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Complex Analysis and Operator Theory

Dominated Compactness Theorem in Banach Function Spaces and its Applications

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Abstract. A famous *dominated compactness theorem* due to Krasnosel'skiĭ states that compactness of a regular linear integral operator in L^p follows from that of a majorant operator. This theorem is extended to the case of the spaces $L^{p(\cdot)}(\Omega, \mu, \varrho)$, $\mu\Omega < \infty$, with variable exponent $p(\cdot)$, where we also admit power type weights ϱ. This extension is obtained as a corollary to a more general similar dominated compactness theorem for arbitrary Banach function spaces for which the dual and associate spaces coincide. The result on compactness in the spaces $L^{p(\cdot)}(\Omega, \mu, \varrho)$ is applied to fractional integral operators over bounded open sets.

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1. Introduction

A well known theorem of Krasnosel'skiĭ, see [12, 13], states that if a linear integral operator $\mathbb{K}_0$ with a positive kernel $\mathcal{K}_0(x, y)$ is compact in L^p, then the same is valid for any linear integral operator with the kernel $\mathcal{K}(x, y)$ satisfying the condition

$$|\mathcal{K}(x, y)| \le \mathcal{K}_0(x, y)\,. \tag{1.1}$$

In relation with various questions of operator theory within the frameworks of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega, \varrho)$ there arose a necessity of extension of Krasnosel'skiĭ's theorem to the case of weighted variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega, \varrho)$. We refer to surveying papers [6, 8, 18] on operator theory and harmonic analysis in such spaces in general, and to the paper [9] where the compactness of potential operators in variable exponent spaces was proved.

Observe that a study of compactness of operators in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega, \varrho)$ may be made also via interpolation of the property of compactness, at the least in non-weighted case. An extension of the well-known theorem of M. A. Krasnosel'skiĭ [11] on the interpolation of the compactness property in L^p-spaces to the case of variable $p(x)$ was proved in [17].

In some applications, Krasnosel'skiĭ type theorem on compactness, based on straightforward comparison (1.1) of kernels, is more direct. We give an extension of this Krasnosel'skiĭ's theorem in the context of more general setting of Banach function spaces (BFS). We refer to [2] for BFS. Note that the study of various problems related to compactness of operators in Banach function spaces has a long history, we refer e.g. to the paper [16] and books [15, 20].

As a corollary, we obtain such an extension for variable exponent spaces $L^{p(\cdot)}(\Omega, \varrho)$ with a power type weight. The latter is used to show the compactness of potential type operators in these spaces.

The main results, at the least from the point of view of further applications, are given in Section 4, in Theorems 4.1 and 4.5, in terms of weighted generalized Lebesgue spaces $L^{p(\cdot)}(\Omega, \varrho)$ with variable exponent. However, Theorem 4.1 is obtained as a corollary to a more general Theorem 3.12 (dominated compactness theorem), proved in Section 3 for arbitrary Banach function spaces, for which the dual and associate spaces coincide. Section 2 contains necessary preliminaries.

The authors are thankful to the anonymous referee who called their attention to the paper [14] on the difference between continuous and absolutely continuous norms in Banach function space, and his comments which helped to improve the presentation in the paper.

2. Preliminaries

Let (Ω, μ) be a measure space and $\mathcal{M}(\Omega, \mu)$ a space of measurable functions on Ω.

Definition 2.1 ([2]). A normed linear space $\mathsf{X} = (\mathsf{X}(\Omega, \mu), \|\cdot\|_{\mathsf{X}})$ of functions $f : \Omega \to \mathbb{R}^1$ is called a *Banach function space* if the following conditions are satisfied:

(P1) the norm $\|f\|_{\mathsf{X}}$ $(0 \leq \|f\|_{\mathsf{X}} \leq \infty)$ is defined for all $f \in \mathcal{M}(\Omega, \mu)$;
(P2) $\|f\|_{\mathsf{X}} = 0$ if and only if $f(x) = 0$ μ-a.e. on Ω;
(P3) $\|f\|_{\mathsf{X}} = \||f|\|_{\mathsf{X}}$ for all $f \in \mathsf{X}$;
(P4) if $E \subset \Omega$ with $\mu E < \infty$, then $\|\chi_E\|_{\mathsf{X}} < \infty$;
(P5) if $f_n \in \mathcal{M}(\Omega, \mu)$ and $0 \leq f_n \uparrow f$ μ-a.e. on Ω, then

$$\|f_n\|_{\mathsf{X}} \uparrow \|f\|_{\mathsf{X}},$$

(*strong Fatou property*);
(P6) given $E \subset \Omega$ with $\mu E < \infty$, there exists a positive constant C_E such that

$$\int_E |f(x)| d\mu(x) \leq C_E \|f\|_{\mathsf{X}}.$$

The fundamentals of Banach function spaces can be found in [2].

In what follows, the set Ω will be always assumed to be a finite measure set, i.e., $\mu(\Omega) < \infty$.

The property of Banach function spaces given in Lemma 2.2 is known, see [2, p. 6 and 4], its proof being direct: let $f_\infty = \text{s-lim } f_n$, then by (P6) we have

$$\mu\big(\{x \in \Omega : |f_\infty(x) - f_n(x)| > \varepsilon\}\big) \le \frac{1}{\varepsilon}\int_\Omega |f_\infty(x) - f_n(x)| d\mu(x)$$
$$\le \frac{C_\Omega}{\varepsilon}\|f_\infty - f_n\|_{\mathsf{X}} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Lemma 2.2. *Let* X *be a Banach function space with* $\mu\Omega < \infty$. *Then strong convergence implies convergence in measure.*

Definition 2.3. We say that a function $f \in \mathsf{X}$ possesses *absolutely continuous norm*, if

$$\lim_{\mu(D)\to 0} \|P_D f\| = 0$$

where

$$P_D f(x) = \begin{cases} f(x), & x \in D\,; \\ 0, & x \notin D\,. \end{cases}$$

By X^a we denote the set of all $f \in \mathsf{X}$ which have absolutely continuous norm. Observe that in the case $\mathsf{X}^a \ne \mathsf{X}$, the space X^a is not a Banach function space, as shown in [14] (in [14] it is shown that any (closed) subspace of a Banach function space is not a Banach function space). In the case $\mathsf{X}^a = \mathsf{X}$ we say that the space X has absolutely continuous norm.

Lemma 2.4 ([2, p. 16]). *The set* X^a *is a closed subspace of* X.

Definition 2.5 ([2]). We define the *associate space* X' of a Banach function space X as the set of all measurable functions $g \in \mathcal{M}(\Omega, \mu)$ such that the following norm is finite

$$\|g\|_{\mathsf{X}'} = \sup\left\{\int_\Omega |fg| d\mu : f \in \mathsf{X}, \|f\|_{\mathsf{X}} \le 1\right\}. \tag{2.1}$$

Lemma 2.6 ([2]). *The dual Banach space* X^* *of a Banach function space* X *is isometrically isomorphic to the associate space* X' *if and only if* $\mathsf{X}^a = \mathsf{X}$.

3. Dominated compactness theorem in Banach function spaces

3.1. General preliminaries

Definition 3.1. A family $\mathcal{X}$ of functions in the space X is said to have *equi-absolutely continuous norms*, if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\mu(D) < \delta(\varepsilon)$ implies the inequality $\|P_D f\|_{\mathsf{X}} < \varepsilon$ for all $f \in \mathcal{X}$.

Definition 3.2. A bounded linear operator $T : \mathsf{X} \to \mathsf{Y}$ is *compact in measure* if the image $\{Tu_n\}$ of any bounded sequence $\{u_n\}$ of X contains a Cauchy subsequence with respect to measure, i.e., if $\|u_n\|_{\mathsf{X}} \le C$, then there exists a subsequence $\{u_{n_k}\}$ such that $\forall\varepsilon > 0, \forall\delta > 0$, there exists an $N(\varepsilon, \delta)$ such that

$$\mu_Y\big(\{s \in \Omega : |Tu_{n_k}(s) - Tu_{m_k}(s)| > \varepsilon\}\big) < \delta \quad \text{for all} \quad n_k, m_k > N(\varepsilon, \delta)\,.$$

The following theorem is a kind of version of the known statements from [16] and [13], see also [2, p. 31]. We give its proof for the completeness of the presentation.

Theorem 3.3. *Let* X *and* Y *be Banach function spaces and* T *a bounded linear operator acting from* Y *into* X^a*. The operator* T *is compact if and only if it is compact in measure and the set* $\{Tf : \|f\|_{\mathsf{Y}} \leq 1\}$ *has equi-absolutely continuous norms.*

Proof. Let T be compact. By Lemma 2.2, we only need to check the equi-absolute continuity of the norm in the set $\{Tf : \|f\|_{\mathsf{Y}} \leq 1\}$. Assume, to the contrary, that there exists a sequence of $f_n \in \mathsf{Y}$ belonging to the unit ball and a sequence of sets $D_n \subset \Omega$ such that $\mu(D_n) \to 0$ when $n \to \infty$ and $\|P_{D_n} T f_n\|_{\mathsf{X}} \geq \varepsilon_0 > 0$ for all n. By the compactness of T, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\|Tf_{n_k} - g\|_{\mathsf{X}} \to 0$ when $n \to \infty$, with $g \in \mathsf{X}$, thus $\|Tf_{n_k} - g\|_{\mathsf{X}} < \varepsilon_0/2$ for $n_k > N_1$. By the fact that each function Tf_n has the absolutely continuous norm, by Lemma 2.4 g has absolutely continuous norm as well, thus $\|P_{D_n} g\|_{\mathsf{X}} < \varepsilon_0/2$ for $n > N_2$. Then for $n_k > \max\{N_1, N_2\}$, we have $\|P_{D_{n_k}} T f_{n_k}\|_{\mathsf{X}} < \varepsilon_0$ arriving at a contradiction.

Let now T be compact in measure and the set $\{Tf : \|f\|_{\mathsf{Y}} \leq 1\}$ have equi-absolutely continuous norms. Given $\varepsilon > 0$, define ε_0 as $0 < \varepsilon_0 < \varepsilon/(2 + \mu\Omega)$. For any such $\varepsilon_0 > 0$, by the equi-absolute continuity of norms, there exists $\delta(\varepsilon_0) > 0$ such that for all the sets D with $\mu(D) < \delta(\varepsilon_0)$ we have $\|P_D T f\|_{\mathsf{X}} < \varepsilon_0$ whenever f belongs to the unit ball of Y. Let $\|f_n\|_{\mathsf{X}} \leq 1$. We denote $E_{n,m}(\varepsilon) = \{s \in \Omega : |Tf_n(s) - Tf_m(s)| > \varepsilon\}$. By the compactness in measure of the operator T, there exists $\{f_{n_k}\}$ and $N(\varepsilon_0, \delta(\varepsilon_0))$ such that

$$\mu\big(E_{n_k,m_k}(\varepsilon_0)\big) < \delta(\varepsilon_0) \quad \text{when} \quad n_k, m_k > N\big(\varepsilon_0, \delta(\varepsilon_0)\big) .$$

Then we have

$$\begin{aligned} \|Tf_{n_k} - Tf_{m_k}\|_{\mathsf{X}} &\leq \|P_{E_{n_k,m_k}(\varepsilon_0)}(Tf_{n_k} - Tf_{m_k})\|_{\mathsf{X}} \\ &\quad + \|P_{\Omega \setminus E_{n_k,m_k}(\varepsilon_0)}(Tf_{n_k} - Tf_{m_k})\|_{\mathsf{X}} \\ &\leq \|P_{E_{n_k,m_k}(\varepsilon_0)} T f_{n_k}\|_{\mathsf{X}} + \|P_{E_{n_k,m_k}(\varepsilon_0)} T f_{m_k}\|_{\mathsf{X}} + \varepsilon_0 \mu\Omega \\ &\leq \varepsilon_0(2 + \mu\Omega) \\ &< \varepsilon \end{aligned}$$

which proves the compactness of T. □

Theorem 3.4. *Let* X *and* Y *be Banach function spaces and* T *be a bounded linear operator acting from* Y *to* X^a*. The operator* T *is compact if and only if it is compact in measure and*

$$\lim_{\mu(D) \to 0} \|P_D T\|_{\mathsf{Y} \to \mathsf{X}} = 0 . \tag{3.1}$$

Proof. "If" part. By (3.1) the range of operator T on each ball has equi-absolutely continuous norms. Then the result follows from Theorem 3.3.

"Only if" part. By Theorem 3.3, T is compact in measure. Suppose, to the contrary, that (3.1) is not valid. Then there exists a sequence $\{f_n\}$ of functions with $\|f_n\|_{\mathsf{Y}} \leq 1$ and a sequence of sets D_n with measure converging to zero as $n \to \infty$ such that

$$\|P_{D_n} T f_n\|_{\mathsf{X}} \geq \varepsilon_0 > 0\,, \quad \forall n \in \mathbb{N}\,, \tag{3.2}$$

which contradicts the equi-absolutely continuity of the norms of the elements $\{Tf : \|f\|_{\mathsf{Y}} \leq 1\}$. □

3.2. Regular operators

In this section, when extending the Krasnosel'skiĭ theorem on compactness of regular integral operators in L^p to the case of arbitrary Banach function spaces, we mainly follow ideas of book [13]. We consider linear integral operators of the form

$$\mathbb{K}f(x) = \int_\Omega \mathcal{K}(x,y) f(y) d\mu(y) \tag{3.3}$$

where it is always assumed that the kernel $\mathcal{K}(x,y)$ is measurable and integrable in y on Ω for almost all $x \in \Omega$.

Definition 3.5. An operator $\mathbb{K}$ acting from a space X into a space Y is called a *regular linear integral operator from* X *to* Y, if the operator $|\mathbb{K}|$ defined by

$$|\mathbb{K}|f(x) := \int_\Omega |\mathcal{K}(x,y)| f(y) d\mu(y)$$

is bounded from X to Y.

Definition 3.6. Let Ψ be a linear subspace of the space X^*. A sequence $\{x_n\} \in \mathsf{X}$ is called Ψ-weakly convergent, if, for each $\psi \in \Psi$, the sequence $\{\psi(x_n)\}$ converges.

Lemma 3.7. *Let Ψ be a linear subspace of X^*. If Ψ is separable, then X is Ψ-weakly compact.*

Proof. We wish to prove that given $\{f_n\}$ with $\|f_n\| \leq 1$, there exists a subsequence $\{f_{n_k}\}$ such that $\{\psi(f_{n_k})\}$ is a Cauchy sequence, where $\psi \in \Psi$.

Such a subsequence may be constructed inductively, basing on the fact that Ψ is separable, so that it has a countable dense set

$$\Phi = \{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n, \ldots\}\,.$$

First, we note that given a linear functional $\psi \in \mathsf{X}^*$ and a sequence $\{f_n\}$ in the unit ball of X, there exists a subsequence $\{f_{n_k}\}$ such that $\{\psi(f_{n_k})\}$ is convergent. (Just note that the set $\{\psi(f_{n_k})\}$ is a bounded set of $\mathbb{R}^1$ and use the Bolzano–Weierstraß theorem). Then from $\{f_n\}$ we can find $\{f_n^1\}$ such that $\{\varphi_1(f_n^1)\}$ converges. From $\{f_n^1\}$ we find a subsequence $\{f_n^2\}$ such that $\{\varphi_2(f_n^2)\}$ converges and similarly, we can find $\{f_n^k\}$, a subsequence of $\{f_n^{k-1}\}$, such that $\{\varphi_k(f_n^k)\}$ converges, *ad infinitum*; by Cantor diagonal process we choose the subsequence $\{\varkappa_n\}$, i.e., $\{\varkappa_n\} = \{f_n^n\}$. Note that, for any $\varphi_k \in \Phi$, $\{\varphi_k(\varkappa_n)\}$ is convergent.

Finally, given $\varepsilon > 0$ choose from Φ an appropriate φ_N such that $\|\psi - \varphi_N\| < \varepsilon/3$, and an appropriate M such that $n, m > M$ implies $|\varphi_N(\varkappa_n) - \varphi_N(\varkappa_m)| < \varepsilon/3$, then we have:

$$\begin{aligned}|\psi(\varkappa_n) - \psi(\varkappa_m)| &\leq |\psi(\varkappa_n) - \varphi_N(\varkappa_n)| + |\varphi_N(\varkappa_n) - \varphi_N(\varkappa_m)| \\ &\quad + |\varphi_N(\varkappa_m) - \psi(\varkappa_m)| \\ &< 2\|\psi - \varphi_N\| + \varepsilon/3 \\ &< \varepsilon\,.\end{aligned}$$

□

Corollary 3.8. *The space $L_\infty(\Omega, \mu)$ is L_1-weakly compact.*

Proof. Indeed, it suffices to mention that $L_1(\Omega, \mu)$, a subspace of $L^*_\infty(\Omega, \mu)$, is separable. □

Theorem 3.9. *A regular linear integral operator $\mathbb{K}$ acting from L_∞ into a space X^a is compact.*

Proof. Let $u(x) = \int_\Omega |\mathcal{K}(x, y)| d\mu(y)$. Since $\mathbb{K}$ is a regular operator from L_∞ into X^a, we have $u \in \mathsf{X}^a$. Then $|\mathbb{K}|f(x) \leq u(x)\|f\|_\infty$ for all $f \in L_\infty$. By the properties of the norm, we obtain

$$\|P_D\mathbb{K}\|_{L_\infty \to \mathsf{X}} \leq \|P_D u\|_{\mathsf{X}}$$

thus proving that

$$\lim_{\mu(D)\to 0} \|P_D\mathbb{K}\|_{L_\infty \to \mathsf{X}} = 0\,. \tag{3.4}$$

For almost all $x \in \Omega$, the functional

$$F_x(f) = \int_\Omega \mathcal{K}(x, y) f(y) d\mu(y)$$

is a continuous linear functional on L_∞ for those x when $u(x)$ is finite. Since L_∞ is L_1-weakly compact by Corollary 3.8, from each bounded sequence $\{f_n\}$ in L_∞ there may be derived a subsequence $\{f_{n_k}\}$ such that $F_x(f_{n_k})$ converges for almost all $x \in \Omega$, that is, the sequence of numbers $\mathbb{K}f_{n_k}(x)$ converges, which implies that $\mathbb{K}$ transforms each ball in L_∞ into a set of functions compact in measure. By (3.4) and the compactness in measure, the result follows from Theorem 3.4. □

Theorem 3.10. *Let $\mathsf{X} = \mathsf{X}^a$. A regular linear integral operator $\mathbb{K}$ acting from a space X into L_1 is compact.*

Proof. By the Schauder theorem on the compactness of the dual operator, see [5, 19], the required compactness is equivalent to the compactness of the operator $\mathbb{K}^*$ from L_∞ to $\mathsf{X}^* = \mathsf{X}'$. According to Theorem 3.9, it suffices to check that the operator $\mathbb{K}^*$ acts boundedly from L_∞ to $(\mathsf{X}')^a$. This is valid, if the operator $\mathbb{K}$ is bounded from $[(\mathsf{X}')^a]^*$ to L_1. The latter holds by the assumption of the theorem, since it is known that $[(\mathsf{X}')^a]^* = (\mathsf{X}')'$, see [2, p. 23, Corollary 4.2], and $\mathsf{X}'' = \mathsf{X}$, see [2, p. 10, Theorem 2.7]. □

Theorem 3.11. *Let $\mathsf{X} = \mathsf{X}^a$. A regular linear integral operator $\mathbb{K}$ acting from a space X into a space Y^a is compact in measure.*

Proof. This follows from the fact that $\mathbb{K}$ acts from X^a into L_1. Then by Theorem 3.10, the operator is compact and therefore, it is compact in measure. □

Let

$$\mathbb{K}_0 f(x) = \int_\Omega \mathcal{K}_0(x,y) f(y) d\mu(y)\,, \quad \mathcal{K}_0(x,y) \geq 0\,. \tag{3.5}$$

In the case

$$|\mathcal{K}(x,y)| \leq \mathcal{K}_0(x,y)\,, \quad (x,y) \in \Omega \times \Omega\,, \tag{3.6}$$

we say that the operator $\mathbb{K}_0$ is a majorant of the operator $\mathbb{K}$.

Theorem 3.12. *Let* $\mathsf{X} = \mathsf{X}^a$. *Let condition* (3.6) *be fulfilled and suppose that the operator* $\mathbb{K}_0$ *acts from a space* X *into a space* Y^a *and is compact. Then* $\mathbb{K}$ *is also a compact operator acting from* X *into* Y^a.

Proof. We have

$$\begin{aligned} \lim_{\mu(D)\to 0} \|P_D \mathbb{K}\|_{\mathsf{X}\to\mathsf{Y}} &= \lim_{\mu(D)\to 0} \sup_{\|f\|_{\mathsf{X}} \leq 1} \|P_D \mathbb{K} f\|_{\mathsf{Y}} \\ &\leq \lim_{\mu(D)\to 0} \sup_{\|f\|_{\mathsf{X}} \leq 1} \|P_D \mathbb{K}_0(|f|)\|_{\mathsf{Y}} \\ &\leq \lim_{\mu(D)\to 0} \|P_D \mathbb{K}_0\|_{\mathsf{X}\to\mathsf{Y}} = 0\,. \end{aligned}$$

Then the operator $\mathbb{K}$ is compact in measure by Theorem 3.11. Therefore, its compactness follows from Theorem 3.4. □

4. Application to variable exponent Lebesgue spaces

Let Ω be an open set in $\mathbb{R}^n$ and $d\mu(x) = dx$ the Lebesgue measure. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$, where $1 \leq \operatorname{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$, is the set of functions for which the following modular

$$I_p(f) := \int_\Omega |f(x)|^{p(x)} dx$$

is finite. This is a Banach function space with respect to the norm

$$\| f \|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1 \right\},$$

see [7]. The modular $I_p(f)$ and the norm $\|f\|_{p(\cdot)}$ are related to each other by

$$\|f\|_{p(\cdot)}^{\sigma} \leq I_p(f) \leq \|f\|_{p(\cdot)}^{\theta} \tag{4.1}$$

where

$$\sigma = \begin{cases} \operatorname{ess\,inf}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \geq 1\,; \\ \operatorname{ess\,sup}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \leq 1\,; \end{cases} \quad \text{and} \quad \theta = \begin{cases} \operatorname{ess\,inf}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \leq 1\,; \\ \operatorname{ess\,sup}_{x\in\Omega} p(x), & \|f\|_{p(\cdot)} \geq 1\,. \end{cases}$$

The basics on the spaces $L^{p(\cdot)}(\Omega)$ may be found in [10].

In the sequel we assume that $p(x)$ satisfies the following standard conditions

$$1 < \operatorname*{ess\,inf}_{x\in\Omega} p(x) \le p(x) \le \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty \tag{4.2}$$

and

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2}, \quad x, y \in \Omega. \tag{4.3}$$

The continuous imbedding

$$L^{p(x)} \hookrightarrow L^{r(x)}, \quad 1 \le r(x) \le p(x) \le \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty \tag{4.4}$$

is valid if $|\Omega| < \infty$.

By $q(x)$ we denote the conjugate exponent: $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$.

Let ϱ be an almost everywhere positive integrable function, called weight. The *weighted variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega, \varrho)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} = \|\varrho f\|_{p(\cdot)} < \infty .$$

The Hölder inequality holds in the form

$$\left| \int_\Omega f(x)g(x)\, dx \right| \le k \|f\|_{L^{p(\cdot)}(\Omega,\varrho)} \|g\|_{L^{q(\cdot)}(\Omega,1/\varrho)} . \tag{4.5}$$

We deal with the class $W_{p(\cdot)}(\Omega)$ of weights related to the exponent $p(x)$ in the following way. We say that $\varrho \in W_{p(\cdot)}(\Omega)$, if ϱ is a finite product of power weights of the form

$$\varrho(x) = \prod_{k=1}^{N} |x - x_k|^{\beta_k}, \quad x_k \in \Omega, \tag{4.6}$$

where

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{q(x_k)}, \quad k = 1, 2, \dots, N. \tag{4.7}$$

Theorem 4.1. *Let* $\mathbb{K}$ *and* $\mathbb{K}_0$ *be regular linear integral operators as defined in* (3.3) *and* (3.5)*, acting from* $L^{p_1(\cdot)}(\Omega, \varrho_1)$ *into* $L^{p_2(\cdot)}(\Omega, \varrho_2)$*, where* $|\Omega| < \infty$*, let* $p_1(\cdot)$ *and* $p_2(\cdot)$ *satisfy conditions* (4.2)–(4.3) *and* $\varrho_j \in W_{p_j(\cdot)}(\Omega), j = 1, 2$*. If*

$$|\mathcal{K}(x,t)| \le \mathcal{K}_0(x,t),$$

and the majorizing operator $\mathbb{K}_0$ *is compact from* $L^{p_1(\cdot)}(\Omega, \varrho_1)$ *to* $L^{p_2(\cdot)}(\Omega, \varrho_2)$*, then the operator* $\mathbb{K}$ *is also compact.*

Proof. The statement of the theorem follows from Theorem 3.12, since $L^{p_j(\cdot)}(\Omega,\varrho_j)$ $(j = 1, 2)$ are Banach function spaces with absolutely continuous norms. The latter follows from Lemma 2.6, since for the case of variable exponent Lebesgue spaces the associate space and dual space coincide under the assumptions of the theorem. □

From this theorem we will derive a statement on compactness of potential type operators in weighted variable exponent Lebesgue spaces. To this end, we first need Lemma 4.3 below, in which we approximate integral operators

$$\mathbb{K}f(x) = \int_\Omega \mathcal{K}(x,y)f(y)dy\,, \quad x \in \Omega\,, \tag{4.8}$$

by similar operators $\mathbb{K}_m f(x) = \int_\Omega \mathcal{K}_m(x,y)f(y)dy$, with degenerate kernels $\mathcal{K}_m(x,y)$ of the form

$$\mathcal{K}_m(x,y) = \sum_{k=1}^m a_k(x)b_k(y)\,. \tag{4.9}$$

We also make use of the following fact for the mixed norm spaces.

Remark 4.2. Let $A_k \subset \Omega$ be pairwise disjoint bounded open sets and let $B_k \subset \Omega$ satisfy the same property. Functions of the form

$$\ell_m(x,y) = \sum_{k=1}^m C_k \chi_{A_k}(x)\chi_{B_k}(y)\,, \tag{4.10}$$

where $C_k = const$, are dense in every mixed norm space $L^P[L^Q](\Omega\times\Omega)$ for all the constant exponents P and $Q, 1 \le P < \infty, 1 \le Q < \infty$. We refer to [1,3,4] for mixed norm spaces.

Lemma 4.3. *Let Ω be a bounded open set, let $p \in C(\Omega), 1 \le p(x) \le P < \infty$ and $\varrho \in W_{p(\,\cdot\,)}(\Omega)$ and let $\mathcal{K}(x,y) \in L^\infty(\Omega\times\Omega)$. Then there exists a sequence of bounded operators $\mathbb{K}_m$ with degenerate kernels $\mathcal{K}_m(x,y)$ of form* (4.9) *such that*

$$\|\mathbb{K} - \mathbb{K}_m\|_{L^{p(\cdot)}(\Omega,\varrho)\to L^{p(\cdot)}(\Omega,\varrho)} \to 0 \tag{4.11}$$

as $m \to \infty$.

Proof. Without loss of generality we can consider a single weight

$$\varrho(x) = |x - x_0|^\beta\,.$$

(The case of the products of such weights is reduced to the case of a single weight by the known standard arguments with the unity partition $1 = \sum_{k=1}^n \omega_k(x)$, where the C^∞-functions $\omega_k(x)$ are supported in a neighborhood of the point x_k and are identical zero in neighborhoods of other points x_j with $j \ne k$).

To find the approximations $\mathcal{K}_m(x,y)$, we proceed as follows. By Hölder inequality (4.5), we obtain

$$|(\mathbb{K} - \mathbb{K}_m)f(x)| \le c\,\|\varrho f\|_{p(\,\cdot\,)} \left\| \frac{1}{\varrho}\big[\mathcal{K}(x,\,\cdot\,) - \mathcal{K}_m(x,\,\cdot\,)\big]\right\|_{q(\,\cdot\,)} \tag{4.12}$$

and then

$$\|(\mathbb{K} - \mathbb{K}_m)f\|_{L^{p(\cdot)}(\Omega,\varrho)} \le c\,\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} \left\| \varrho \left\| \frac{1}{\varrho}\,(\mathcal{K} - \mathcal{K}_m)\right\|_{q(\,\cdot\,)}\right\|_{p(\,\cdot\,)}. \tag{4.13}$$

Let $B_\delta = B_\delta(x_0) = \{x \in \Omega : |x - x_0| < \delta\}$, where $\delta > 0$ will be later chosen sufficiently small. We find it convenient to introduce the notation

$$\Omega_1 = \Omega \backslash B_\delta\,, \quad \Omega_2 = B_\delta\,, \quad \Omega_{ij} = \Omega_i \times \Omega_j\,, \quad i,j = 1,2$$

and

$$\begin{aligned} P_1 &= \sup_{x\in\Omega} p(x)\,, & Q_1 &= \sup_{x\in\Omega} q(x)\,, \\ P_2 &= \sup_{x\in B_\delta} p(x)\,, & Q_2 &= \sup_{x\in B_\delta} q(x)\,. \end{aligned}$$

We split the weights

$$\varrho(x) = c(x)\chi_{\Omega_1}(x) + \varrho(x)\chi_{\Omega_2}(x) \quad \text{and} \quad \frac{1}{\varrho(y)} = d(y)\chi_{\Omega_1}(y) + \frac{\chi_{\Omega_2}(y)}{\varrho(y)} \tag{4.14}$$

where $c(x)$ and $d(y)$ are bounded functions.

According to the splitting in (4.14), from (4.13) we have

$$\begin{aligned} \|(\mathbb{K} - \mathbb{K}_m)f\|_{L^{p(\cdot)}(\Omega,\varrho)} \le c\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} \Bigg(& \big\| \|(\mathcal{K} - \mathcal{K}_m)\|_{L^{q(\cdot)}(\Omega_1)} \big\|_{L^{p(\cdot)}(\Omega_1)} \\ & + \left\| \left\| \frac{1}{\varrho}(\mathcal{K} - \mathcal{K}_m) \right\|_{L^{q(\cdot)}(\Omega_2)} \right\|_{L^{p(\cdot)}(\Omega_1)} \\ & + \big\| \varrho \|(\mathcal{K} - \mathcal{K}_m)\|_{L^{q(\cdot)}(\Omega_1)} \big\|_{L^{p(\cdot)}(\Omega_2)} \\ & + \left\| \varrho \left\| \frac{1}{\varrho}(\mathcal{K} - \mathcal{K}_m) \right\|_{L^{q(\cdot)}(\Omega_2)} \right\|_{L^{p(\cdot)}(\Omega_2)} \Bigg). \end{aligned}$$

By imbedding (4.4) we then obtain

$$\begin{aligned} \|\mathbb{K} - \mathbb{K}_m\|_{L^{p(\cdot)}(\Omega,\varrho) \to L^{p(\cdot)}(\Omega,\varrho)} \le C & \big\| \|(\mathcal{K} - \mathcal{K}_m)\|_{L^{Q_1}(\Omega_1)} \big\|_{L^{P_1}(\Omega_1)} \\ & + \left\| \left\| \frac{1}{\varrho}\,(\mathcal{K} - \mathcal{K}_m) \right\|_{L^{Q_2}(\Omega_2)} \right\|_{L^{P_1}(\Omega_1)} \\ & + \big\| \varrho \|(\mathcal{K} - \mathcal{K}_m)\|_{L^{Q_1}(\Omega_1)} \big\|_{L^{P_2}(\Omega_2)} \\ & + \left\| \varrho \left\| \frac{1}{\varrho}\,(\mathcal{K} - \mathcal{K}_m) \right\|_{L^{Q_2}(\Omega_2)} \right\|_{L^{P2}(\Omega_2)}. \end{aligned}$$

Under the notation

$$\begin{aligned} k^{11}(x,y) &= \mathcal{K}(x,y)\chi_{\Omega_1}(x)\chi_{\Omega_1}(y)\,, & k^{12}(x,y) &= \frac{\mathcal{K}(x,y)}{\varrho(y)}\chi_{\Omega_1}(x)\chi_{\Omega_2}(y)\,, \\ k^{21}(x,y) &= \varrho(x)\mathcal{K}(x,y)\chi_{\Omega_2}(x)\chi_{\Omega_1}(y)\,, & k^{22}(x,y) &= \frac{\varrho(x)}{\varrho(y)}\mathcal{K}(x,y)\chi_{\Omega_2}(x)\chi_{\Omega_2}(y)\,, \end{aligned}$$

and similar notation for $k_m^{ij}(x,y)$, this turns into

$$\|\mathbb{K}-\mathbb{K}_m\|_{L^{p(\cdot)}(\Omega,\varrho)\to L^{p(\cdot)}(\Omega,\varrho)} \le C\sum_{i,j=1}^{2}\Big\|\|k^{ij}-k_m^{ij}\|_{L^{Q_j}(\Omega_j)}\Big\|_{L^{P_i}(\Omega_i)}. \tag{4.15}$$

Since $\varrho\in W_{p(\cdot)}(\Omega)$, we have $-\frac{n}{p(x_0)}<\beta_0<\frac{n}{q(x_0)}$. By continuity of $p(x)$ we may choose $\delta>0$ sufficiently small so that

$$-\frac{n}{P_2}<\beta_0<\frac{n}{Q_2}. \tag{4.16}$$

Then

$$k^{ij}(x,y)\in L^{P_i}[L^{Q_j}](\Omega_i\times\Omega_j),$$

which was the main goal of the splitting we made. Therefore, by Remark 4.2 every function $k^{ij}(x,y), i,j=1,2$, may be approximated in $L^{P_i}[L^{Q_j}](\Omega_i\times\Omega_j)$ by degenerate functions of form (4.10). We chose $k_m^{ij}(x,y)$ as these approximations. Then

$$\sum_{i,j=1}^{2}\Big\|\|k^{ij}-k_m^{ij}\|_{L^{Q_j}(\Omega_j)}\Big\|_{L^{P_i}(\Omega_i)}\to 0$$

as $m\to\infty$ and we arrive at (4.11) under the choice $\mathcal{K}_m(x,y)$

$$\begin{aligned}\mathcal{K}_m(x,y)&=k_m^{11}(x,y)\chi_{\Omega_1}(x)\chi_{\Omega_1}(y)+\varrho(y)k_m^{12}(x,y)\chi_{\Omega_1}(x)\chi_{\Omega_2}(y)\\&\quad+\frac{k_m^{21}(x,y)}{\varrho(x)}\chi_{\Omega_2}(x)\chi_{\Omega_1}(y)+\frac{\varrho(y)}{\varrho(x)}k_m^{22}(x,y)\chi_{\Omega_2}(x)\chi_{\Omega_2}(y).\end{aligned}$$ □

Corollary 4.4. *Let $\Omega, p(x)$ and $\varrho(x)$ satisfy the assumptions of Lemma 4.3. Integral operators with bounded kernel are compact in $L^{p(\cdot)}(\Omega,\varrho)$.*

Theorem 4.5. *Let*

$$\left(I_A^{\alpha(\cdot)}\varphi\right)(x)=\int_\Omega\frac{A(x,y)}{|x-y|^{n-\alpha(y)}}\,\varphi(y)dy,$$

where Ω is a bounded open set. Under the conditions

$$A(x,y)\in L^\infty(\Omega\times\Omega)\quad\textit{and}\quad \alpha_0:=\operatorname*{ess\,inf}_{y\in\Omega}\alpha(y)>0$$

the operator $I_A^{\alpha(\cdot)}$ is compact in the space $L^{p(\cdot)}(\Omega,\varrho)$, if $p(x)$ satisfies assumptions (4.2) and (4.3) and $\varrho\in W_{p(\cdot)}(\Omega)$.

Proof. In view of Theorem 4.1, it suffices to prove the compactness of the operator $I_A^{\alpha(\cdot)}$ with $A=const$. Since Ω is bounded and $\operatorname{ess\,inf}\alpha(x)>0$, by the same theorem we may also assume that $\alpha(y)=\alpha=const>0$. The compactness of the operator $I_A^{\alpha(\cdot)}$ with $A=const$ and $\alpha(y)=\alpha_0>0$ under conditions (4.2) and (4.3) was proved in [9] in the non-weighted case $\varrho=const$.

The weighted case may be dealt with via Hedberg's trick and Corollary 4.4. We represent the operator$I^{\alpha_0} := I_A^{\alpha_0}|_{A=1}$ as

$$\begin{aligned} I^{\alpha_0} f(x) &= \int_{|x-y|<\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha_0}} + \int_{|x-y|>\varepsilon} \frac{|f(y)|dy}{|x-y|^{n-\alpha_0}} \\ &= \mathbb{A}_\varepsilon f(x) + \mathbb{B}_\varepsilon f(x) \end{aligned} \tag{4.17}$$

under the usual assumption that $f(y) \equiv 0$ for $y \notin \Omega$. The operator $\mathbb{B}_\varepsilon$ is compact by Corollary 4.4. As is well known,

$$|\mathbb{A}_\varepsilon f(x)| \leq c\varepsilon^{\alpha_0} (Mf)(x)\,. \tag{4.18}$$

Therefore, by (4.18)

$$\begin{aligned} \left\|I^{\alpha_0} - \mathbb{B}_\varepsilon\right\|_{L^{p(\cdot)}(\Omega,\varrho)\to L^{p(\cdot)}(\Omega,\varrho)} &= \left\|\mathbb{A}_\varepsilon\right\|_{L^{p(\cdot)}(\Omega,\varrho)\to L^{p(\cdot)}(\Omega,\varrho)} \\ &\leq c\varepsilon^{\alpha_0} \left\|M\right\|_{L^{p(\cdot)}(\Omega,\varrho)\to L^{p(\cdot)}(\Omega,\varrho)} \xrightarrow[\varepsilon\to 0]{} 0 \end{aligned}$$

in view of the boundedness of the maximal operator in $L^{p(\cdot)}(\Omega, \varrho)$, see [9], so that I^{α_0} is a compact operator as well. □

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