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Fractional and hypersingular operators in variable exponent spaces on metric measure spaces

Alexandre Almeida and Stefan Samko

Abstract. We prove the continuity of potential type operators and hypersingular operators in variable Lebesgue and Sobolev spaces on a metric measure space $(\mathcal{X}, d, \mu)$. Two variants of such operators are considered, according to the regularity admitted on the measure μ.

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1. Introduction

Lebesgue and Sobolev spaces with variable exponent have attracted the interest of many researchers during the last years (see, for instance, [40] for the basic properties). In particular, there was an important progress concerning the study of classical operators of Harmonic Analysis in these spaces. We refer to the surveying papers [11], [29], [48] for details on the development of this theory and a more recent paper [9], where the boundedness of various operators was obtained by extrapolation techniques.

In this paper, we study fractional integrals of variable order in variable exponent Lebesgue spaces on doubling and non-doubling measure metric spaces. We prove two versions of Sobolev-type theorems with variable exponents. Various versions of such theorems for constant p were proved in [18], [15], [16], [17], [30], [31]. We also give boundedness statements for corresponding fractional maximal operators. Potential operators on variable Lebesgue spaces were firstly studied in [45].

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We also mention the papers [32], [33], [34], [10], [42], [12] where these operators were considered within the frameworks of variable exponent spaces on Euclidean domains, and paper [23] where the non-weighted boundedness of operators of form

$$\mathfrak{I}^{\alpha} f(x) = \int_{\Omega} \frac{[d(x,y)]^{\alpha}}{\mu B(x, d(x,y))} f(y)\, d\mu(y), \quad \alpha > 0. \tag{1.1}$$

was proved on metric spaces, paper [37] where it was proved for potentials on Carleson curves, and papers [38], [39], where weighted boundedness was obtained on doubling metric measure spaces for potentials of the form

$$I_X^{\alpha} f(x) = \int_X \frac{f(y)\, d\mu(y)}{\mu B(x, d(x,y))^{1-\gamma}}. \tag{1.2}$$

In this paper we deal with potential operators of form (1.1), where we admit variable exponent $\alpha(x)$, and also potentials of a different form, see (3.1). We also consider hypersingular integrals of variable order of Sobolev functions with variable exponent on metric measure spaces. We refer to the books [49] and [47] for hypersingular integrals in general. Hypersingular operators in the variable exponent setting were firstly studied by the authors in [1], [2], [46] in related to the problem of inversion and characterization of the Riesz potentials in the Euclidean case, and in [3] there where studied some mapping properties of hypersingular integrals in the context of variable exponent Sobolev spaces on metric measure spaces.

In the case of constant exponents, hypersingular integrals on metric spaces, jointly with potential operators, were considered in [16], [17].

This paper is structured as follows. Notation and basic definitions on variable exponent spaces on metric measure spaces are given in Section 2. Fractional integrals are studied in Section 3. The main results are the Sobolev-type Theorems 3.6 and 3.2 given for doubling and non-doubling metric spaces, respectively. Section 4 deals with fractional maximal functions. Hypersingular integrals are studied in Section 5. In this section sufficient conditions are given for the boundedness and pointwise convergence of hypersingular integrals of variable Hajłasz-Sobolev functions.

2. Preliminaries

In the sequel, we use the notation

$$\varphi_+ := \operatorname*{ess\,sup}_{x\in\mathcal{X}} \varphi(x) \quad \text{and} \quad \varphi_- := \operatorname*{ess\,inf}_{x\in\mathcal{X}} \varphi(x), \tag{2.1}$$

where φ is non-negative function defined on a quasi-metric measure space $\mathcal{X} = (\mathcal{X}, d, \mu)$.

By c (or C) we denote generic positive constants which may take different values at different occurrences. Sometimes we add subscripts to the constant (e.g. c_0, c_1, ...) or emphasize its dependence on certain parameters ($c(\alpha)$ or c_α means that c depends on α, etc).

2.1. Metric measure spaces

By a *(quasi-)metric measure space* we mean a triple $(\mathcal{X}, d, \mu)$, where $\mathcal{X}$ is a non-empty set, $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a (quasi-)metric on $\mathcal{X}$ and μ is a non-negative Borel measure. The quasi-metric d is assumed to satisfy the standard conditions:

$$d(x,y) \geq 0, \quad d(x,y) = 0 \iff x = y, \quad d(x,y) = d(y,x),$$
$$d(x,y) \leq a_0[d(x,z) + d(z,y)], \quad a_0 \geq 1. \tag{2.2}$$

$(\mathcal{X}, d, \mu)$ is said a *space of homogeneous type* if, in addition, the *doubling condition*

$$\mu B(x, 2r) \leq C_\mu \, \mu B(x,r), \quad C_\mu > 1, \tag{2.3}$$

holds for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})$, where $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$ denotes the open ball centered at x and of radius r. For simplicity, we shall write $\mathcal{X}$ instead of $(\mathcal{X}, d, \mu)$ if no ambiguity arises.

As is known (see for instance [21], Lemma 14.6), from (2.3) there follows the property

$$\frac{\mu B(x,\varrho)}{\mu B(y,r)} \leq C \left(\frac{\varrho}{r}\right)^N, \quad N = \log_2 C_\mu, \tag{2.4}$$

for all the balls $B(x,\varrho)$ and $B(y,r)$ with $0 < r \leq \varrho$ and $y \in B(x,r)$, where $C > 0$ does not depend on r, ϱ and x. From (2.4) we have

$$\mu B(x,r) \geq c_0 \, r^N, \qquad x \in \Omega, \; 0 < r \leq \ell, \tag{2.5}$$

for any $\ell < \infty$ and any open set $\Omega \subseteq \mathcal{X}$ on which $\inf\limits_{x \in \Omega} \mu B(x,\ell) > 0$. Condition (2.5) is also known as the *lower Ahlfors regularity condition*. We will sometimes suppose that the *upper Ahlfors regularity condition* (also called the *non-doubling condition*) holds on Ω: there exists $n > 0$ such that

$$\mu B(x,r) \leq c_1 r^n, \tag{2.6}$$

where $c_1 > 0$ does not depend on $x \in \Omega$ and $r \in (0, \operatorname{diam}(\Omega))$, and n need not to be an integer.

We refer to [7], [8],[13], [18], [21], [26] for general properties of metric measure spaces.

2.2. Variable exponent spaces

Let $p : \mathcal{X} \to [1, \infty)$ be a μ-measurable function. Everywhere below we assume that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \quad x \in \mathcal{X}, \tag{2.7}$$

according to the notation in (2.1). By $L^{p(\cdot)}(\mathcal{X})$ we denote the space of all μ-measurable functions f on $\mathcal{X}$ such that the modular

$$I_{p(\cdot)}(f) = I_{p(\cdot),\mathcal{X}}(f) := \int_{\mathcal{X}} |f(x)|^{p(x)} d\mu(x)$$

is finite. This is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),\mathcal{X}} := \inf \left\{ \lambda > 0 : \; I_{p(\cdot)} \left(\frac{f}{\lambda}\right) \leq 1 \right\}. \tag{2.8}$$

Observe that

$$\|f\|_{p(\cdot)} = \|f^a\|^{\frac{1}{a}}_{\frac{p(\cdot)}{a}} \tag{2.9}$$

for any $0 < a \leq p_-$. As usual, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$ and it is defined pointwise by $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \mathcal{X}$.

For completeness, we recall here some basic properties for the spaces $L^{p(\cdot)}(\mathcal{X})$. The Hölder inequality is valid in the form

$$\int_{\mathcal{X}} |f(x)g(x)|\, d\mu(x) \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|f\|_{p(\cdot)}\, \|g\|_{p'(\cdot)}. \tag{2.10}$$

The embedding

$$L^{q(\cdot)}(\mathcal{X}) \hookrightarrow L^{p(\cdot)}(\mathcal{X}) \tag{2.11}$$

holds for $1 \leq p(x) \leq q(x) \leq q^+ < \infty$, when $\mu(\mathcal{X}) < \infty$.

Often the exponent $p(\cdot)$ is supposed to satisfy the local logarithmic condition

$$|p(x) - p(y)| \leq \frac{A_0}{\frac{1}{\ln d(x,y)}}, \quad d(x,y) \leq 1/2, \quad x, y \in \mathcal{X}, \tag{2.12}$$

from which we derive

$$|p(x) - p(y)| \leq \frac{2RA_0}{\frac{2R}{\ln d(x,y)}}, \quad d(x,y) \leq R, \quad x, y \in \mathcal{X}. \tag{2.13}$$

Assumption (2.12) is known in the literature as *Dini-Lipschitz condition* or log-*Hölder continuity*.

Variable exponent Lebesgue spaces on general metric measure spaces have been considered in [14], [23], [24], [27], [41] and more recently in [35], [36], where the maximal operator was studied in the weighted case. Recall that the maximal operator of a locally integrable function in $\mathcal{X}$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

The non-weighted $L^{p(\cdot)}$-boundedness of $\mathcal{M}$ on bounded homogeneous spaces was proved in [24] and [27] under the logarithmic condition (2.12) on the exponent.

2.3. Hajłasz-Sobolev spaces with variable exponent

Let $1 < p_- \leq p_+ < \infty$. We say that a function $f \in L^{p(\cdot)}(\mathcal{X})$ belongs to the *Hajłasz-Sobolev space* $M^{1,p(\cdot)}(\mathcal{X})$, if there exists a non-negative function $g \in L^{p(\cdot)}(\mathcal{X})$ such that the inequality

$$|f(x) - f(y)| \leq d(x,y)\, [g(x) + g(y)] \tag{2.14}$$

holds μ-almost everywhere in $\mathcal{X}$. In this case, g is called a *generalized gradient* of f. $M^{1,p(\cdot)}(\mathcal{X})$ is a Banach space with respect to the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{M^{1,p(\cdot)}(\mathcal{X})} := \|f\|_{p(\cdot)} + \inf \|g\|_{p(\cdot)}, \tag{2.15}$$

where the infimum is taken over all generalized gradients of f.

The spaces $M^{1,p}$ were first introduced by P. Hajłasz [19] in the case of constant exponents $p(x) \equiv p$, as a generalization of the classical Sobolev spaces $W^{1,p}$ to the general setting of the metric measure spaces. If $\mathcal{X} = \Omega$ is a bounded domain with Lipschitz boundary (or $\Omega = \mathbb{R}^n$), endowed with the Euclidean distance and the Lebesgue measure, then $M^{1,p}(\Omega)$ coincides with $W^{1,p}(\Omega)$. Recall that the oscillation of a Sobolev function may be estimated by the maximal function of its gradient. In other words, every function $f \in W^{1,p}(\Omega)$ satisfies (2.14) by taking $\mathcal{M}(|\nabla f|)$ as a generalized gradient (see, for instance, [6], [22], [28], for details and applications, and [3] where this property was also discussed for variable exponents).

Hajłasz-Sobolev spaces with variable exponent have been considered in [25], [23] and, more recently, in [5], where embeddings of such a spaces into Hölder classes of variable order were obtained. In [25] it was proved that $M^{1,p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n)$ if the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, which generalizes a result from [19] for constant p. Here $W^{1,p(\cdot)}(\mathbb{R}^n)$ denotes the usual Sobolev space equipped with the norm

$$\|f\|_{1,p(\cdot)} := \|f\|_{p(\cdot)} + \||\nabla f|\|_{p(\cdot)},$$

where ∇f denotes the (weak) gradient of f.

3. Fractional operators

Let Ω be an open set in $\mathcal{X}$. In the sequel, $B(x,r)$ will stand for $B(x,r) \cap \Omega$ for simplicity.

Fractional integrals over quasi-metric measure spaces are known to be considered in different forms. Let $\alpha(\cdot)$ be a μ-measurable positive function on Ω. We find it convenient to introduce the following notation

$$I_m^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(y)\, d\mu(y)}{[d(x,y)]^{m-\alpha(x)}}, \quad m > 0, \tag{3.1}$$

and

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \int_\Omega \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} f(y)\, d\mu(y). \tag{3.2}$$

We will be mainly interested in the case where $m = n$ is the "upper dimension" from (2.6). Obviously,

$$\mathfrak{I}^{\alpha(\cdot)} f(x) \le \frac{1}{c_0} I_N^{\alpha(\cdot)} f(x), \quad f \ge 0, \tag{3.3}$$

in case the measure μ satisfies the lower Ahlfors condition (2.5). Similarly,

$$I_n^{\alpha(\cdot)} f(x) \le c_1 \mathfrak{I}^{\alpha(\cdot)} f(x), \quad f \ge 0,$$

when (2.6) holds. In the case $\mathcal{X}$ has constant dimension, in the sense that $C_1 r^N \le \mu B(x,r) \le C_2 r^N$, the integrals $\mathfrak{I}^{\alpha(\cdot)} f(x)$ and $I_N^{\alpha(\cdot)} f(x)$ are equivalent. In the general case, where $n \le N$, the operator $\mathfrak{I}^{\alpha(\cdot)}$ is better suited for spaces $\mathcal{X}$ with lower Ahlfors bound, and $I_n^{\alpha(\cdot)}$ is better adjusted for spaces with upper Ahlfors bound.

In the case of constant exponents, the fractional operators $\mathfrak{I}^\alpha$ and I_n^α were widely studied, see e.g. the book [18] for $\mathfrak{I}^\alpha$, and the book [13] and papers [15], [16], [17] for I_n^α.

In the next theorem, for functions on doubling measure spaces with upper bound (2.6), we deal with the "quasi-Sobolev" exponent $\widetilde{q} = \widetilde{q}(n, N)$ defined by

$$\frac{1}{\widetilde{q}(x)} = \frac{1}{q(x)} \cdot \frac{1}{1 - \alpha(x)p(x)\left(\frac{1}{n} - \frac{1}{N}\right)}, \tag{3.4}$$

where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$. Observe that the denominator $1 - \alpha(x)p(x)\left(\frac{1}{n} - \frac{1}{N}\right) = \frac{\alpha(x)p(x)}{N} + 1 - \frac{\alpha(x)p(x)}{n}$ in (3.4) is bounded from below under the conditions

$$\alpha_- > 0, \quad \alpha_+ < \frac{n}{p_+}, \tag{3.5}$$

which will be assumed to be satisfied.

The inequalities

$$p(x) < \widetilde{q}(x) \le q(x)$$

are valid, where the right-hand side inequality is obvious, while the left-hand side one is easily checked by direct verification. Observe that in Theorem 3.2 we impose neither the log-condition, nor any condition of continuity on $\alpha(\cdot)$, so that $\alpha(\cdot)$ may be any bounded function satisfying conditions (3.5); hence $\widetilde{q}(\cdot)$ may be discontinuous everywhere, where $\alpha(\cdot)$ is.

Below we need the following auxiliary result which was given in [23] (see also [4] for an alternative proof).

Lemma 3.1. *Let $\mathcal{X}$ be bounded, the measure μ satisfy condition* (2.5) *and $p(\cdot)$ satisfy condition* (2.12)*. Then*

$$\left\|\chi_{B(x,r)}\right\|_{p(\cdot)} \le c\, [\mu B(x,r)]^{\frac{1}{p(x)}} \tag{3.6}$$

with $c > 0$ not depending on $x \in \mathcal{X}$ and $r > 0$.

Theorem 3.2. *Let Ω be a bounded open set in $\mathcal{X}$, on which the measure is doubling, and let the upper Ahlfors condition* (2.6) *be satisfied, and $p(\cdot)$ be log-Hölder continuous on Ω. Let also conditions* (3.5) *be satisfied. Then*

$$\|I_n^{\alpha(\cdot)} f\|_{\widetilde{q}(\cdot),\Omega} \le C\, \|f\|_{p(\cdot),\Omega}, \tag{3.7}$$

where $\widetilde{q}(\cdot)$ is defined by (3.4) *with the exponent N from* (2.4)*, $0 < n \le N < \infty$.*

Proof. We have to show that

$$\int_\Omega \left|I_n^{\alpha(\cdot)} f(x)\right|^{\widetilde{q}(x)} d\mu(x) \le C < \infty \tag{3.8}$$

when $\|f\|_{p(\cdot),\Omega} \le 1$. We will make use of the Hedberg approach to show that

$$\left[I_n^{\alpha(\cdot)} f(x)\right]^{\widetilde{q}(x)} \le C[\mathcal{M}f(x)]^{p(x)}. \tag{3.9}$$

To this end, we make use of the standard splitting

$$I_n^{\alpha(\cdot)}f(x) = \int\limits_{B(x,r)} \frac{f(y)\,d\mu(y)}{[d(x,y)]^{n-\alpha(x)}} + \int\limits_{\Omega\backslash B(x,r)} \frac{f(y)\,d\mu(y)}{[d(x,y)]^{n-\alpha(x)}} =: \mathcal{A}_r(x) + \mathcal{B}_r(x) \tag{3.10}$$

where $0 < r < \operatorname{diam}(\Omega)$. For $\mathcal{A}_r(x)$, via the standard binary decomposition

$$\mathcal{A}_r(x) = \sum_{k=0}^{\infty} \int\limits_{B(x,2^{-k}r)\backslash B(x,2^{-k-1}r)} \frac{f(y)\,d\mu(y)}{[d(x,y)]^{n-\alpha(x)}},$$

we obtain

$$\mathcal{A}_r(x) \le c_1 \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)}-1}\mathcal{M}f(x), \tag{3.11}$$

where c_1 is the constant from (2.6). By (3.5), from (3.11) we get

$$\mathcal{A}_r(x) \le c\, r^{\alpha(x)}\mathcal{M}f(x), \tag{3.12}$$

with some absolute constant $c > 0$ not depending on x and r.

For the term $\mathcal{B}_r(x)$ we make use of the Hölder inequality and obtain

$$\mathcal{B}_r(x) \le \|f\|_{p(\cdot)}\|\chi_{\Omega\backslash B(x,r)}(y)d(x,y)^{\alpha(x)-n}\|_{p'(\cdot)} \le \|\chi_{\Omega\backslash B(x,r)}(y)d(x,y)^{\alpha(x)-n}\|_{p'(\cdot)},$$

the norm being taken with respect to y. In the following estimation of the last norm, we adjust the arguments in the proof of Theorem C from [37], where potential operators were studied on Carleson curves, to our general case. By (2.9) we have

$$\mathcal{B}_r(x) \le \left\| \frac{\chi_{\Omega\backslash B(x,r)}(\cdot)}{[d(x,\cdot)]^n} \right\|_{\frac{n-\alpha(x)}{n}p'(\cdot)}^{\frac{n-\alpha(x)}{n}}. \tag{3.13}$$

The following pointwise estimate

$$\frac{\chi_{\Omega\backslash B(x,r)}(y)}{[d(x,y)]^n} \le C\mathcal{M}\left[\frac{\chi_{B(x,r)}}{\mu B(x,r)}\right](y), \tag{3.14}$$

is valid, with $C > 0$ not depending on x, y and r. Inequality (3.14) should be checked for $y \in \Omega\backslash B(x,r)$. We have

$$\mathcal{M}\left[\frac{\chi_{B(x,r)}}{\mu B(x,r)}\right](y) \ge \sup_{\delta>0} \frac{\mu\{B(x,r)\cap B(y,\delta)\}}{[\mu B(x,r)][\mu B(y,\delta)]} \ge \frac{\mu\{B(x,r)\cap B(y,\delta_0)\}}{[\mu B(x,r)][\mu B(y,\delta_0)]}$$

with an arbitrary $\delta_0 > 0$. We choose it so that $2a_0 d(x,y) \le \delta_0 \le 3a_0 d(x,y)$, where a_0 is the constant from (2.2). Then $B(x,r) \subset B(y,\delta_0)$ and consequently $\mu\{B(x,r)\cap B(y,\delta)\} = \mu B(x,r)$. Therefore,

$$\mathcal{M}\left[\frac{\chi_{B(x,r)}}{\mu B(x,r)}\right](y) \ge \frac{1}{\mu B(y,\delta_0)} \ge \frac{1}{c_1\delta_0^n} \ge \frac{C}{[d(x,y)]^n}, \qquad y \in \Omega\backslash B(x,r),$$

where $C = \frac{1}{c_1(3a_0)^n}$, which proves (3.14).

From (3.13) and (3.14) we obtain

$$\mathcal{B}_r(x) \le \frac{C}{[\mu B(x,r)]^{\frac{n-\alpha(x)}{n}}} \left\| \mathcal{M}\left[\chi_{B(x,r)}\right] \right\|_{\frac{n-\alpha(x)}{n} p'(\cdot)}^{\frac{n-\alpha(x)}{n}}.$$

By (3.5) we have $\inf\limits_{y\in\Omega} \frac{n-\alpha(x)}{n} p'(y) > 1$. Therefore, by the boundedness of the maximal operator, valid under the assumptions of our theorem, see [24], Theorem 4.3, we get

$$\mathcal{B}_r(x) \le \frac{C}{[\mu B(x,r)]^{\frac{n-\alpha(x)}{n}}} \left\| \chi_{B(x,r)} \right\|_{\frac{n-\alpha(x)}{n} p'(\cdot)}^{\frac{n-\alpha(x)}{n}} = \frac{C \left\| \chi_{B(x,r)} \right\|_{p'(\cdot)}}{[\mu B(x,r)]^{\frac{n-\alpha(x)}{n}}}.$$

By means of Lemma 3.1 and condition (2.5) we conclude that

$$\mathcal{B}_r(x) \le \frac{C}{[\mu B(x,r)]^{\frac{1}{p(x)} - \frac{\alpha(x)}{n}}} \le \frac{C}{r^{N\left[\frac{1}{p(x)} - \frac{\alpha(x)}{n}\right]}}. \tag{3.15}$$

Therefore, from (3.10), (3.12) and (3.15) we obtain

$$I_n^{\alpha(\cdot)} f(x) \le C \left\{ r^{\alpha(x)} \mathcal{M}f(x) + r^{N\left[\frac{\alpha(x)}{n} - \frac{1}{p(x)}\right]} \right\}. \tag{3.16}$$

Optimizing the right-hand side with $r = [\mathcal{M}f(x)]^{-\frac{1}{\alpha(x)+\frac{N}{q(x)}}}$, after easy calculations we arrive at (3.9). The proof is complete. □

Remark 3.3. In the case of constant exponents p and α, the statement of Theorem 3.2 is known to be valid without the doubling condition and with optimal Sobolev exponent q instead of the "quasi-Sobolev" exponent $\widetilde{q}$, see Theorem 3.2 and Corollary 3.3 in [15]. The progress in [15] was based on the weak estimate for the potential I_n^α and the Marcinkiewicz interpolation theorem. The latter tool is still absent in the theory of variable exponent Lebesgue spaces $L^{p(\cdot)}$: the validity of the Marcinkiewicz theorem remains an open question. The direct estimation of the potential via the maximal function by means of the Hedberg approach, used in the proof of Theorem 3.2, led us to the "quasi-Sobolev" exponent $\widetilde{q}(\cdot)$.

Corollary 3.4. *Under conditions of Theorem* 3.2*, the operator* $I_n^{\alpha(\cdot)}$ *is bounded in the space* $L^{p(\cdot)}(\Omega)$.

Proof. It suffices to refer to the fact that $p(x) < \widetilde{q}(x)$ and Ω is bounded. □

As regards the operator $\mathfrak{I}^\alpha$, the following statement was proved in [23] for constant orders α.

Theorem 3.5. *Let* Ω *be a bounded open set in* $\mathcal{X}$ *and* $(\mathcal{X}, \mu, d)$ *satisfy one of the following conditions:*
i) $\mathcal{X} \subset \mathbb{R}^m$ *(m natural) and the measure* μ *satisfy the lower Alhfors condition* (2.5)*;*
ii) the measure μ *is doubling.*

Then the operator $\mathfrak{I}^\alpha$, $0 < \alpha < N$, *is bounded from* $L^{p(\cdot)}(\Omega)$ *to* $L^{q(\cdot)}(\Omega)$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{N}$, *under the assumptions that* $1 < p_- \leq p_+ < \frac{N}{\alpha}$ *and* $p(\cdot)$ *is log-continuous.*

Note that a somewhat more general statement was obtained in [23]: the "dimension" N was allowed to be variable and a certain modification J^α of the operator $\mathfrak{I}^\alpha$ was considered, $|\mathfrak{I}^\alpha f| \leq J^\alpha(|f|)$, see Theorem 4.8 and Remark 4.9 in [23].

Statement *ii)* of Theorem 3.5 can be generalized to the case of variable $\alpha(\cdot)$ as follows.

Theorem 3.6. *Let* Ω *be a bounded open set in* $\mathcal{X}$ *and* μ *be doubling. Let also* $p(\cdot)$ *satisfy the logarithmic condition* (2.12) *and* $\alpha(\cdot)$ *satisfy the assumptions*

$$\alpha_- > 0 \qquad \text{and} \qquad \sup_{x\in\Omega} \alpha(x)p(x) < N \tag{3.17}$$

(with N *from* (2.4)*). Then* $\mathfrak{I}^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot)}(\Omega)$ *into* $L^{q(\cdot)}(\Omega)$, *where* $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{N}$.

Proof. Let $\|f\|_{p(\cdot),\Omega} \leq 1$. As in the proof of Theorem 3.2, we follow the Hedberg approach and write

$$\left|\mathfrak{I}^{\alpha(\cdot)} f(x)\right| \leq A_r(x) + B_r(x),$$

where

$$A_r(x) := \int\limits_{B(x,r)} \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} |f(y)|\, d\mu(y)$$

and

$$B_r(x) = \int\limits_{\Omega\setminus B(x,r)} \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} |f(y)|\, d\mu(y),$$

$0 < r < \operatorname{diam}(\Omega)$ being arbitrary. The standard decomposition technique gives

$$A_r(x) \leq c\, r^{\alpha(x)}\, \mathcal{M} f(x) \tag{3.18}$$

taking into account that $\alpha_- > 0$ and μ is doubling on Ω.

As regards the term $B_r(x)$, we have

$$\begin{aligned} B_r(x) &= \sum_{k=0}^{\infty} \int\limits_{2^k r\le d(x,y)<2^{k+1}r} \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x,d(x,y))}|f(y)|\,d\mu(y) \\ &\le c\sum_{k=0}^{\infty} \frac{(2^{k+1}r)^{\alpha(x)}}{\mu B(x,2^k r)} \left\|\chi_{B(x,2^{k+1}r)}\right\|_{p'(\cdot),\Omega} \\ &\le c\sum_{k=0}^{\infty} \frac{(2^{k+1}r)^{\alpha(x)}}{\mu B(x,2^k r)} \left[\mu B(x,2^{k+1}r)\right]^{\frac{1}{p'(x)}} \\ &\le c\sum_{k=0}^{\infty} \frac{(2^{k+1}r)^{\alpha(x)}}{[\mu B(x,2^k r)]^{\frac{1}{p(x)}}} \end{aligned}$$

where we used the Hölder inequality, Lemma 3.1 and the doubling property of μ, successively. Since $\alpha(\cdot)$ is bounded and μ satisfies the lower Ahlfors condition (2.5), we get

$$B_r(x) \le c\sum_{k=0}^{\infty}(2^k r)^{\alpha(x)-\frac{N}{p(x)}}.$$

By the second assumption in (3.17), $\inf\limits_{x\in\Omega}(N-\alpha(x)p(x))>0$ so that

$$B_r(x) \le c\, r^{\alpha(x)-\frac{N}{p(x)}} \tag{3.19}$$

with $c>0$ independent of x and r. Combining estimates (3.18) and (3.19), we choose $r=[\mathcal{M}f(x)]^{-\frac{p(x)}{N}}$ and obtain

$$\left|\mathfrak{I}^{\alpha(\cdot)}f(x)\right| \le c\,[\mathcal{M}f(x)]^{\frac{p(x)}{q(x)}},$$

from which we complete the proof by making use of the boundedness of the maximal operator on $L^{p(\cdot)}(\Omega)$. □

Corollary 3.7. *Under conditions of Theorem* 3.6*, the operator* $\mathfrak{I}^{\alpha(\cdot)}$ *is bounded in the space* $L^{p(\cdot)}(\Omega)$.

4. On fractional maximal operators

Let $\alpha:\mathcal{X}\to(0,\infty)$ be a μ-measurable function. The fractional maximal function $\mathcal{M}_{\alpha(\cdot)}f$ of a locally integrable function f is defined by

$$\mathcal{M}_{\alpha(\cdot)}f(x)=\sup_{r>0}\frac{r^{\alpha(x)}}{\mu B(x,r)}\int_{B(x,r)}|f(y)|\,d\mu(y),$$

where the order α is admitted to be variable, with $0\le\alpha(x)\le\alpha_+<\infty$, $x\in\mathcal{X}$. In the limiting case $\alpha(x)\equiv 0$, we have the Hardy-Littlewood maximal operator $\mathcal{M}=\mathcal{M}_0$.

In the following lemma we make use of the condition

$$\frac{\mu B(x,\varrho)}{\mu B(x,r)} \le C\left(\frac{\varrho}{r}\right)^{\alpha(x)} \quad \text{for all} \quad \varrho < r, \quad \alpha(x) \ge 0, \tag{4.1}$$

where the constant $C > 0$ is assumed to be not dependent on ϱ, r and x. Note that in the cases where μ is doubling, from (2.4) it follows that (4.1) is only possible if

$$\alpha(x) \le N.$$

We find it convenient to say that the measure μ *has order* α *of growth*, if condition (4.1) is fulfilled. Note that measures μ satisfying the *halving condition*

$$\mu B\left(x, \frac{r}{2}\right) \le c_\mu(x)\, \mu B(x,r), \quad 0 < c_\mu(x) < 1, \tag{4.2}$$

have the order of growth $\alpha(x)$ with $\alpha(x) = \log_2 \frac{1}{c_\mu(x)}$. Inequality (4.1) follows from (4.2) by repetition similarly as (2.4) is derived from (2.3).

Lemma 4.1. *Let $\mathcal{X}$ be an arbitrary metric measure space satisfying condition* (4.1)*. Then the pointwise estimate*

$$\mathcal{M}_{\alpha(\cdot)} f(x) \le C\, \mathfrak{I}^{\alpha(\cdot)} f(x), \quad f \ge 0, \tag{4.3}$$

holds, with the same constant C from (4.1)*.*

Proof. The proof is obvious: by condition (4.1) we have

$$\frac{r^{\alpha(x)}}{\mu B(x,r)} \int_{B(x,r)} |f(y)|\, d\mu(y) \le C \int_{B(x,r)} \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} |f(y)|\, d\mu(y),$$

from which (4.3) follows. □

Theorem 4.2. *Let the measure μ be doubling and have order of growth $\alpha(x)$. If $p(\cdot)$ is log-Hölder continuous, $\alpha_- > 0$ and $\sup\limits_{x\in\Omega} \alpha(x)p(x) < N$, then*

$$\|\mathcal{M}_{\alpha(\cdot)} f\|_{q(\cdot)} \le c\, \|f\|_{p(\cdot)}, \quad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{N}. \tag{4.4}$$

Proof. In view of the inequality (4.3), it suffices to have the boundedness of the fractional operator $\mathfrak{I}^{\alpha(\cdot)}$, which follows from Theorem 3.6. □

5. Hypersingular operators on spaces $\mathbf{M^{1,p(\cdot)}(\Omega)}$

Let Ω be a bounded open set in $\mathcal{X}$. Similarly to (3.1) and (3.2) we can consider two forms of hypersingular integrals:

$$D^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x) - f(y)}{[d(x,y)]^{n+\alpha(x)}}\, d\mu(y), \quad x \in \Omega, \tag{5.1}$$

where $n > 0$ is from (2.6), and

$$\mathfrak{D}^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x) - f(y)}{\mu B(x, d(x,y))[d(x,y)]^{\alpha(x)}}\, d\mu(y), \quad x \in \Omega. \tag{5.2}$$

We admit variable order $\alpha = \alpha(x)$, $0 < \alpha(x) < 1$, $x \in \Omega$. A detailed information about hypersingular integrals (of constant order) of functions defined in $\mathbb{R}^m$ can be found in [47] and [49]; variable order hypersingular integrals were studied in [46]. Hypersingular integrals of constant order on metric measure spaces were considered in [16], [17] within the frameworks of Lipschitz (Hölder) function spaces.

5.1. Preliminaries

The following two lemmas on the estimation of the oscillation of Sobolev functions were proved in [5] and generalize former results from [20] given for constant $\alpha = \beta$ and p.

Lemma 5.1. *Let $\mathcal{X}$ satisfy the doubling condition* (2.3) *and let $f \in M^{1,p(\cdot)}(\mathcal{X})$ and $g \in L^{p(\cdot)}(\mathcal{X})$ be a generalized gradient of f. If $0 \le \alpha_+ < 1$, $0 \le \beta_+ < 1$, then*

$$|f(x) - f(y)| \le C(\mu, \alpha, \beta) \left[d(x,y)^{1-\alpha(x)} \mathcal{M}_{\alpha(\cdot)} g(x) + d(x,y)^{1-\beta(y)} \mathcal{M}_{\beta(\cdot)} g(y)\right] \tag{5.3}$$

μ-almost everywhere.

Lemma 5.2. *Let $\mathcal{X}$ be bounded and let μ be doubling. Suppose also that $p(\cdot)$ satisfies* (2.12) *with $p_- > N$. If $f \in M^{1,p(\cdot)}(\mathcal{X})$ and g is a generalized gradient of f, then there exists $C > 0$ such that*

$$|f(x) - f(y)| \le C \, \|g\|_{p(\cdot)} \, d(x,y)^{1 - \frac{N}{\max\{p(x), p(y)\}}} \tag{5.4}$$

for every $x, y \in \mathcal{X}$ with $d(x,y) \le 1$.

5.2. The case of the operator $D^{\alpha(\cdot)}$

The next theorem is an extension of a result in [3], from the Euclidean case to the case of metric measure spaces. Recall that assumption that the measure μ has order of growth $1 - \alpha(x)$, used in Theorem 5.3, is fulfilled if the measure μ satisfies the halving condition (4.2) with $c_\mu(x) = 2^{\alpha(x)-1}$.

Theorem 5.3. *Let the measure μ be doubling and satisfying the upper Ahlfors condition* (2.6)*. Let $0 < \alpha_- \le \alpha(x) \le \alpha_+ < 1$ and let $p(\cdot)$ satisfy condition* (2.12) *and the assumption*

$$\sup_{x \in \Omega} p(x)[1 - \alpha(x)] < n. \tag{5.5}$$

If the measure μ has the order of growth $1 - \alpha(x)$, then the operator $D^{\alpha(\cdot)}$ is bounded from $M^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ with

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\lambda(x)}{n} \tag{5.6}$$

where $\lambda(\cdot)$ is any log-Hölder continuous function such that

$$\lambda_- > 0 \quad \textit{and} \quad \sup_{x \in \Omega} [\lambda(x) + \alpha(x)] < 1. \tag{5.7}$$

Proof. By Lemma 5.1 and the log-Hölder continuity of $\lambda(\cdot)$, we have

$$\begin{aligned}\left|D^{\alpha(\cdot)}f(x)\right| &\leq \int_\Omega \frac{|f(x)-f(y)|}{[d(x,y)]^{n+\alpha(x)}}\,d\mu(y)\\ &\leq c\int_\Omega \frac{\mathcal{M}_{\lambda(\cdot)}g(x)+\mathcal{M}_{\lambda(\cdot)}g(y)}{[d(x,y)]^{n+\alpha(x)+\lambda(x)-1}}\,d\mu(y)\end{aligned}$$

for μ-almost all $x\in\Omega$, with $c>0$ not depending on x and f. Note that $\lambda_+<1$ in view of (5.7) and condition $\alpha_->0$.

Put $\beta(x)=1-\alpha(x)-\lambda(x)$. Then $0<1-(\alpha+\lambda)_+\leq\beta(x)<1-\alpha_-$. We have

$$\left|D^{\alpha(\cdot)}f(x)\right|\leq c\int_\Omega \frac{\mathcal{M}_{\lambda(\cdot)}g(x)}{[d(x,y)]^{n-\beta(x)}}\,d\mu(y)+c\int_\Omega \frac{\mathcal{M}_{\lambda(\cdot)}g(y)}{[d(x,y)]^{n-\beta(x)}}\,d\mu(y).$$

Since Ω is bounded and $\beta_->0$, the fractional integral $\int_\Omega \frac{d\mu(y)}{[d(x,y)]^{n-\beta(x)}}$ of a constant is a bounded function, which is known, see for instance [15], Lemma 2.1, and is easily verified via the standard decomposition

$$B(x,1)=\bigcup_{k=0}^{\infty}\{y:2^{-k-1}\leq d(x,y)<2^{-k}\}. \tag{5.8}$$

Therefore,

$$\left|D^{\alpha(\cdot)}f(x)\right|\leq c\,\mathcal{M}_{\lambda(\cdot)}g(x)+c\,I_n^{\beta(\cdot)}\left[\mathcal{M}_{\lambda(\cdot)}g\right](x),$$

where $I_n^{\beta(\cdot)}$ is the fractional operator of type (3.1). Hence

$$\left\|D^{\alpha(\cdot)}f\right\|_{q(\cdot),\Omega}\leq c\left\|\mathcal{M}_{\lambda(\cdot)}g\right\|_{q(\cdot),\Omega}+c\left\|I_n^{\beta(\cdot)}\left[\mathcal{M}_{\lambda(\cdot)}g\right]\right\|_{q(\cdot),\Omega}. \tag{5.9}$$

In view of the conditions $\beta_->0$ and $\beta_+<\frac{n}{q_+}$, the operator $I_n^{\beta(\cdot)}$ is bounded in the space $L^{q(\cdot)}(\Omega)$ by Corollary 3.4. Therefore,

$$\left\|D^{\alpha(\cdot)}f\right\|_{q(\cdot),\Omega}\leq c\left\|\mathcal{M}_{\lambda(\cdot)}g\right\|_{q(\cdot),\Omega}. \tag{5.10}$$

By Theorem 4.2 we then have

$$\left\|D^{\alpha(\cdot)}f\right\|_{q(\cdot),\Omega}\leq c\,\|g\|_{p(\cdot),\Omega}\leq c\,\|f\|_{1,p(\cdot),\Omega},$$

that theorem being applicable since $\lambda_->0$ and

$$\sup_{x\in\Omega}\lambda(x)p(x)\leq\sup_{x\in\Omega}[1-\alpha(x)]p(x)<n,$$

according to (5.7) and (5.5). Note also that from condition (5.7) it follows that growth of order $1-\alpha(x)$ implies that of order $\lambda(x)$.

Thus the boundedness of $D^{\alpha(\cdot)}$ from $M^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ has been proved for $q(\cdot)$ of form (5.6). □

For constant exponents the following statement holds.

Corollary 5.4. *Let α and μ be as in Theorem* 5.3 *and suppose that* $1 < p < \frac{n}{1-\alpha}$. *Then there exists* $c > 0$ *such that*

$$\|D^\alpha f\|_{q,\Omega} \le c\,\|f\|_{1,p,\Omega}, \quad f \in M^{1,p}(\Omega),$$

for any exponent q fulfilling

$$p < q < \frac{np}{n-(1-\alpha)p}. \tag{5.11}$$

Lemma 5.2 allows us to derive the following conclusion on the pointwise convergence of the hypersingular integral.

Theorem 5.5. *Let α and μ be as in Theorem* 5.3. *Under condition* (2.12)*, the hypersingular integral* $D^{\alpha(\cdot)}$*, with* $0 < \alpha_- \le \alpha(x) < 1$, $x \in \Omega$*, of functions in* $M^{1,p(\cdot)}(\Omega)$ *converges at all those points* $x \in \Omega$ *where* $p(x)(1-\alpha(x)) > n$.

Proof. The pointwise convergence of the hypersingular integral is an immediate consequence of (5.4). We only observe that the assumption $p(x)(1-\alpha(x)) > n$ implies $p_- > n$. □

5.3. The case of the operator $\mathfrak{D}^{\alpha(\cdot)}$

We continue the study of hypersingular integrals of Hajłasz-Sobolev functions but now of form (5.2). We are able to get the corresponding results for the hypersingular integrals of this form in the case when the difference $N - n$ between the dimensions N and n is small.

Theorem 5.6. *Let the measure μ be doubling and satisfying the upper Ahlfors condition* (2.6)*. Let also* $0 < \alpha_- \le \alpha(x) \le \alpha_+ < 1$ *and suppose that* $N - n < 1 - \alpha_+$*. Let also* $p(\cdot)$ *satisfy condition* (2.12) *and the assumption*

$$\sup_{x\in\Omega} p(x)[1-\alpha(x)] < N. \tag{5.12}$$

If the measure μ has the order of growth $1-\alpha(x)$*, then the operator* $\mathfrak{D}^{\alpha(\cdot)}$ *is bounded from* $M^{1,p(\cdot)}(\Omega)$ *into* $L^{q(\cdot)}(\Omega)$ *with*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\lambda(x)}{N}, \tag{5.13}$$

where $\lambda(\cdot)$ is any log-Hölder continuous function such that

$$\lambda_- > 0 \quad \text{and} \quad \sup_{x\in\Omega}[\lambda(x)+\alpha(x)] < 1-(N-n). \tag{5.14}$$

Proof. We make use of (5.3) and the logarithmic condition on $\lambda(\cdot)$ and arrive at

$$\left|\mathfrak{D}^{\alpha(\cdot)} f(x)\right| \le c\,\mathcal{M}_{\lambda(\cdot)}g(x)\int_\Omega \frac{[d(x,y)]^{\beta(x)}}{\mu B(x,d(x,y))}\,d\mu(y) + c\,\mathfrak{I}^{\beta(\cdot)}\left[\mathcal{M}_{\lambda(\cdot)}g\right](x),$$

where $\beta(x) = 1-\alpha(x)-\lambda(x)$ and $\mathfrak{I}^{\beta(\cdot)}$ is the fractional operator of type (3.2). Since μ is doubling, one gets

$$\int_\Omega \frac{[d(x,y)]^{\beta(x)}}{\mu B(x,d(x,y))}\,d\mu(y) \le C\int_\Omega \frac{d\mu(y)}{[d(x,y)]^{N-\beta(x)}}$$

But the latter fractional integral is a bounded function. This can be checked through the standard decomposition (5.8), where we take into account the upper Ahlfors bound (2.6) and the condition $\beta_- > N - n$, in view of (5.14) and $\beta(x) \geq 1 - (\alpha + \lambda)_+$.

The proof can now be completed following similar steps of the proof of Theorem 5.3. $\square$

In the case of constant exponents α and p, we have the following result.

Corollary 5.7. *Let α and μ be as in Theorem* 5.6 *and $1 - \alpha < N < n + 1 - \alpha$. If $1 < p < \frac{N}{1-\alpha}$, then there exists $c > 0$ such that*

$$\|\mathfrak{D}^\alpha f\|_{q,\Omega} \leq c\,\|f\|_{1,p,\Omega}, \quad f \in M^{1,p}(\Omega),$$

for any exponent q fulfilling

$$p < q < \frac{Np}{N - (1-\alpha)p + (N-n)p}. \tag{5.15}$$

As in the case of operator D^α, we can also derive conditions on the convergence of hypersingular integrals of the form (5.2).

Theorem 5.8. *Let α and μ be as in Theorem* 5.6. *Under condition* (2.12)*, the hypersingular integral $\mathfrak{D}^{\alpha(\cdot)}$, with $0 < \alpha_- \leq \alpha(x) < 1$, $x \in \Omega$, of functions in $M^{1,p(\cdot)}(\Omega)$ converges at all those points where*

$$1 - \alpha(x) - \frac{N}{p(x)} > N - n.$$

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Alexandre Almeida
Departamento de Matemática
Universidade de Aveiro
3810-193 Aveiro
Portugal
e-mail: jaralmeida@ua.pt

Stefan Samko
Faculdade de Ciências e Tecnologia
Universidade do Algarve
Campus de Gambelas
8005-139 Faro
Portugal
e-mail: ssamko@ualg.pt