\sup_{x\in\Omega} \beta(x) < \infty, \quad \inf_{x\in\Omega} \beta(x)p(x) > n. \tag{11}$$

Then the estimate

$$\left\|\frac{\chi_{\mathbb{R}^n\setminus B(x,r)}(\cdot)}{|x-\cdot|^{\beta(x)}}\right\|_{p(\cdot)} \leq C\, r^{\frac{n}{p(x)} - \beta(x)} \tag{12}$$

is valid, where the constant $C > 0$ depends on $\sup\limits_{x\in\Omega} \beta(x)$ and $\inf\limits_{x\in\Omega} [\beta(x)p(x) - n]$, but does not depend on x and r.

We note that the logarithmic condition (9) is usually called the log-*Hölder continuity* or the *Dini–Lipschitz condition.*

3. Variable Exponent Morrey Spaces

3.1. Definition. Let $\lambda(\cdot)$ be a measurable function on Ω with values in $[0,n]$. We define the variable Morrey space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ as the set of all integrable functions f on Ω such that

$$I_{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in\Omega,\ r>0} r^{-\lambda(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy < \infty. \tag{13}$$

The norm in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ may be introduced in two forms,

$$\|f\|_1 = \inf\left\{\eta > 0 :\ I_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\}$$

and

$$\|f\|_2 = \sup_{x\in\Omega,\ r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)},$$

which actually coincide, as shown in Lemma 3. First we need the following lemma.

Lemma 2. *For every $f \in L^{p(\cdot),\lambda(\cdot)}(\Omega)$, the inequalities*

$$\|f\|_i^{p_+} \le I_{p(\cdot),\lambda(\cdot)}(f) \le \|f\|_i^{p_-} \quad \textit{if} \quad \|f\|_i \le 1, \tag{14}$$

$$\|f\|_i^{p_-} \le I_{p(\cdot),\lambda(\cdot)}(f) \le \|f\|_i^{p_+} \quad \textit{if} \quad \|f\|_i \ge 1 \tag{15}$$

are valid, $i = 1, 2$.

Proof. Let

$$F(x,r;\eta) = \frac{1}{r^{\lambda(x)}} \int\limits_{B(x,r)} \left|\frac{f(y)}{\eta}\right|^{p(y)} dy. \tag{16}$$

For every $(x,r) \in \Omega\times(0,d_\Omega)$, the function $F(x,r;\eta)$ is decreasing in $\eta \in (0,\infty)$. We have

$$\sup_{x\in\Omega,\ r>0} F(x,r;1) = I_{p(\cdot),\lambda(\cdot)}(f) \tag{17}$$

and by the definition of the norm $\|\cdot\|_1$,

$$\sup_{x\in\Omega,\ r>0} F(x,r;\|f\|_1) = 1. \tag{18}$$

Then from (17)–(18), by the monotonicity of $F(x,r;\eta)$ in η, inequalities (14)–(15) with $i = 1$ follow. To cover the case $i = 2$, that is, the case of the norm

$$\|f\|_2 = \sup_{x\in\Omega,\ r>0} \|g_{x,r}\|_{p(\cdot)},$$

where $g_{x,r} = r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)}(\cdot)$, we make use of inequalities (2)–(3) of the $L^{p(\cdot)}$-norm and have

$$\|g_{x,r}\|_{p(\cdot)}^{p_+} \le I_{p(\cdot)}(g_{x,r}) \le \|g_{x,r}\|_{p(\cdot)}^{p_-} \quad \text{if} \quad \|g_{x,r}\|_{p(\cdot)} \le 1,$$

and similarly for the case $\|g_{x,r}\|_{p(\cdot)} \ge 1$. Taking the supremum with respect to x and r, we obtain (14)–(15) for $i = 2$. □

Lemma 3. *For every $f \in L^{p(\cdot),\lambda(\cdot)}(\Omega)$ we have*

$$\|f\|_2 = \|f\|_1.$$

Proof. We note that

$$\|f\|_2 = \sup_{x\in\Omega,\, r>0} \{\mu_{x,r} > 0 : \ F(x,r,\mu_{x,r}) = 1\},$$

where $F(x,r;\eta)$ is function (16). From the equality $F(x,r;\mu_{x,r}) = 1$ and the inequality $F(x,r;\|f\|_1) \le 1$ following from (18), by the monotonicity of the function $F(x,r;\eta)$ with respect to η, we conclude that

$$\|f\|_2 \le \|f\|_1.$$

From relations (14) we easily derive

$$\|f\|_1 \le \begin{cases} \|f\|_2^{\frac{p_-}{p_+}} & \text{if } \ \|f\|_1 \le 1, \\ \|f\|_2 & \text{if } \ \|f\|_1 \ge 1, \ \|f\|_2 \le 1, \\ \|f\|_2^{\frac{p_+}{p_-}} & \text{if } \ \|f\|_1 \ge 1, \ \|f\|_2 \ge 1. \end{cases}$$

Substituting here $\frac{f}{\|f\|_2}$ instead of f, we obtain $\left\|\frac{f}{\|f\|_2}\right\|_1 \le 1$, that is, $\|f\|_1 \le \|f\|_2$, which completes the proof. □

By the coincidence of the norms we put

$$\|f\|_{p(\cdot),\lambda(\cdot)} := \|f\|_1 = \|f\|_2.$$

Remark 1. When the open set Ω is bounded, the supremum defining the norm $\|\cdot\|_2$ is always reached for values of r less than d_Ω. Indeed, if $r \ge d_\Omega$ we have

$$\left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)} \le \left\| d_\Omega^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,d_\Omega)} \right\|_{p(\cdot)}.$$

Lemma 5 below provides another equivalent norm on $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ when $|\Omega| < \infty$. Basically, it states that in case $\lambda(\cdot)$ is log-continuous, there is no difference in taking the parameter λ depending on x or y. Lemma 5 is an immediate consequence of the following simple lemma.

Lemma 4. *Let Ω be a bounded open set and $\lambda(\cdot)$ satisfy the logarithmic condition*

$$|\lambda(x) - \lambda(y)| \le \frac{A_\lambda}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2}, \quad x, y \in \Omega. \tag{19}$$

Then

$$\frac{1}{C} r^{-\lambda(y)} \le r^{-\lambda(x)} \le C r^{-\lambda(y)} \tag{20}$$

for all $x, y \in \Omega$ such that $|x-y| \le r$, with the constant $C = e^{A_\lambda}$ not depending on x, y and r.

Proof. Since the set Ω is bounded and $\lambda(\cdot)$ is a bounded function, it suffices to check (20) for small $r \leq 1$. It is easy to see that (20) is equivalent to

$$|\lambda(x) - \lambda(y)| \, \ln \frac{1}{r} \leq C_1 \ := \ln C = A_\lambda, \tag{21}$$

which is valid since $\ln \frac{1}{r} \leq \ln \frac{1}{|x-y|}$. □

Lemma 5. *If Ω is bounded and $\lambda(\cdot)$ is* log-*Hölder continuous, then the functional*

$$\|f\|_3 := \sup_{x\in\Omega,\, r>0} \left\| r^{-\frac{\lambda(\cdot)}{p(\cdot)}} f \, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)} \tag{22}$$

defines an equivalent norm in $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.

The definitions above recover the classical Morrey spaces (see, for example, [17], Ch. 4), that is, $L^{p(\cdot),\lambda(\cdot)}(\Omega) = L^{p,\lambda}(\Omega)$ if $p(x) \equiv p$ and $\lambda(x) \equiv \lambda$ are constant. Furthermore, if $\lambda(x) \equiv 0$, then $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ coincides with the Lebesgue space $L^{p(\cdot)}(\Omega)$.

3.2. Embeddings of variable Morrey spaces. In Lemma 7 we prove embeddings of the Morrey spaces with variable $p(\cdot)$ and $\lambda(\cdot)$, known for constant exponents (see [17], Theorem 4.3.6 or [10], Ch. III, Proposition 1.1). To this end, we first need the estimate given in the following lemma which was obtained in [11] in the framework of general metric measure spaces setting; we give its another proof for the sake of completeness. This estimate serves better for our goals than the known ([6]) estimate $\left\|\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)} \leq |B(x,r)|^{\frac{1}{p_B}}$, with $\frac{1}{p_B} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \frac{dy}{p(y)}$.

Lemma 6. *Let Ω be a bounded open set in a metric measure space (X, d, μ) where the measure μ satisfies the lower Ahlfors condition $\mu B(x,r) \geq c\, r^\delta$, $\delta > 0$, and let $p(\cdot)$ satisfy the log-condition on Ω. Then*

$$\left\|\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)} \leq C \ [\mu B(x,r)]^{\frac{1}{p(x)}} \tag{23}$$

with $C > 0$ not depending on $x \in \Omega$ and $r > 0$ (for small r one may take $C = e^{\frac{A\delta}{p_-^2}}$, where $A > 0$ is the constant from the log-condition).

Proof. Let $x \in \Omega$ and $0 < r < d_\Omega$. Since $p(\cdot)$ is log-Hölder continuous and the lower Ahlfors condition holds, it is easy to check that

$$\frac{1}{C}\, \mu B(x,r) \leq [\mu B(x,r)]^{\frac{p(y)}{p(x)}} \leq C\, \mu B(x,r) \tag{24}$$

for all $y \in \widetilde{B}(x,r)$, where $C \geq 1$ does not depend on x, y and r. Hence for $C_1 = C^{\frac{1}{p_-}}$ we have

$$\int\limits_{\widetilde{B}(x,r)} \frac{d\mu(y)}{C_1^{p(y)} \, [\mu B(x,r)]^{\frac{p(y)}{p(x)}}} \leq \int\limits_{\widetilde{B}(x,r)} \frac{d\mu(y)}{\mu B(x,r)} \leq 1.$$

Then

$$\left\|\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)} = \inf\left\{\eta > 0 : \int\limits_{\widetilde{B}(x,r)} \eta^{-p(y)}\, d\mu(y) \le 1\right\} \le C_1\,[\mu B(x,r)]^{\frac{1}{p(x)}}. \quad \square$$

Lemma 7. *Let* Ω *be bounded,* $0 \le \lambda(x) \le n$ *and* $0 \le \mu(x) \le n$*. If* $p(\cdot)$ *and* $q(\cdot)$ *are* log-*Hölder continuous,* $p(x) \le q(x)$ *and*

$$\frac{n-\lambda(x)}{p(x)} \ge \frac{n-\mu(x)}{q(x)}, \tag{25}$$

then

$$L^{q(\cdot),\mu(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(\Omega). \tag{26}$$

Proof. Let $\|f\|_{q(\cdot),\mu(\cdot)} \le 1$. This is equivalent to assuming that $I_{q(\cdot),\mu(\cdot)}(f) \le 1$ (see Lemma 2). We have to show that $I_{p(\cdot),\lambda(\cdot)}(f) \le C$ for some $C > 0$ not depending on f. Let $x \in \Omega$ and $r \in (0, d_\Omega)$. Applying the Hölder inequality (6) with the exponent $p_1(x) = \frac{q(x)}{p(x)}$, we get

$$r^{-\lambda(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy \le 2\, r^{-\lambda(x)} \left\|f^{p(\cdot)}\chi_{\widetilde{B}(x,r)}\right\|_{p_1(\cdot)} \left\|\chi_{\widetilde{B}(x,r)}\right\|_{p_1'(\cdot)}. \tag{27}$$

By Lemma 6 we have

$$\left\|\chi_{\widetilde{B}(x,r)}\right\|_{p_1'(\cdot)} \le C\, r^{n\left(1-\frac{p(x)}{q(x)}\right)}. \tag{28}$$

For the norm $\left\|f^{p(\cdot)}\chi_{\widetilde{B}(x,r)}\right\|_{p_1(\cdot)}$ we have the estimate

$$\begin{aligned}\left\|f^{p(\cdot)}\chi_{\widetilde{B}(x,r)}\right\|_{p_1(\cdot)} &= \inf\left\{\eta > 0 : \int\limits_{\widetilde{B}(x,r)} |f(y)|^{q(y)}\, \eta^{-\frac{q(y)}{p(y)}} dy \le 1\right\} \\ &\le A^{p_+}\, r^{\mu(x)\frac{p(x)}{q(x)}},\end{aligned} \tag{29}$$

where A is the constant from the inequality

$$\frac{1}{A}\, r^{\frac{\mu(x)}{q(y)}} \le r^{\frac{\mu(x)p(x)}{q(x)p(y)}} \le A\, r^{\frac{\mu(x)}{q(y)}} \tag{30}$$

($A \ge 1$ not depending on x, y, r). Indeed, by (30),

$$\begin{aligned}\int\limits_{\widetilde{B}(x,r)} \left(|f(y)|\, \left[A^{p_+}\, r^{\mu(x)\frac{p(x)}{q(x)}}\right]^{-\frac{1}{p(y)}}\right)^{q(y)} dy &\le \int\limits_{\widetilde{B}(x,r)} \left(A^{-1}\, |f(y)|\, r^{-\frac{\mu(x)p(x)}{q(x)p(y)}}\right)^{q(y)} dy \\ &\le r^{-\mu(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{q(y)}\, dy \le 1\end{aligned}$$

which proves (29). Making use of estimates (28) and (29) in (27), we get

$$\int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy \le C\, r^{n-\lambda(x)-\frac{p(x)}{q(x)}[n-\mu(x)]},$$

which is dominated by $r^{\mu(x)}$ under condition (25). Then $I_{p(\cdot),\lambda(\cdot)}(f) \le c$, with c not depending on x and r. Therefore $\|f\|_{p(\cdot),\lambda(\cdot)} \le C$ (see Lemma 2), which proves embedding (26).

To complete the proof, it remains to note that estimate (30) is a consequence of the log-Hölder continuity of $\frac{p(\cdot)}{q(\cdot)}$. □

4. The Maximal Operator in Variable Exponent Morrey Spaces

Following the notation above we put $\lambda_+ := \operatorname*{ess\,sup}_{x\in\Omega} \lambda(x)$. In the sequel we suppose that

$$0 \le \lambda(x) \le \lambda_+ < n, \quad x \in \Omega. \tag{31}$$

For the constant exponents $p(x) \equiv p$ and $\lambda(x) \equiv \lambda$ the following theorem was proved in [4].

Theorem 2. *Let Ω be a bounded open set in $\mathbb{R}^n$. Under conditions* (31), (8) *and* (9), *the maximal operator M is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

Proof. We have to show that

$$I_{p(\cdot),\lambda(\cdot)}(Mf) \le C \quad \text{for all } f \text{ with } \|f\|_{p(\cdot),\lambda(\cdot)} \le c, \tag{32}$$

where $c > 0$ and $C = C(c)$ does not depend on f. We continue the function f by zero beyond the set Ω whenever necessary, and obtain

$$\int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\, dy = \int\limits_{\mathbb{R}^n} \left((Mf(y))^{\frac{p(y)}{p_-}} \right)^{p_-} \chi_{\widetilde{B}(x,r)}(y) dy.$$

Since Ω is bounded, $\|f\|_{p,\lambda} \ge c\,\|f\|_p$ with some $c > 0$ depending only on d_Ω and λ_+. Then the pointwise estimate (10) is applicable, which yields

$$\int\limits_{\widetilde{B}(x,r)} [Mf(y)]^{p(y)} dy \le C \left(\int\limits_{\Omega} \left[M\left(|f(\cdot)|^{\frac{p(\cdot)}{p_-}} \right) \right]^{p_-} \chi_{\widetilde{B}(x,r)}(y)\, dy + \int\limits_{\Omega} \chi_{\widetilde{B}(x,r)}(y)\, dy \right).$$

By the Fefferman–Stein inequality (for constant $p \in (1,\infty)$)

$$\int\limits_{\mathbb{R}^n} (Mg)(y)^p\, h(y)\, dy \le \int\limits_{\mathbb{R}^n} g(y)^p\, (Mh)(y)\, dy,$$

valid for all non-negative functions g, h (see [9]), we get

$$\int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\, dy \le C \left(\int\limits_{\Omega} |f(y)|^{p(y)} M\chi_{\widetilde{B}(x,r)}(y)\, dy + r^n \right).$$

The following estimate is known

$$M\chi_{B(x,r)}(y) \le \frac{4^n\, r^n}{(|x-y|+r)^n}, \quad x,y\in\mathbb{R}^n, \quad r>0,$$

see [3], Lemma 2. Therefore

$$\int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\,dy$$

$$\le C\left(\int\limits_{\widetilde{B}(x,2r)} |f(y)|^{p(y)}dy + \sum_{j=1}^{\infty} \int\limits_{\widetilde{B}(x,2^{j+1}r)\setminus\widetilde{B}(x,2^j r)} \frac{r^n\,|f(y)|^{p(y)}}{(|x-y|+r)^n}\,dy + r^n\right)$$

$$\le C\left(r^{\lambda(x)} + \sum_{j=1}^{\infty}\frac{(2^{j+1}r)^{\lambda(x)}}{(2^j+1)^n} + r^n\right) \le C\left(r^{\lambda(x)}+r^n\right) \le C\,r^{\lambda(x)},$$

which proves the uniform estimate $I_{p(\cdot),\lambda(\cdot)}(f)\le C$ and completes the proof. □

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) := \sup_{r>0}\frac{1}{|B(x,r)|}\int\limits_{\widetilde{B}(x,r)}\left|f(y)-f_{\widetilde{B}(x,r)}\right|\,dy,$$

where

$$f_{\widetilde{B}(x,r)} = \frac{1}{|\widetilde{B}(x,r)|}\int\limits_{\widetilde{B}(x,r)} f(z)\,dz$$

is the mean value of f over $\widetilde{B}(x,r)$. From the boundedness of the maximal operator M and the pointwise inequality

$$M^\sharp f(x)\le 2Mf(x), \quad x\in\Omega,$$

we can derive the following statement.

Corollary 1. *Under the same conditions of Theorem 2, the sharp maximal operator $M^\sharp$ is bounded in $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

5. Potential Operators in Variable Morrey Spaces

Below we need to assume that $\alpha(\cdot)$ also satisfies the log-condition

$$|\alpha(x)-\alpha(y)| \le \frac{C}{-\ln|x-y|}, \quad |x-y|\le\frac{1}{2}, \quad x,y\in\Omega. \tag{33}$$

The next theorem in the case of constant p and α was proved in [1].

Theorem 3. *Let Ω be bounded. Under conditions (8), (9), (33) and the conditions*

$$\inf_{x\in\Omega}\alpha(x)>0, \quad \sup_{x\in\Omega}\,[\lambda(x)+\alpha(x)p(x)]<n, \tag{34}$$

the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\lambda(\cdot)}(\Omega)$, where $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n-\lambda(x)}$.

Proof. Let $\|f\|_{p(\cdot),\lambda(\cdot)} \leq 1$. As always, we continue the function f by zero beyond the set Ω. We use the standard decomposition

$$I^{\alpha(\cdot)}f(x) = \left(\int\limits_{B(x,2r)} + \int\limits_{\mathbb{R}^n\setminus B(x,2r)} \right) f(y)|x-y|^{\alpha(x)-n}dy =: F(x,r)+G(x,r). \quad (35)$$

The pointwise estimate

$$|F(x,r)| \leq C\, r^{\alpha(x)} Mf(x) \quad (36)$$

with a constant $C>0$ not depending on f and x is well known in the case of constant α and is also valid for variable $\alpha(\cdot)$, under the condition $\inf\limits_{x\in\Omega} \alpha(x) > 0$ (see [20], formula (56)). For $G(x,r)$ we have

$$|G(x,r)| \leq C \sum_{j=1}^{\infty} \int\limits_{\widetilde{B}(x,2^{j+1}r)\setminus \widetilde{B}(x,2^j r)} |f(y)|\, (2^j r)^{-\frac{\lambda(x)}{p(y)}}\, |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(y)}}\, dy\,,$$

where the series turns out to be a finite sum $\sum\limits_{j=1}^{N}$ for any fixed $r>0$, with $N=N(r)$ tending to infinity as $r\to 0$. Since the set Ω is bounded and $p(\cdot)$ satisfies the log-condition, we also have

$$|G(x,r)| \leq C \sum_{j=1}^{\infty} \int\limits_{\widetilde{B}(x,2^{j+1}r)\setminus \widetilde{B}(x,2^j r)} |f(y)|\, (2^j r)^{-\frac{\lambda(x)}{p(y)}}\, |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}}\, dy. \quad (37)$$

Applying the Hölder inequality, we get

$$|G(x,r)| \leq C \sum_{j=1}^{\infty} \left\| |x-y|^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}} \right\|_{p'(\cdot),\mathbb{R}^n\setminus B(x,2^j r)} \left\| (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \right\|_{p(\cdot),B(x,2^{j+1}r)}. \quad (38)$$

The factor $\left\| (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \right\|_{p(\cdot),B(x,2^{j+1}r)}$ is uniformly bounded. Indeed, to see this, in view of (4)–(5) it suffices to show the boundedness of the corresponding modular, i.e. that

$$I_{p(\cdot)}\left((2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{(B(x,2^{j+1}r)} \right) \leq C < \infty,$$

which is valid, since

$$I_{p(\cdot)}\left((2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{(B(x,2^{j+1}r)} \right) \leq (2^{j+1}r)^{-\lambda(x)} \int\limits_{B(x,2^{j+1}r)} |f(y)|^{p(y)}\, dy \leq C < \infty$$

by the definition in (13). Therefore, from (38), by Theorem 1 we obtain

$$|G(x,r)| \leq C_1 \sum_{j=1}^{\infty} (2^j r)^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}} \leq C_2\, r^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}, \quad (39)$$

where the series defining $C_2 = C_1 \sum\limits_{j=1}^{\infty} 2^{-aj}$, $a = \frac{1}{p_+} \inf\limits_{x\in\Omega}[n-\lambda(x)-\alpha(x)p(x)] > 0$, is convergent.

Thus, from (36) and (39) we have

$$\left|I^{\alpha(\cdot)}f(x)\right| \le C\, r^{\alpha(x)} Mf(x) + C_2\, r^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}.$$

Minimizing with respect to r, at $r = (Mf(x))^{-\frac{p(x)}{n-\lambda(x)}}$ we get

$$\left|I^{\alpha(\cdot)}f(x)\right| \le c\, (Mf(x))^{\frac{p(x)}{q(x)}}.$$

Hence, by Theorem 2, we have

$$\int\limits_{\widetilde{B}(x,r)} \left|I^{\alpha(\cdot)}f(y)\right|^{q(y)} dy \le c \int\limits_{\widetilde{B}(x,r)} (Mf(y))^{p(y)}\, dy \le c\, r^{\lambda(x)},$$

which completes the proof of the theorem. □

The statement of the following corollary in the case of constant exponents p, λ and α is known, see [8], Lemma 4, and [2], Corollary 4.4. Note that in the case of constant p, λ and α, the norm equivalence of $I^\alpha f$ and $M^\alpha f$ in Morrey spaces is also known, see [2].

Corollary 2. *Under the assumptions of Theorem* 3 *the fractional maximal operator* $M^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ *to* $L^{q(\cdot),\lambda(\cdot)}(\Omega)$.

Proof. The result follows from Theorem 3 in view of the pointwise estimate

$$M^{\alpha(\cdot)}f(x) \le c\, I^{\alpha(\cdot)}(|f|)(x), \quad 0 < \alpha(x) < n, \tag{40}$$

where $c > 0$ does not depend on f and x. This inequality, well known for constant α, is also valid for variable $\alpha(x)$ with

$$c = \sup_{x\in\Omega} \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\frac{\alpha(x)}{n}} < \infty,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. To prove (40), we observe that $I^{\alpha(\cdot)}(|f|)(x) \ge \int\limits_{\widetilde{B}(x,r)} \frac{|f(y)|\,dy}{|x-y|^{n-\alpha(x)}}$ for any $x \in \Omega$ and $r > 0$. Since $|B(x,r)| = \frac{|\mathbb{S}^{n-1}|}{n} r^n$, we have

$$\frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy \le \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\frac{\alpha(x)}{n}} \int\limits_{\widetilde{B}(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}}\, dy,$$

whence (40) follows. □

The following statement holds by Theorem 3 and embedding (26).

Theorem 4. *Let the set* Ω *be bounded and* $p(\cdot)$ *satisfy conditions* (8) *and* (9). *Assume also that* $\alpha(\cdot)$ *and* $\lambda(\cdot)$ *are* log-*Hölder continuous and condition (34) is satisfied. Then the operator* $I^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ *to* $L^{q(\cdot),\mu(\cdot)}(\Omega)$, *where $q(\cdot)$ is any exponent satisfying log-condition* (9) *and the condition* $1 \leq q(x) \leq \frac{p(x)[n-\lambda(x)]}{n-[\lambda(x)+\alpha(x)p(x)]}$, *and $\mu(\cdot)$ is defined by the condition*

$$\frac{n-\mu(x)}{q(x)} = \frac{n-\lambda(x)}{p(x)} - \alpha(x).$$

In particular, one may take

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \quad \text{and} \quad \mu(x) = \frac{n\lambda(x)}{n-\alpha(x)p(x)}. \tag{41}$$

For constant exponents the statement of Theorem 4 in the case of (41) may be found, for example, in [4] (see Corollary on page 277).

Similarly to Corollary 2 we derive the following result.

Corollary 3. *Under the assumptions of Theorem* 4*, the fractional maximal operator $M^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\mu(\cdot)}(\Omega)$.*

6. Potential Operators: the Limiting Case

In this section we study the limiting case in (34), that is, we consider the critical exponent

$$p(x) = \frac{n-\lambda(x)}{\alpha(x)}. \tag{42}$$

Theorem 5. *Let Ω be bounded, $p(\cdot)$ satisfy conditions* (8), (9) *and* $\inf_{x\in\Omega} \alpha(x) > 0$. *In the case of exponent* (42) *the operator $M^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{\infty}(\Omega)$:*

$$\|M^{\alpha(\cdot)} f\|_{\infty} \leq C\, \|f\|_{p(\cdot),\lambda(\cdot)}. \tag{43}$$

Proof. Let $x \in \Omega$ and $r > 0$. By the log-condition, property (20), the Hölder inequality and estimate (23), we get successively

$$\begin{aligned}
|B(x,r)|^{\frac{\alpha(x)}{n}-1} \int\limits_{\widetilde{B}(x,r)} |f(y)|\, dy &= |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int\limits_{\widetilde{B}(x,r)} r^{\frac{\lambda(x)}{p(y)}}\, r^{-\frac{\lambda(x)}{p(y)}}\, |f(y)|\, dy \\
&\leq C\, |B(x,r)|^{\frac{\alpha(x)}{n}-1}\, r^{\frac{\lambda(x)}{p(x)}} \int\limits_{\widetilde{B}(x,r)} r^{-\frac{\lambda(x)}{p(y)}}\, |f(y)|\, dy \\
&\leq C r^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}}\, f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)} \left\| \chi_{\widetilde{B}(x,r)} \right\|_{p'(\cdot)} \\
&\leq C\, r^{\alpha(x)-n+\frac{\lambda(x)}{p(x)}}\, \|f\|_{p(\cdot),\lambda(\cdot)}\, |B(x,r)|^{\frac{1}{p'(x)}} \leq C\, \|f\|_{p(\cdot),\lambda(\cdot)}. \qquad \square
\end{aligned}$$

Remark 2. Since the set Ω is bounded, inequality (43) obviously holds also in the super-critical case $p(x) > \frac{n-\lambda(x)}{\alpha(x)}$.

In the limiting situation, the mapping properties of the Riesz potential operator $I^{\alpha(\cdot)}$ and the fractional maximal operator $M^{\alpha(\cdot)}$ are slightly different. For constant exponents, it is well known that a result similar to Theorem 5 holds for

I^α only if we replace $L^\infty(\Omega)$ by the space $BMO = \{f : M^\sharp f \in L^\infty\}$ equipped with the norm

$$\|f\|_{BMO} := \|M^\sharp f\|_\infty$$

(supposing that we do not distinguish functions differing by a constant).

The similar $L^{p(\cdot),\lambda(\cdot)} \to BMO$-boundedness holds also in the variable exponent setting. In the case of constant exponents this was proved by S. Spanne and published in [19] (see Theorem 5.4 in [19]). To extend this boundedness to the case of variable $p(\cdot)$ and $\lambda(\cdot)$ we make use of the pointwise estimate (see [1], Proposition 3.3)

$$M^\sharp(I^\alpha f)(x) \leq c\, M^\alpha f(x), \quad x \in \Omega. \tag{44}$$

Then from (44) and Theorem 5 the following statement follows immediately.

Theorem 6. *Let $\lambda(x) \geq 0$, $0 < \alpha < n$, $\sup\limits_{x\in\Omega} \lambda(x) < n - \alpha$, and let $p(x) = \frac{n-\lambda(x)}{\alpha}$. Then under condition* (9) *the operator I^α is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $BMO(\Omega)$.*

Remark 3. Within the framework of variable exponent spaces the result of Theorem 6 seems to be new even in the case of variable Lebesgue spaces, that is, in the case, where $\lambda(x) \equiv 0$.

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