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Weighted estimates of generalized potentials in variable exponent Lebesgue spaces on homogeneous spaces

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Abstract. For generalized potential operators with the kernel $\frac{a[\varrho(x,y)]}{[\varrho(x,y)]^N}$ on bounded measure metric space (X, μ, ϱ) with doubling measure μ satisfying the upper growth condition $\mu B(x,r) \leq Cr^N, N \in (0,\infty)$, we prove weighted estimates in the case of radial type power weight $w = [\varrho(x,x_0)]^\nu$. Under some natural assumptions on $a(r)$ in terms of almost monotonicity we prove that such potential operators are bounded from the weighted variable exponent Lebesgue space $L^{p(\cdot)}(X, w, \mu)$ into a certain weighted Musielak-Orlicz space $L^\Phi(X, w^{\frac{1}{p(x_0)}}, \mu)$ with the N-function $\Phi(x,r)$ defined by the exponent $p(x)$ and the function $a(r)$.

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1. Introduction

The Lebesgue spaces $L^{p(\cdot)}$ with variable exponent were intensively investigated during the last years, we refer to the papers [35], [23] for the basic properties of these spaces. The growing interest to such spaces is caused by applications to various problems, for instance, in image restoration, fluid dynamics, elasticity theory and differential equations with non-standard growth conditions (see e.g. [2], [28], [36]). The spaces $L^{p(\cdot)}$ with variable exponent are special cases of Orlicz-Musielak spaces, see [24] for these spaces. We refer to [3], where the maximal operator was studied in the context of Orlicz-Musielak spaces. A significant progress has already been made been made in the study of classical

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integral operators in the context of the $L^{p(\cdot)}$ spaces, see for instance the surveying papers [4], [18] and[34].

The spaces $L^{p(\cdot)}$ on measure quasimetric spaces and maximal and potential operators in such spaces were studied in [1], [6], [15], [16], [14], [17], [21].

We study the generalized Riesz potential operators

$$I_a f\left(x\right) := \int\limits_X \mathcal{K}(x,y) f(y) d\mu(y), \quad \mathcal{K}(x,y) = \frac{a\left(\varrho\left(x,y\right)\right)}{\left[\varrho\left(x,y\right)\right]^N} \tag{1.1}$$

over a bounded measure space X with quasimetric ϱ, where N is the upper Ahlfors dimension of X. In [13], under some assumptions on the function $a(\varrho)$ there was proved a Sobolev-type theorem on the boundedness of the operator I_a from $L^{p(\cdot)}(X)$ into a certain Orlicz-Musielak space. In this paper we extend this result to the weighted case. We deal with the case of power weights

$$w(x) = [\varrho(x,x_0)]^{\nu}, \quad x_0 \in X.$$

Note that the interest to the case of power weights is caused not only by the fact that such weights are first of all important in various applications, but also because in the case of variable exponents it is a problem to derive the result for concrete weights from the existing forms of general conditions on weights. (Recall that even in the case of constant exponents the belongness of these or other special weights to the Muckenhoupt type classes was first not checked directly, but obtained from the necessity of the Muckenhoupt condition).

An extension to the weighted case proved to be a non-easy task within the frameworks of variable exponents even for power weights, the difficulties being caused both by the variability of the exponent and non-homogeneity of the kernel. This extension is based on the technique of weighted norm estimation of kernels of truncated potentials given and applied in [29], [32], [31], [33], which is developed in this paper for non-homogeneous kernels.

The generalized Riesz potential operators I_a attracted attention last years, we refer in particular to [12], [25], where such potentials were studied in Orlicz spaces in the case $X = \mathbb{R}^n$ and Euclidean metric, and to [26], where homogeneous spaces with constant dimension were admitted. We refer also to [27] for the study of the similar generalized potentials in the Euclidean setting in rearrangement invariant spaces. For "standard" potentials (that is, potentials with the kernel of the form $\frac{1}{d(x,y)^{N-\alpha}}$ or $\frac{d(x,y)^{\alpha}}{B(x,d(x,y))}$) on metric measure spaces, we refer to [5], [7], [8], [9], [10], [11], [19], [20] and references therein.

The main results are formulated in Section 2 and proved in Section 5. The main technical tool is provided by Lemma 4.1 in Section 3.

2. Formulation of the main result

In the sequel (X, ϱ, μ) always stands for a bounded quasimetric space with quasidistance $\varrho(x,y) = \varrho(y,x)$:

$$\varrho(x,y) \le k[\varrho(x,z) + \varrho(z,y)], \quad k \ge 1 \tag{2.1}$$

and Borel regular measure μ. We denote $d = diam\, X$. The measure μ is supposed to satisfy the growth condition

$$\mu\left(B\left(x,r\right)\right) < Kr^N. \tag{2.2}$$

Definition 2.1. A function $\Phi : X \times [0, \infty) \to [0, +\infty)$ is said to be an N-function, if

1. for every $x \in X$ the function $\Phi\left(x,t\right)$ is convex, nondecreasing and continuous in $t \in [0, \infty)$,
2. $\Phi\left(x,0\right) = 0,\ \ \Phi\left(x,t\right) > 0$ for every $t > 0$,
3. $\Phi\left(x,t\right)$ is a μ-measurable function of x for every $t \geq 0$.

Definition 2.2. Let Φ be an N-function and w a weight. The weighted Orlicz-Musielak space $L^{\Phi}(X,w)$ is defined as the set of all real-valued μ-measurable functions f on X such that

$$\int\limits_X \Phi\left(x, \frac{w(x)f(x)}{\lambda}\right) d\mu(x) < \infty$$

for some $\lambda > 0$. We equip it with the norm

$$\|f\|_{\Phi,w} = \inf\left\{\lambda > 0 : \int\limits_X \Phi\left(x, \frac{w(x)f(x)}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

In particular, $\Phi(x,t) = t^{p(x)}$, where $1 \leq p(x) < \infty$, is an N-function and the corresponding space is the variable exponent Lebesgue space $L^{p(\cdot)}(X,w)$.

Everywhere in the sequel, when dealing with the space $L^{p(\cdot)}(X,w)$, we suppose that

$$1 < p_- \leq p(x) \leq p_+ < +\infty, \tag{2.3}$$

$$|p(x) - p(y)| \leq \frac{A}{ln\frac{1}{\varrho(x,y)}}, \qquad \varrho(x,y) < \frac{1}{2} \tag{2.4}$$

and denote

$$w^{\nu} = [\varrho(x,x_0)]^{\nu}, \qquad x_0 \in X.$$

The function $a : [0,d] \to [0,\infty)$ is assumed to satisfy the assumptions

1) $a(r)$ is continuous, almost increasing, positive for $r > 0$ and $a(0) = 0$,

2) $\int\limits_0^d \frac{a(r)}{r} dr < \infty$.

We denote

$$A(r) = \int\limits_0^r \frac{a(t)}{t}\, dt.$$

In the following theorem we make use of the notion of the lower dimension of X defined by

$$\underline{\mathfrak{dim}}(X) = \sup_{t>1} \frac{\ln\left(\liminf\limits_{r\to 0}\ \inf\limits_{x\in X} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t}.$$

as introduced in [30]. It is clear that $\underline{\mathfrak{dim}}(X) = N$ in the cases where X has constant dimension N, that is, $c_1 r^N \le \mu B(x,r) \le c_2 r^N$. In general, if X has the property that

$$0 < \underline{\mathfrak{dim}}(X) < \infty,$$

then X satisfies the growth condition with every

$$0 < N < \underline{\mathfrak{dim}}(X). \tag{2.5}$$

This follows from the inequality

$$\mu B(x,r) \le C r^{\underline{\mathfrak{dim}}(X)-\varepsilon}, \tag{2.6}$$

where $\varepsilon > 0$ is arbitrarily small and $C = C(\varepsilon) > 0$ does not depend on x, which is easily derived from the results in [30], Subsection 2.1.

Theorem 2.3. *Let (X, ϱ, μ) be quasimetric space with doubling measure and positive finite lower dimension $\underline{\mathfrak{dim}}(X)$, and let p fulfill assumptions* (2.3)-(2.4) *and*

$$0 \le \nu < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_0)}.$$

Suppose that there exists a $\beta \in \left(0, \frac{\underline{\mathfrak{dim}}(X)}{p_+}\right)$ such that

$$\frac{a(r)}{r^\beta} \quad \text{is almost decreasing.} \tag{2.7}$$

Then the operator I_a is bounded from the space $L^{p(\cdot)}(X, w^\nu)$ into the weighted Orlicz-Musielak space $L^\Phi(X, w^{\nu_1})$, where $\nu_1 = \frac{\nu}{p(x_0)}$ and the N-function Φ is defined by its inverse (for every fixed $x \in X$)

$$\Phi^{-1}(x,r) = \int_0^r A\left(t^{-\frac{1}{N}}\right) t^{-\frac{1}{p'(x)}}\, dt. \tag{2.8}$$

The proof of Theorem 2.3 will be based on Lemma 4.1 and the following statement proved in [21], [22].

Theorem 2.4. *Let X be a bounded doubling measure quasimetric space and $p(x)$ satisfy assumptions* (2.3)-(2.4). *The maximal operator*

$$Mf(x) = \sup_{r>0} \frac{1}{B(x,r)} \int_{B(x,r)} |f(y)| d\mu(y)$$

is bounded in $L^{p(\cdot)}(X, w^\nu)$, if $-\frac{\underline{\mathfrak{dim}}(X)}{p(x_0)} < \nu < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_0)}$.

We will also use the following lemma proved in [13] (see Lemma 4.9 in [13]).

Lemma 2.5. *Let $p(x)$ satisfy condition* (2.3) *and $a(r)$ be a non-negative almost increasing continuous on $[0,d], 0 < d < \infty$ function such that the function $\frac{a(t)}{t^{\frac{N}{p_+}-\varepsilon}}$ is almost* *decreasing for some* $\varepsilon > 0$*. Then there exist constants* $C_1 > 0, C_2 > 0$ *not depending on* x *and* r *such that*

$$C_1 \frac{A(r)}{r^{\frac{N}{p(x)}}} \le \Phi^{-1}\left(x, \frac{1}{r^N}\right) \le C_2 \frac{A(r)}{r^{\frac{N}{p(x)}}}. \tag{2.9}$$

3. Auxiliary estimates

To prove our weighted generalized Sobolev-type theorem for the potential I_a via Hedberg approach, we need to estimate the integral

$$\mathcal{J}(x,r) := \int\limits_{X\setminus B(x,r)} \left(\frac{a\left(\varrho\left(x,y\right)\right)}{\varrho\left(x,y\right)^N}\right)^{p(x)} \varrho(y,x_0)^b d\mu\left(y\right), \quad x_0 \in X.$$

Lemma 3.1. *Let* X *satisfy the growth condition* (2.2)*, let the function* $a(r)$ *be non-negative and almost decreasing on* $[0,d]$ *and* $\gamma(x)$ *be an arbitrary bounded function on* Ω*. Then the estimate*

$$\int\limits_{X\setminus B(x,r)} \left(\frac{a\left[\varrho(x,y)\right]}{\varrho(x,y)^{\gamma(x)}}\right)^{p(x)} d\mu(y) \le C \int\limits_r^d t^{N-1}\left[\frac{a(t)}{t^{\gamma(x)}}\right]^{p(x)} dt, \quad 0 < r < \frac{d}{2}, \tag{3.1}$$

holds, where $C > 0$ *does not depend on* x *and* r*.*

Lemma 3.1 was proved in [13] in the case $\gamma(x) = N$, the proof based on the binary decomposition is the same for an arbitrary bounded $\gamma(x)$ in view of the monotonicity of the power function $t^{\gamma(x)}$.

Lemma 3.2. *Let* X *satisfy the growth condition* (2.2)*, Suppose that the function* $a(r) : (0,d) \to (0,+\infty)$ *is almost increasing and the function* $\frac{a(r)}{r^N}$ *is almost decreasing. Then for* $0 < r < \frac{d}{2}$ *the estimate*

$$\mathcal{J}(x,r) \le C\mathcal{G}(x,r) := C \begin{cases} \int\limits_r^d t^{N-1}\left[\frac{a(t)}{t^N}\right]^{p(x)} t^b\, dt, & \textit{if} \ \ \varrho(x_0,x) \le r \\ \varrho(x_0,x)^b \int\limits_r^d t^{N-1}\left[\frac{a(t)}{t^N}\right]^{p(x)} dt, & \textit{if} \ \ \varrho(x_0,x) > r \end{cases} \tag{3.2}$$

holds, where $p : X \to (1,+\infty)$*,* $1 \le p(x) < p_+ < +\infty$*,* $b > -N$ *amd* $C > 0$ *does not depend on* x *and* r*.*

Proof. Consider separately the cases $\varrho(x_0,x) \le \frac{r}{2k}$, $\frac{r}{2k} \le \varrho(x_0,x) \le 2kr$, $\varrho(x_0,x) \ge 2kr$, where k is the constant from the triangle inequality (2.1).

The case $\varrho(x_0,x) \le \frac{r}{2k}$**.**

We have $\dfrac{\varrho(x_0,y)}{\varrho(x,y)} \le \dfrac{k\left(\varrho(x,y)+\varrho(x_0,x)\right)}{\varrho(x,y)} \le k\left(1+\dfrac{\varrho(x_0,x)}{r}\right) \le 2k$ and $\dfrac{\varrho(x_0,y)}{\varrho(x,y)} \ge \dfrac{1}{k} - \dfrac{\varrho(x_0,x)}{r} \ge \dfrac{1}{2k}$. Hence $\dfrac{1}{2k} \le \dfrac{\varrho(x_0,y)}{\varrho(x,y)} \le 2k$. Consequently,

$$\mathcal{J}(x,r) \le C \int\limits_{X\setminus B(x,r)} \left(\frac{a\left(\varrho\left(x,y\right)\right)}{\varrho\left(x,y\right)^{N-\frac{b}{p(x)}}}\right)^{p(x)} d\mu\left(y\right).$$

Then by Lemma 3.1

$$\mathcal{J}(x,r) \le C \int\limits_r^d t^{N-1}\left[\frac{a(t)}{t^N}\right]^{p(x)} t^b\, dt.$$

Therefore

$$\mathcal{J}(x,r) \le C\mathcal{G}(x,r), \quad \varrho(x_0,x) \le \frac{r}{2k} \tag{3.3}$$

The case $\frac{r}{2k} \le \varrho(x_0,x) \le 2kr$.

We split the integration in $\mathcal{J}(x,r)$ as follows

$$\mathcal{J}(x,r) := \int\limits_{r<\varrho(x,y)<2k\varrho(x_0,x)} \left(\frac{a\left(\varrho\left(x,y\right)\right)}{\varrho\left(x,y\right)^N}\right)^{p(x)} \varrho(x_0,y)^b d\mu\left(y\right)$$

$$+ \int\limits_{\varrho(x,y)>2k\varrho(x_0,x)} \left(\frac{a\left(\varrho\left(x,y\right)\right)}{\varrho\left(x,y\right)^N}\right)^{p(x)} \varrho(x_0,y)^b d\mu\left(y\right) := J_1 + J_2.$$

Since $\frac{a(r)}{r^N}$ is almost decreasing, we obtain

$$J_1 \le C\left(\frac{a\left(r\right)}{r^N}\right)^{p(x)} \int\limits_{r<\varrho(x,y)<2k\varrho(x_0,x)} \varrho(x_0,y)^b d\mu\left(y\right)$$

When $\varrho(x,y) > r$ and $\varrho(x_0,x) < 2kr$, then $\varrho(x_0,y) \le k\left(\varrho(x,y)+\varrho(x_0,x)\right) \le k\left(\varrho(x,y)+2kr\right) \le 3k^2\varrho(x,y)$. Consequently,

$$\le C\left(\frac{a\left(r\right)}{r^N}\right)^{p(x)} \int\limits_{\substack{\varrho(x,y)<2k\varrho(x_0,x)\\ \varrho(x_0,y)\le 3k^2\varrho(x,y)}} \varrho(x_0,y)^b d\mu\left(y\right)$$

$$\le C\left(\frac{a\left(r\right)}{r^N}\right)^{p(x)} \int\limits_{\varrho(x_0,y)\le 6k^3\varrho(x_0,x)} \varrho(x_0,y)^b d\mu\left(y\right)$$

We make use of the known estimate

$$\int\limits_{\varrho(x,y)\le R} \varrho(x,y)^b d\mu(y) \le CR^{b+N}, \quad b > -N \tag{3.4}$$

valid for quasimetric spaces with the growth condition (2.2), see for instance [8], Lemma 1 (actually $C = \frac{K2^N}{2^{N+b}-1}$ in (3.4), where K is the constant from (2.2)), which yields

$$J_1 \leq C\left(\frac{a\,(r)}{r^N}\right)^{p(x)} \varrho(x_0,x)^{b+N}.$$

It is easily seen that then

$$J_1 \leq Ca\,(r)^{p(x)}\,\varrho(x_0,x)^{b+N}\int\limits_r^d t^{-Np(x)-1}dt \leq C\int\limits_r^d t^{N-1}\left[\frac{a(t)}{t^N}\right]^{p(x)} t^b\,dt,$$

so that

$$J_1 \leq C\mathcal{G}(x,r).$$

The estimate for $J_2 = \mathcal{J}(x, 2k\varrho(x_0,x))$ is contained in (3.3) with $r = 2k\varrho(x_0,x)$. Hence

$$\mathcal{J}(x,r) \leq C\mathcal{G}(x,r), \qquad \frac{r}{2k} \leq \varrho(x_0,x) \leq 2kr \tag{3.5}$$

The case $\varrho(x_0,x) \geq 2kr$**.**
We have

$$\mathcal{J}(x,r) = \int\limits_{r<\varrho(x,y)<\frac{\varrho(x_0,x)}{2k}} \left(\frac{a\,(\varrho\,(x,y))}{\varrho\,(x,y)^N}\right)^{p(x)} \varrho(x_0,y)^b d\mu\,(y)$$

$$+\int\limits_{\varrho(x,y)>\frac{\varrho(x_0,x)}{2k}} \left(\frac{a\,(\varrho\,(x,y))}{\varrho\,(x,y)^N}\right)^{p(x)} \varrho(x_0,y)^b d\mu\,(y) = J_3 + J_4.$$

For the term J_3 we have $\varrho(x_0,y) \geq \frac{1}{k}\varrho(x_0,x) - \varrho(x,y) \geq \frac{1}{k}\varrho(x_0,x) - \frac{1}{2k}\varrho(x_0,x) = \frac{1}{2k}\varrho(x_0,x)$ and $\varrho(x_0,y) \leq k(\varrho(x_0,x)+\varrho(x,y)) < 2k\varrho(x_0,x)$. Then

$$J_3 \leq C\varrho(x_0,x)^b \int\limits_{r<\varrho(x,y)<\frac{1}{2k}\varrho(x_0,x)} \left(\frac{a\,(\varrho\,(x,y))}{\varrho\,(x,y)^N}\right)^{p(x)} d\mu\,(y)\,.$$

By Lemma 3.1 we then obtain

$$J_3 \leq C\varrho(x_0,x)^b\int\limits_r^d t^{N-1}\left[\frac{a(t)}{t^N}\right]^{p(x)} dt = C\mathcal{G}(x,r).$$

The term J_4, coincides with $\mathcal{J}\left(x, \frac{\varrho(x_0,x)}{2k}\right)$ and its estimate is contained in the preceding case $\frac{r}{2k} \leq \varrho(x_0,x) \leq 2kr$. Therefore,

$$\mathcal{J}(x,r) \leq C\mathcal{G}(x,r), \qquad \varrho(x_0,x) \geq 2kr. \tag{3.6}$$

Gathering estimates (3.3), (3.5), (3.6), we arrive at (3.2). □

4. Main lemma

We need to estimate the norm

$$\eta_{p,\gamma}(x,r) = \left\| \frac{a\left(\varrho(x,y)\right)}{\varrho(x,y)^N} \right\|_{L^{p(\cdot)}\left(X\setminus B(x,r), w^{\frac{\gamma-N}{p(x_0)}}\right)}, \tag{4.1}$$

where $w^{\frac{\gamma-N}{p(x_0)}}(y) = \varrho(x_0,y)^{\frac{\gamma-N}{p(x_0)}}$ and $\gamma > 0$.

Lemma 4.1. *Let (X, ϱ, μ) be a bounded quasimetric space with Borel regular measure μ: satisfying the growth condition* (2.2)*, $d = diam\ X$ and let p satisfy assumptions* (2.3)-(2.4)*. Suppose that the function $a(r) : (0, d) \to (1, +\infty)$ is almost increasing, there exists $0 < \beta < \min\left(\frac{N}{(p_-)'}, N - \frac{\gamma}{p_-}\right)$ such that*

$$\frac{a(r)}{r^\beta} \quad \textit{is almost decreasing}. \tag{4.2}$$

Then

$$\eta_{p,\gamma}(x,r) \le C\frac{a(r)}{r^{\frac{N}{p'(x)}}}\left[\max\left(r, \varrho(x_0,x)\right)\right]^{\frac{\gamma-N}{p(x)}} \quad \textit{for} \quad 0 < r < \frac{d}{2}. \tag{4.3}$$

Proof. By definition of the norm

$$\int\limits_{\substack{y\in X\\ \varrho(x,y)>r}} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N \eta_{p,\gamma}}\right)^{p(y)} \varrho(x_0,y)^{\gamma-N} d\mu(y) = 1. \tag{4.4}$$

1st step. Values $\eta_{p,\gamma} \ge 1$ are only of interest. This follows from the fact that the right hand side of (4.3) is bounded from below.

$$\frac{a(r)}{r^{\frac{N}{p'(x)}}}\left[\max\left(r, \varrho(x_0,x)\right)\right]^{\frac{\gamma-N}{p(x)}} \ge \frac{a(r)}{r^{\frac{N}{p'(x)}}} \min\left(r^{\frac{\gamma-N}{p(x)}}, d^{\frac{\gamma-N}{p(x)}}\right)$$

$$= \min\left(\frac{a(r)}{r^{N-\frac{\gamma}{p(x)}}}, d^{\frac{\gamma-N}{p(x)}}\frac{a(r)}{r^{\frac{N}{p'(x)}}}\right) \ge C\frac{a(r)}{r^\beta} \ge C > 0,$$

the last inequality following from the fact that $\frac{a(r)}{r^\beta}$ is almost decreasing on $[0, d]$.

2nd step. Small values of r, say $0 < r < \frac{1}{2}$, are only of interest. To show that this assumption is possible, we have to check that the right-hand side of (4.3) is bounded from below and $\eta_{p,\gamma}(x,r)$ is bounded from above when $r \ge \frac{1}{2}$.

Let $r \ge \frac{1}{2}$. From the fact that $\eta_{p,\gamma} \ge 1$ it follows that

$$\int\limits_{\substack{y\in X:\\ \varrho(x,y)>1}} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N}\right)^{p(y)} \frac{1}{\eta_{p,\gamma}} \varrho(x_0,y)^{\gamma-N} d\mu(y) \ge 1.$$

Hence

$$\eta_{p,\gamma} \leq \int\limits_{\substack{y\in X:\\ \varrho(x,y)>1}} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N}\right)^{p(y)} \varrho(x_0,y)^{\gamma-N} d\mu(y) < C.$$

3rd step. Rough estimate. First we derive a weaker estimate

$$\eta_{p,\gamma}(x,r) \leq Cr^{-N}a(r) \tag{4.5}$$

which will be used later to obtain the final estimate (4.3). From (4.4) we have

$$1 \leq \int\limits_{\substack{y\in X\\ \varrho(x,y)>r}} \left[\left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N\eta_{p,\gamma}}\right)^{p_-} + \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N\eta_{p,\gamma}}\right)^{p_+}\right] \varrho(x_0,y)^{\gamma-N} d\mu(y)$$

Since $\varrho(x,y) > r$, we obtain

$$1 \leq C\left[\left(\frac{a(r)}{r^N\eta_{p,\gamma}}\right)^{p_-} + \left(\frac{a(r)}{r^N\eta_{p,\gamma}}\right)^{p_+}\right] \int\limits_X \varrho(x_0,y)^{\gamma-N} d\mu(y)$$

$$\leq C\left[\left(\frac{a(r)}{r^N\eta_{p,\gamma}}\right)^{p_-} + \left(\frac{a(r)}{r^N\eta_{p,\gamma}}\right)^{p_+}\right],$$

where the convergence of the integral $\int\limits_X \varrho(x_0,y)^{\gamma-N} d\mu(y)$ with $\gamma > 0$ (see (3.4)) was taken into account.

If $\frac{a(r)}{r^N\eta_{p,\gamma}} \geq 1$, there is nothing to prove. When $\frac{a(r)}{r^N\eta_{p,\gamma}} < 1$, we obtain $1 \leq 2C\left(\frac{a(r)}{r^N\eta_{p,\gamma}}\right)^{p_-}$, which proves the estimate.

4th step. We split integration in (4.4) as follows

$$1 = \sum_{i=1}^{3} \int\limits_{X_i(x)} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N\eta_{p,\gamma}}\right)^{p(y)} \varrho(x_0,y)^{\gamma-N} d\mu(y) := I_1 + I_2 + I_3,$$

where

$$X_1(x) = \left\{y \in X : r < \varrho(x,y) < \frac{1}{2},\, K(x,y) > \eta_{p,\gamma}\right\},$$

$$X_2(x) = \left\{y \in X : r < \varrho(x,y) < \frac{1}{2},\, K(x,y) < \eta_{p,\gamma}\right\},$$

$$X_3(x) = \left\{y \in X : \varrho(x,y) > \frac{1}{2}\right\}.$$

5th step. Estimation of I_1. We have

$$I_1 = \int\limits_{X_1(x)} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N\eta_{p,\gamma}}\right)^{p(x)} \varrho(x_0,y)^{\gamma-N} u_r(x,y) d\mu(y),$$

where

$$u_r(x,y) = \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N \eta_{p,\gamma}}\right)^{p(y)-p(x)}.$$

We show that the function $u_r(x,y)$ is bounded from below and above uniformly in x, y and r. To this end, we make use of (2.4) and following estimations in [33], p. 432, and obtain

$$|\ln u_r(x,y)| \leq C \frac{\ln \dfrac{a(\varrho(x,y))}{\varrho(x,y)^N \eta_{p,\gamma}}}{\ln \dfrac{1}{\varrho(x,y)}} = C \frac{\ln \dfrac{a(\varrho(x,y))}{\varrho(x,y)^N} - \ln \eta_{p,\gamma}}{\ln \dfrac{1}{\varrho(x,y)}},$$

where we took into account that $\dfrac{a(\varrho)}{\varrho^N \eta_{p,\gamma}} \geq 1$. Therefore,

$$|\ln u_r(x,y)| \leq C \frac{\ln \dfrac{a(\varrho(x,y))}{\varrho(x,y)^N}}{\ln \dfrac{1}{\varrho(x,y)}} \leq C \frac{|\ln a(\varrho(x,y)| + N \ln \dfrac{1}{\varrho(x,y)}}{\ln \dfrac{1}{\varrho(x,y)}} \leq C.$$

Then

$$I_1 \leq \frac{C}{\eta_{p,\gamma}^{p(x)}} \int\limits_{X_1(x)} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N}\right)^{p(x)} \varrho(x_0,y)^{\gamma-N} d\mu(y).$$

By Lemma 3.2 we get

$$I_1 \leq C\mathcal{F}(x,r,p(x)), \tag{4.6}$$

where

$$\mathcal{F}(x,r,q) = \begin{cases} \int\limits_r^d t^{\gamma-1} \left[\frac{a(t)}{\eta_{p,\gamma} t^N}\right]^q dt, & \text{if } \varrho(x_0,x) \leq r \\ \varrho(x_0,x)^{\gamma-N} \int\limits_r^d t^{N-1} \left[\frac{a(t)}{\eta_{p,\gamma} t^N}\right]^q dt, & \text{if } \varrho(x_0,x) > r \end{cases}$$

6th step. Estimation of I_2. For I_2 we obtain

$$I_2 \leq \int\limits_{r<\varrho<1} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N \eta_{p,\gamma}}\right)^{p_-} \varrho(x_0,y)^{\gamma-N} d\mu(y)$$

$$\leq \frac{1}{\eta_{p,\gamma}^{p_-}} \int\limits_{r<\varrho<1} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N}\right)^{p_-} \varrho(x_0,y)^{\gamma-N} d\mu(y)$$

and the application of Lemma 3.2 gives

$$I_2 \leq C\mathcal{F}(x,r,p_-). \tag{4.7}$$

7th step. Estimation of I_3. For I_3 we have

$$I_3 \le \frac{C}{\eta_{p,\gamma}^{p_-}} \int\limits_{\varrho>\frac{1}{2}} \left(\frac{\frac{a(\varrho(x,y))}{\sup\limits_{t\in(0,d)} a(t)}}{(2\varrho(x,y))^N} \right)^{p(y)} \varrho(x_0,y)^{\gamma-N} d\mu(y)$$

$$\le \frac{C}{\eta_{p,\gamma}^{p_-}} \int\limits_{\varrho>\frac{1}{2}} \left(\frac{\frac{a(\varrho(x,y))}{\sup\limits_{t\in(0,d)} a(t)}}{(2\varrho(x,y))^N} \right)^{p_-} \frac{d\mu(y)}{\varrho(x_0,y)^{N-\gamma}} \le \frac{C}{\eta_{p,\gamma}^{p_-}} \int\limits_{\varrho>\frac{1}{2}} \left(\frac{a(\varrho(x,y))}{\varrho(x,y)^N} \right)^{p_-} \frac{d\mu(y)}{\varrho(x_0,y)^{N-\gamma}},$$

where the last integral is convergent and uniformly bounded with respect to x by Lemma 3.2. Hence

$$I_3 \le \frac{C}{\eta_{p,\gamma}^{p_-}}. \tag{4.8}$$

8th step. By (4.6), (4.7), (4.8) we have

$$1 \le C \left[\mathcal{F}(x,r,p(x)) + \mathcal{F}(x,r,p_-) + \frac{1}{\eta_{p,\gamma}^{p_-}} \right].$$

We may consider $\eta_{p,\gamma}(x,r)$ only for those x,r for which $\eta_{p,\gamma}(x,r)$ is sufficiently large: $\eta_{p,\gamma}(x,r) \ge (2C)^{\frac{1}{p_-}}$, where C is the constant from the last inequality. For such x,r we have $\frac{C}{\eta_{p,\gamma}} \le \frac{1}{2}$ and we then obtain

$$\frac{1}{2} \le C \left[\mathcal{F}(x,r,p(x)) + \mathcal{F}(x,r,p_-) \right]. \tag{4.9}$$

Taking into account that $\dfrac{Ca(t)}{t^N \eta_{p,\gamma}} \ge 1$ by (4.5), we have

$$\mathcal{F}(x,r,p_-) \le C\mathcal{F}(x,r,p(x))$$

Then (4.9) yields the inequality $1 \le C\mathcal{F}(x,r,p(x))$, that is,

$$\eta_{p,\gamma}^{p(x)} \le C \begin{cases} \int\limits_r^d t^{\gamma-1} \left[\frac{a(t)}{t^N}\right]^{p(x)} dt, & \text{if } \ \varrho(x_0,x) \le r \\ \varrho(x_0,x)^{\gamma-N} \int\limits_r^d t^{N-1} \left[\frac{a(t)}{t^N}\right]^{p(x)} dt, & \text{if } \ \varrho(x_0,x) > r \end{cases} \tag{4.10}$$

9th step. Final estimate of $\eta_{p,\gamma}$. Write (4.10) in the next form

$$\eta_{p,\gamma}^{p(x)} \le C \begin{cases} \int\limits_r^d t^{\beta p(x)-Np(x)+\gamma-1} \left[\frac{a(t)}{t^\beta}\right]^{p(x)} dt, & \text{if } \ \varrho(x_0,x) \le r \\ \varrho(x_0,x)^{\gamma-N} \int\limits_r^d t^{\beta p(x)-Np(x)+N-1} \left[\frac{a(t)}{t^\beta}\right]^{p(x)} dt, & \text{if } \ \varrho(x_0,x) > r \end{cases}.$$

By (4.2) we have

$$\eta_{p,\gamma}^{p(x)} \le C\left[\frac{a(r)}{r^{\beta}}\right]^{p(x)} \begin{cases} \int\limits_r^d t^{\beta p(x)-Np(x)+\gamma-1}\,dt, & \text{if } \varrho(x_0,x)\le r \\ \varrho(x_0,x)^{\gamma-N}\int\limits_r^d t^{\beta p(x)-Np(x)+N-1}\,dt, & \text{if } \varrho(x_0,x)>r \end{cases}.$$

Since $0<\beta<\min\left(\frac{N}{(p_-)'}, N-\frac{\gamma}{p_-}\right)$ we have $\beta p(x)-Np(x)+m<0$, where m can take two values: N or γ. Then

$$\int\limits_r^d t^{\beta p(x)-Np(x)+m-1}dt \le Cr^{\beta p(x)-Np(x)+m}.$$

Therefore

$$\eta_{p,\gamma}^{p(x)} \le C\left[\frac{a(r)}{r^{N}}\right]^{p(x)} \begin{cases} r^{\gamma}, & \text{if } \varrho(x_0,x)\le r \\ \varrho(x_0,x)^{\gamma-N}r^N, & \text{if } \varrho(x_0,x)>r \end{cases}$$

$$= C\left[\frac{a(r)}{r^{\frac{N}{p'(x)}}}\right]^{p(x)}\left[\max(\varrho(x_0,x),r)\right]^{\gamma-N},$$

which proves (4.3).

□

5. Proof of the main result

Proof. As usual, we may suppose that $f(x)\ge 0$ and $\|f\|_{L^{p(\cdot)}(X,w^{\nu})}\le 1$ and show that

$$\int\limits_X \Phi\left[x, w(x)I^a f(x)\right]d\mu(x)\le C<\infty. \tag{5.1}$$

In accordance with Hedberg's trick, we split $I_a f(x)$ as follows

$$I_a f(x)=\int\limits_{B(x,r)}\frac{a[\varrho(x,y)]}{\varrho(x,y)^N}f(y)d\mu(y)+\int\limits_{X\setminus B(x,r)}\frac{a\left(\varrho\left(x,y\right)\right)}{\varrho\left(x,y\right)^N}f(y)d\mu\left(y\right)$$

$$=\mathcal{A}_r(x)+\mathcal{B}_r(x).$$

The estimation of the first term via the maximal function well known in the case $a(r)=r^{\alpha}$, now holds in the form

$$\mathcal{A}_r(x)\le CA(r)Mf(x), \quad A(r)=\int\limits_0^r\frac{a(t)}{t}\,dt, \tag{5.2}$$

see [13], Subsection 4.4.

For $\mathcal{B}_r(x)$, by Hölder inequality

$$\left| \int\limits_X f(x)g(x)d\mu(x) \right| \leq k\|f\|_{L^{p(\cdot)}(X,\varrho)}\|g\|_{L^{p'(\cdot)}(X,\varrho^{-1})}$$

for variable exponents, we obtain

$$\mathcal{B}_r(x) \leq C\,\|f\|_{L^{p(\cdot)}(X\setminus B(x,r),w^{\nu})} \left\| \frac{a\left(\varrho\left(x,y\right)\right)}{\varrho\left(x,y\right)^{N}} \right\|_{L^{p'(\cdot)}(X\setminus B(x,r),w^{-\nu})},$$

where we denote $w = \varrho(\cdot, x_0)$ for brevity. Under notation (4.1) we obtain

$$\mathcal{B}_r(x) \leq C\eta_{p',\gamma}(x,r) \quad \text{with} \quad \gamma = N - \nu p' x_0),$$

with N from the growth condition (one may take $N < \underline{\mathfrak{dim}}(X)$ arbitrarily close to $\underline{\mathfrak{dim}}(X)$ according to (2.5)-(2.6)).

We apply Lemma 4.1 and obtain

$$\begin{aligned} \eta_{p',\gamma}(x,r) &\leq C\frac{a(r)}{r^{\frac{N}{p(x)}}}[\max(r, \varrho(x_0,x))]^{-\nu} \\ &\leq C\frac{a(r)}{r^{\frac{N}{p(x)}}}\varrho(x_0,x)^{-\nu} \sim C\frac{a(r)}{r^{\frac{N}{p(x)}}}\varrho(x_0,x)^{-\nu} \end{aligned}$$

Therefore

$$\mathcal{B}_r(x) \leq C\frac{a(r)}{r^{\frac{N}{p(x)}}}\varrho(x_0,x)^{-\nu}$$

and

$$\begin{aligned} I_a f(x) &\leq C\left[A(r)Mf(x) + \frac{a(r)}{r^{\frac{N}{p(x)}}}\varrho(x_0,x)^{-\nu}\right] \\ &\leq CA(r)\left[Mf(x) + \frac{1}{r^{\frac{N}{p(x)}}}\varrho(x_0,x)^{-\nu}\right], \end{aligned}$$

where we used the fact that $a(r \leq CA(r))$ which follows from (4.2). Consequently, by Lemma 2.5 we get

$$I_a f(x) \leq C\Phi^{-1}\left(x, \frac{1}{r^N}\right)\left[r^{\frac{N}{p(x)}}Mf(x) + \varrho(x,x_0)^{-\nu}\right]$$

Now we choose $r = \dfrac{1}{\varrho(x_0,x)^{\frac{\nu p(x)}{N}}Mf(x)^{\frac{p(x)}{N}}}$ and get

$$\frac{1}{C}[\varrho(x,x_0)]^{\nu}I_a f(x) \leq \Phi^{-1}\left(x, [\varrho(x,x_0)]^{\nu p(x)}Mf(x)^{p(x)}\right).$$

Hence

$$\int\limits_X \Phi\left(x, \frac{1}{C}[\varrho(x,x_0)]^{\nu}I_a f(x)\right)d\mu(x) \leq \int\limits_X [\varrho(x,x_0)]^{\nu p(x)}Mf(x)^{p(x)}d\mu(x).$$

Then the application of Theorem 2.4 completes the proof of (5.1), if we take into account property (2.5) . □

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