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Embeddings of variable Hajłasz–Sobolev spaces into Hölder spaces of variable order ☆

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ABSTRACT

Pointwise estimates in variable exponent Sobolev spaces on quasi-metric measure spaces are investigated. Based on such estimates, Sobolev embeddings into Hölder spaces with variable order are obtained. This extends some known results to the variable exponent setting.

1. Introduction

Lebesgue and Sobolev spaces with variable exponent have been intensively studied during the last decade. We only mention the surveying papers [9,31,38], where many references may be found. In particular, embeddings of Sobolev spaces started to be investigated since the beginning of the theory of these spaces, mainly those into Lebesgue spaces over Euclidean domains (cf. [8,11,12]). In [13,22] corresponding generalizations were investigated within the frameworks of the measure metric spaces. We also refer to [35], where the continuity of Sobolev functions was proved in the limiting case.

The case when the exponent is greater than the dimension of the Euclidean space $\mathbb{R}^n$ was less studied. The first attempt to get Sobolev embeddings into Hölder classes of variable order was done in [10]. Later a capacity approach was used in [21] to obtain embeddings into the space of continuous functions. Based on certain pointwise inequalities involving the oscillation of Sobolev functions, the authors [2] proved embeddings into variable Hölder spaces on bounded domains with Lipschitz boundary. More recently, Hölder quasicontinuity of Sobolev functions was studied in [25], including estimates for the exceptional set in terms of capacities.

In this paper we derive more general results, namely we obtain embeddings of variable exponent Hajłasz–Sobolev spaces into Hölder classes of variable order on a bounded (quasi-)metric measure space $(\mathcal{X}, d, \mu)$ with doubling condition. Our approach is based on the estimation of Sobolev functions through maximal functions. We refer to papers [4,7,19,20,30], where this way was used in the case of constant exponents.

The paper is organized as follows. After some preliminaries in Section 2 on variable spaces defined on spaces of homogeneous type, in Section 3 we extend some known estimates of the oscillation of Sobolev functions to the variable exponent

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setting. The embeddings of variable Hajłasz–Sobolev spaces into Hölder spaces of variable order on metric measure spaces are proved in Section 4. Sobolev embeddings of higher smoothness are also proved in the Euclidean case.

2. Preliminaries

Everywhere below $\mathcal{X} = (\mathcal{X}, d, \mu)$ is a quasi-metric measure space. For any positive μ-measurable function φ defined on $\mathcal{X}$, φ_- and φ_+ denote the quantities

$$\varphi_+ := \operatorname*{ess\,sup}_{x\in\mathcal{X}} \varphi(x) \quad \text{and} \quad \varphi_- := \operatorname*{ess\,inf}_{x\in\mathcal{X}} \varphi(x). \tag{1}$$

By C (or c) we denote generic positive constants which may have different values at different occurrences. Sometimes we emphasize their dependence on certain parameters (e.g. $C(\alpha)$ or C_α means that C depends on α, etc.).

2.1. Spaces of homogeneous type

By a *space of homogeneous type* we mean a triple $(\mathcal{X}, d, \mu)$, where $\mathcal{X}$ is a non-empty set, $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a quasi-metric on $\mathcal{X}$ and μ is a non-negative Borel measure such that the *doubling condition*

$$\mu B(x, 2r) \leqslant C_\mu \mu B(x, r), \quad C_\mu > 1, \tag{2}$$

holds for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})$, where $B(x, r) = \{y \in \mathcal{X}\colon d(x, y) < r\}$ denotes the open ball centered at x and of radius r. For simplicity, we shall write $\mathcal{X}$ instead of $(\mathcal{X}, d, \mu)$ if no ambiguity arises.

As is well known, by iteration of condition (2) it can be shown that there exists a positive constant C such that

$$\frac{\mu B(x, \varrho)}{\mu B(y, r)} \leqslant C\left(\frac{\varrho}{r}\right)^N, \qquad N = \log_2 C_\mu, \tag{3}$$

for all the balls $B(x, \varrho)$ and $B(y, r)$ with $0 < r \leqslant \varrho$ and $y \in B(x, r)$. From (3) it follows that

$$\mu B(x, r) \geqslant c_0 r^N, \quad x \in \mathcal{X}, \ 0 < r \leqslant \operatorname{diam} \mathcal{X}, \tag{4}$$

in the case $\mathcal{X}$ is bounded. Condition (4) is sometimes called the *lower Ahlfors regularity condition*.

The quasi-metric d is assumed to satisfy the standard conditions:

$$d(x, y) \geqslant 0, \qquad d(x, y) = 0 \iff x = y, \qquad d(x, y) = d(y, x),$$
$$d(x, y) \leqslant a_0\big[d(x, z) + d(z, y)\big], \quad a_0 \geqslant 1.$$

We refer to [5,6,16,26] for general properties of spaces of homogeneous type.

2.2. On variable exponent spaces

Let $p : \mathcal{X} \to [1, \infty)$ be a μ-measurable function. Everywhere below we assume that

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty, \quad x \in \mathcal{X},$$

according to the notation in (1).

By $L^{p(\cdot)}(\mathcal{X})$ we denote the space of all μ-measurable functions f on $\mathcal{X}$ such that the modular

$$I_{p(\cdot)}(f) = I_{p(\cdot),\mathcal{X}}(f) := \int_{\mathcal{X}} |f(x)|^{p(x)}\, d\mu(x)$$

is finite. This is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),\mathcal{X}} := \inf\left\{\lambda > 0\colon\ I_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

Variable exponent Lebesgue spaces over quasi-metric measure spaces have been considered in [13,22,23,29,35] and more recently in [32,33], where the maximal operator was studied on weighted spaces. For completeness, we recall here some basic properties of the spaces $L^{p(\cdot)}(\mathcal{X})$. It is known that the norm $\|\cdot\|_{p(\cdot)}$ and the modular $I_{p(\cdot)}$ are simultaneously greater or simultaneously less than one, from which there follows that

$$c_1 \leqslant \|f\|_{p(\cdot)} \leqslant c_2 \quad \Longrightarrow \quad c_3 \leqslant I_{p(\cdot)}(f) \leqslant c_4$$

and

$$C_1 \leqslant I_{p(\cdot)}(f) \leqslant C_2 \quad \Longrightarrow \quad C_3 \leqslant \|f\|_{p(\cdot)} \leqslant C_4,$$

with $c_3 = \min(c_1^{p_-}, c_1^{p_+})$, $c_4 = \max(c_2^{p_-}, c_2^{p_+})$, $C_3 = \min(C_1^{1/p_-}, C_1^{1/p_+})$ and $C_4 = \max(C_2^{1/p_-}, C_2^{1/p_+})$.

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As usual, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$ and it is defined pointwise by $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \mathcal{X}$. The Hölder inequality is valid in the form

$$\int_{\mathcal{X}} \left| f(x)g(x) \right| d\mu(x) \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

We also note that the embedding

$$L^{q(\cdot)}(\mathcal{X}) \hookrightarrow L^{p(\cdot)}(\mathcal{X})$$

holds for $1 \leqslant p(x) \leqslant q(x) \leqslant q^+ < \infty$, when $\mu(\mathcal{X}) < \infty$.

Often the exponent $p(\cdot)$ is supposed to satisfy the local logarithmic condition

$$\left| p(x) - p(y) \right| \leqslant \frac{A_0}{\ln \frac{1}{d(x,y)}}, \qquad d(x,y) \leqslant 1/2, \quad x, y \in \mathcal{X}, \tag{5}$$

from which we derive

$$\left| p(x) - p(y) \right| \leqslant \frac{2RA_0}{\ln \frac{2R}{d(x,y)}}, \qquad d(x,y) \leqslant R, \quad x, y \in \mathcal{X}.$$

Assumption (5) is known in the literature as *Dini–Lipschitz condition* or log-*Hölder continuity*.

We will also deal with *Hölder spaces* $H^{\lambda(\cdot)}$ of variable order. Hölder functions on metric measure spaces were considered, for instance, in [14,15,34,36] for constant orders λ. Hölder spaces of variable order $\lambda(x)$ were considered in [17,27,28,37] in the one-dimensional Euclidean case and in [39–42] on the unit sphere S^{n-1} in $\mathbb{R}^n$. In the case of variable order we follow a symmetric approach, which was suggested in [37] in the one-dimensional Euclidean case. We say that a bounded function f belongs to $H^{\lambda(\cdot)}(\mathcal{X})$ if there exists $c > 0$ such that

$$\left| f(x) - f(y) \right| \leqslant c\, d(x,y)^{\max\{\lambda(x),\lambda(y)\}}$$

for every $x, y \in \mathcal{X}$, where λ is a μ-measurable function on $\mathcal{X}$ taking values in $(0,1]$. $H^{\lambda(\cdot)}(\mathcal{X})$ is a Banach space with respect to the norm

$$\|f\|_{H^{\lambda(\cdot)}(\mathcal{X})} = \|f\|_\infty + [f]_{\lambda(\cdot)},$$

where

$$[f]_{\lambda(\cdot)} := \sup_{\substack{x,y \in \mathcal{X} \\ 0 < d(x,y) \leqslant 1}} \frac{|f(x) - f(y)|}{d(x,y)^{\max\{\lambda(x),\lambda(y)\}}}.$$

We observe that for $0 < \beta(x) \leqslant \lambda(x) \leqslant 1$, there holds

$$H^{\lambda(\cdot)}(\mathcal{X}) \hookrightarrow H^{\beta(\cdot)}(\mathcal{X}),$$

where "$\hookrightarrow$" means continuous embedding.

2.3. Hajłasz–Sobolev spaces with variable exponent

Let $1 < p_- \leqslant p_+ < \infty$. We say that a function $f \in L^{p(\cdot)}(\mathcal{X})$ belongs to the *Hajłasz–Sobolev space* $M^{1,p(\cdot)}(\mathcal{X})$, if there exists a non-negative function $g \in L^{p(\cdot)}(\mathcal{X})$ such that the inequality

$$\left| f(x) - f(y) \right| \leqslant d(x,y)\left[g(x) + g(y) \right] \tag{6}$$

holds μ-almost everywhere in $\mathcal{X}$. In this case, g is called a *generalized gradient* of f. $M^{1,p(\cdot)}(\mathcal{X})$ is a Banach space with respect to the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{M^{1,p(\cdot)}(\mathcal{X})} := \|f\|_{p(\cdot)} + \inf \|g\|_{p(\cdot)},$$

where the infimum is taken over all generalized gradients of f.

For constant exponents $p(x) \equiv p$, the spaces $M^{1,p}$ were first introduced by P. Hajłasz [18] as a generalization of the classical Sobolev spaces $W^{1,p}$ to the general setting of the quasi-metric measure spaces. If $\mathcal{X} = \Omega$ is a bounded domain with Lipschitz boundary (or $\Omega = \mathbb{R}^n$), endowed with the Euclidean distance and the Lebesgue measure, then $M^{1,p}(\Omega)$ coincides with $W^{1,p}(\Omega)$. Recall that the oscillation of a Sobolev function may be estimated by the maximal function of its gradient. In other words, every function $f \in W^{1,p}(\Omega)$ satisfies (6) by taking $\mathcal{M}(|\nabla f|)$ as a generalized gradient (see, for instance, [4,20,30], for details and applications, and [2] where this property was also discussed for variable exponents).

Hajłasz–Sobolev spaces with variable exponent have been considered in [22,24]. In [24] it was shown that $M^{1,p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n)$ if the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, which generalizes the result from [18] for constant p.

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3. Some pointwise estimates

Let $\alpha : \mathcal{X} \to (0,\infty)$ be a μ-measurable function. We define the fractional sharp maximal function of order $\alpha(\cdot)$ as

$$\mathcal{M}^{\sharp}_{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{r^{-\alpha(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} \left| f(y) - f_{B(x,r)} \right| d\mu(y),$$

where $f_{B(x,r)}$ denotes the average of f over $B(x,r)$, with $f \in L^1_{\mathrm{loc}}(\mathcal{X})$. In the limiting case $\alpha \equiv 0$, $\mathcal{M}^{\sharp}_{\alpha(\cdot)}$ is the well-known Fefferman–Stein function.

According to the notation above we write

$$\alpha_- = \operatorname*{ess\,inf}_{x\in\mathcal{X}} \alpha(x) \quad \text{and} \quad \alpha_+ = \operatorname*{ess\,sup}_{x\in\mathcal{X}} \alpha(x).$$

For constant $\alpha = \beta$ the following inequality was proved in [19, Lemma 3.6], which in turn generalizes Theorem 2.7 in [7], given in the Euclidean setting.

Lemma 1. *Let $\mathcal{X}$ satisfy the doubling condition* (2) *and f be a locally integrable function on $\mathcal{X}$. If*

$$0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < \infty \quad \text{and} \quad 0 < \beta_- \leqslant \beta(x) \leqslant \beta_+ < \infty,$$

then

$$\left| f(x) - f(y) \right| \leqslant C(\mu,\alpha,\beta)\left[d(x,y)^{\alpha(x)} \mathcal{M}^{\sharp}_{\alpha(\cdot)} f(x) + d(x,y)^{\beta(y)} \mathcal{M}^{\sharp}_{\beta(\cdot)} f(y) \right] \tag{7}$$

μ-almost everywhere.

Proof. We skip some details since the proof follows similar arguments of [19]. For a Lebesgue point x we have

$$\left| f(x) - f_{B(x,r)} \right| \leqslant \sum_{j=0}^{\infty} \left| f_{B(x,2^{-(j+1)}r)} - f_{B(x,2^{-j}r)} \right| \leqslant \sum_{j=0}^{\infty} \frac{1}{\mu B(x,2^{-(j+1)}r)} \int\limits_{B(x,2^{-j}r)} \left| f(z) - f_{B(x,2^{-j}r)} \right| d\mu(z).$$

Hence, by the doubling condition (2) we get

$$\left| f(x) - f_{B(x,r)} \right| \leqslant c_\mu \sum_{j=0}^{\infty} \frac{1}{\mu B(x,2^{-j}r)} \int\limits_{B(x,2^{-j}r)} \left| f(z) - f_{B(x,2^{-j}r)} \right| d\mu(z) \leqslant c_\mu c(\alpha) r^{\alpha(x)} \mathcal{M}^{\sharp}_{\alpha(\cdot)} f(x) \tag{8}$$

where $c(\alpha) := \sum_{j=0}^{\infty} 2^{-j\alpha_-} = \frac{2^{\alpha_-}}{2^{\alpha_-}-1}$. On the other hand, similar techniques also yield

$$\left| f(y) - f_{B(x,r)} \right| \leqslant \left| f(y) - f_{B(y,2r)} \right| + \left| f_{B(x,r)} - f_{B(y,2r)} \right| \leqslant c(\mu,\beta) r^{\beta(y)} \mathcal{M}^{\sharp}_{\beta(\cdot)} f(y) \tag{9}$$

when $y \in B(x,r)$ and $\beta_+ < \infty$. Thus, if $x \neq y$ we take $r = 2d(x,y)$ and write

$$\left| f(x) - f(y) \right| \leqslant \left| f(x) - f_{B(x,2d(x,y))} \right| + \left| f(y) - f_{B(x,2d(x,y))} \right|.$$

Now it remains to make use of (8) and (9), where we also observe that both $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded. □

Having in mind some applications, it is of interest to estimate the oscillation of a Hajłasz–Sobolev function in terms of the fractional maximal function of the (generalized) gradient. Recall that the fractional maximal function $\mathcal{M}_{\alpha(\cdot)} f$ of a locally integrable function f is given by

$$\mathcal{M}_{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{r^{\alpha(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} \left| f(y) \right| d\mu(y),$$

where the order α is admitted to be variable, namely α is a μ-measurable function, with $0 \leqslant \alpha(x) \leqslant \alpha_+ < \infty$. In the limiting case $\alpha(x) \equiv 0$ we obtain the well-known Hardy–Littlewood maximal function $\mathcal{M} = \mathcal{M}_0$.

The next lemma is an adaptation of Corollary 3.10 in [19] to variable exponents. We point out that the pointwise inequality (10) has been discussed before for variable exponents in the Euclidean case, see [2, Proposition 3.3].

Lemma 2. *Let $\mathcal{X}$ satisfy the doubling condition* (2) *and let $f \in M^{1,p(\cdot)}(\mathcal{X})$ and $g \in L^{p(\cdot)}(\mathcal{X})$ be a generalized gradient of f. If $0 \leqslant \alpha_+ < 1$, $0 \leqslant \beta_+ < 1$, then*

$$\left| f(x) - f(y) \right| \leqslant C(\mu,\alpha,\beta)\left[d(x,y)^{1-\alpha(x)} \mathcal{M}_{\alpha(\cdot)} g(x) + d(x,y)^{1-\beta(y)} \mathcal{M}_{\beta(\cdot)} g(y) \right] \tag{10}$$

μ-almost everywhere.

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Proof. Taking into account (7), it suffices to show the estimate

$$\mathcal{M}^{\sharp}_{1-\lambda(\cdot)} g(x) \leqslant c\mathcal{M}_{\lambda(\cdot)} g(x), \quad 0 \leqslant \lambda(x) < 1. \tag{11}$$

But (11) follows from the Poincaré type inequality

$$\int_{B(x,r)} \left| f(z) - f_{B(x,r)} \right| d\mu(z) \leqslant cr \int_{B(x,r)} g(z)\, d\mu(z), \quad x \in \mathcal{X},\ r > 0, \tag{12}$$

which is valid for every $f \in M^{1,p(\cdot)}(\mathcal{X})$, where $g \geqslant 0$ is a generalized gradient of f. Indeed, (12) can be obtained just by integrating the both sides of

$$\left| f(y) - f(z) \right| \leqslant d(y,z)\left[g(y) + g(z)\right], \quad \mu-\text{a.e. } y, z \in B(x,r),$$

(see (6)) over the ball $B(x,r)$, first with respect to y and then to z. □

Remark 3. In the previous proof we used partially the statement of Theorem 4.2 in [24]. However, the boundedness of the maximal operator required there is not needed here.

4. Sobolev embeddings into variable exponent Hölder spaces

4.1. Embeddings of variable Hajłasz–Sobolev spaces

Estimate (10) suggests that a function $f \in M^{1,p(\cdot)}(\mathcal{X})$ is Hölder continuous (after a modification on a set of zero measure) if the fractional maximal function of the gradient is bounded. As we will see below, this is the case when the exponent $p(\cdot)$ takes values greater than the "dimension". First we need some auxiliary lemmas.

The following statement was given in [22] (see also [3] for an alternative proof).

Lemma 4. *Let $\mathcal{X}$ be bounded, the measure μ satisfy condition* (4) *and $p(\cdot)$ satisfy condition* (5)*. Then*

$$\|\chi_{B(x,r)}\|_{p(\cdot)} \leqslant c\left[\mu B(x,r)\right]^{\frac{1}{p(x)}} \tag{13}$$

with $c > 0$ not depending on $x \in \mathcal{X}$ and $r > 0$.

Below $N > 0$ denotes the constant from (4).

Lemma 5. *Let $\mathcal{X}$ be bounded and μ satisfy condition* (4)*. Suppose that $p(\cdot)$ is* log*-Hölder continuous. If $f \in L^{p(\cdot)}(\mathcal{X})$, then*

$$\mathcal{M}_{\frac{N}{p(\cdot)}} f(x) \leqslant c\|f\|_{p(\cdot)}, \tag{14}$$

where $c > 0$ is independent of x and f.

Proof. Let $x \in \mathcal{X}$ and $r > 0$. By the Hölder inequality we have

$$\frac{r^{\frac{N}{p(x)}}}{\mu B(x,r)} \int_{B(x,r)} \left|f(y)\right| d\mu(y) \leqslant \frac{2r^{\frac{N}{p(x)}}}{\mu B(x,r)} \|f\|_{p(\cdot)} \|\chi_{B(x,r)}\|_{p'(\cdot)}.$$

From this, we easily arrive at (14) by using the inequality (13) and the assumption (4). □

Theorem 6. *Let $\mathcal{X}$ be bounded and let μ be doubling. Suppose also that $p(\cdot)$ satisfies* (5) *with $p_- > N$. If $f \in M^{1,p(\cdot)}(\mathcal{X})$ and g is a generalized gradient of f, then there exists $C > 0$ such that*

$$\left| f(x) - f(y) \right| \leqslant C\|g\|_{p(\cdot)}\, d(x,y)^{1-\frac{N}{\max\{p(x),p(y)\}}} \tag{15}$$

for every $x, y \in \mathcal{X}$ with $d(x,y) \leqslant 1$.

Proof. After redefining f on a set of zero measure, we make use of (10) with $\alpha(x) = \frac{N}{p(x)}$ and $\beta(y) = \frac{N}{p(y)}$, and get

$$\left| f(x) - f(y) \right| \leqslant C(\mu, N, p)\, d(x,y)^{1-\frac{N}{\min\{p(x),p(y)\}}} \left[\mathcal{M}_{\frac{N}{p(\cdot)}} g(x) + \mathcal{M}_{\frac{N}{p(\cdot)}} g(y)\right]$$

for all $x, y \in \mathcal{X}$. Hence we arrive at (15) taking into account (14). □

The statement of the next theorem was proved in [2] within the frameworks of the Euclidean domains with Lipschitz boundary.

Theorem 7. *Let the set $\mathcal{X}$ be bounded and the measure μ be doubling. If $p(\cdot)$ is* log-Hölder *continuous and $p_- > N$, then*

$$M^{1,p(\cdot)}(\mathcal{X}) \hookrightarrow H^{1-\frac{N}{p(\cdot)}}(\mathcal{X}). \tag{16}$$

Proof. Let $x \in \mathcal{X}$ and $r_0 > 0$ be arbitrary. Recovering the argument from (8), we make use of (11) and we get

$$\begin{aligned} |f(x) - f_{B(x,r_0)}| &\leqslant c r_0^{1-\frac{N}{p(x)}} \mathcal{M}^\sharp_{1-\frac{N}{p(\cdot)}} f(x) \\ &\leqslant c r_0^{1-\frac{N}{p(x)}} \mathcal{M}_{\frac{N}{p(\cdot)}} g(x) \\ &\leqslant c r_0^{1-\frac{N}{p(x)}} \|g\|_{p(\cdot)}, \end{aligned}$$

where in the last inequality we took estimate (14) into account, with $g \in L^{p(\cdot)}(\mathcal{X})$ denoting a gradient of $f \in M^{1,p(\cdot)}(\mathcal{X})$. On the other hand, the Hölder inequality (cf. proof of Lemma 5) yields

$$|f_{B(x,r_0)}| \leqslant c r_0^{-\frac{N}{p(x)}} \|f\|_{p(\cdot)}.$$

Hence, choosing $r_0 = \min\{1, \operatorname{diam}(\mathcal{X})\}$ above, one obtains

$$\|f\|_\infty \leqslant c\|f\|_{1,p(\cdot)}. \tag{17}$$

It remains to show that f is Hölder continuous. To this end, we apply inequality (15) and we get

$$\frac{|f(x) - f(y)|}{d(x,y)^{\max\{1-\frac{N}{p(x)}, 1-\frac{N}{p(y)}\}}} \leqslant c\|g\|_{p(\cdot)} d(x,y)^{\frac{N}{\max\{p(x),p(y)\}} - \frac{N}{\min\{p(x),p(y)\}}}$$

for every $x, y \in \mathcal{X}$, $x \neq y$, with $d(x,y) \leqslant 1$. Since $p(\cdot)$ satisfies the log-condition, then

$$d(x,y)^{\frac{N}{\max\{p(x),p(y)\}}} \sim d(x,y)^{\frac{N}{\min\{p(x),p(y)\}}}.$$

Hence there holds $[f]_{1-\frac{N}{p(\cdot)}} \leqslant c\|g\|_{p(\cdot)}$, from which the embedding (16) follows, having also in mind (17). □

4.2. Further results for the Euclidean case

In the particular case when $\mathcal{X}$ is a bounded domain Ω (with Lipschitz boundary) in the Euclidean space $\mathbb{R}^n$, then

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow H^{1-\frac{n}{p(\cdot)}}(\Omega) \tag{18}$$

where it is assumed that $p(\cdot)$ is log-Hölder continuous and $p_- > n$. In this section we are concerned with corresponding embeddings for higher smoothness. For constant exponents p such embeddings are well known and can be found, for instance, in [1].

Let $\Omega \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}$. Recall that the usual Sobolev space $W^{k,p(\cdot)}(\Omega)$ consists of all functions f for which the (weak) derivatives $D^\beta f$ are in $L^{p(\cdot)}(\Omega)$, for any $0 \leqslant |\beta| \leqslant k$. This is a Banach space with respect to the norm

$$\|f\|_{k,p(\cdot),\Omega} = \sum_{|\beta|\leqslant k} \|D^\beta f\|_{p(\cdot),\Omega}.$$

The next statement was proved in [8].

Theorem 8. *Let $k \in \mathbb{N}$ with $1 \leqslant k < n$. If $p(\cdot)$ is* log-Hölder *continuous and is constant outside some large ball, with $1 < p_- \leqslant p_+ < \frac{n}{k}$, then*

$$W^{k,p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^n), \tag{19}$$

where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{k}{n}$, $x \in \mathbb{R}^n$.

If $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary, then there exists a bounded linear extension operator $\mathcal{E}: W^{k,p(\cdot)}(\Omega) \to W^{k,\tilde{p}(\cdot)}(\mathbb{R}^n)$, such that $\mathcal{E}f(x) = f(x)$ almost everywhere in Ω, for all $f \in W^{k,p(\cdot)}(\Omega)$. The exponent $\tilde{p}(\cdot)$ is an extension of $p(\cdot)$ to the whole $\mathbb{R}^n$ preserving the original bounds and the continuity modulus of $p(\cdot)$. All the details of this construction in the case $k = 1$ can be found in [8, Theorem 4.2 and Corollary 4.3], and [10, Theorem 4.1]; constructions for $k \neq 1$ follow the same way, since the Hestenes method is known to work well with higher derivatives as well. As a consequence we conclude that embedding (19) holds also for bounded open sets Ω with Lipschitz boundary, namely if $p(\cdot)$ satisfies condition (5) in Ω with $1 < p_- \leqslant p_+ < \frac{n}{k}$, then

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega), \qquad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{k}{n}, \quad x \in \Omega. \tag{20}$$

Remark 9. Embedding (20) was proved in [8, Corollary 5.3] in the case $k = 1$, which in turn generalizes a former result from [10] formulated for Lipschitz continuous exponents. We note that embedding (20) was also proved in [12] by assuming the Lipschitz continuity of $p(\cdot)$ and the cone condition in Ω.

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. If $p(\cdot)$ is* log*-Hölder continuous and $(k-1)p_+ < n < kp_-$, then*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow H^{k-\frac{n}{p(\cdot)}}(\Omega). \tag{21}$$

Proof. As in the classical setting of constant exponents, the proof can be reduced to the case $k = 1$ as follows. By (20) we have

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow W^{k-1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),$$

where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{k-1}{n}$, $x \in \Omega$. Thus we also get $W^{k,p(\cdot)}(\Omega) \hookrightarrow W^{1,q(\cdot)}(\Omega)$. Hence it remains to observe that

$$W^{1,q(\cdot)}(\Omega) \hookrightarrow H^{1-\frac{n}{q(\cdot)}}(\Omega),$$

which follows from (18). Indeed, we have $\frac{n}{q_-} = \frac{n}{p_-} - k + 1 < 1$ and $1 - \frac{n}{q(x)} = k - \frac{n}{p(x)}$. □

Remark 11. Since we consider Hölder spaces of orders less than 1, in Theorem 10 we in fact have a restriction $k < \frac{p_+}{p_+ - p_-}$. To avoid this restriction one should make use Hölder spaces of higher order which we do not touch here. Observe that the condition $(k-1)p_+ < n$ of Theorem 10 may be omitted, but then we should just assume that $(k-1)p_+ \neq n$ and embedding (21) written in the form $W^{k,p(\cdot)}(\Omega) \hookrightarrow H^{\{k-\frac{n}{p(\cdot)}\}}(\Omega)$, where $\{k - \frac{n}{p(x)}\}$ stands for the fractional part of $k - \frac{n}{p(x)}$.

Corollary 12. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Let also $p(\cdot)$ be* log*-Hölder continuous with $p_- > \frac{n}{k}$, $k > 1$. Then*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow H^{\lambda(\cdot)}(\Omega),$$

for any function $\lambda(\cdot) \in L^\infty(\Omega)$ such that $\lambda(x) \leqslant k - \frac{n}{p(x)}$ and $\lambda_- > 0$, $\lambda_+ < 1$.

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