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On Some Classical Operators of Variable Order in Variable Exponent Spaces

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To Vladimir Maz'ya on the occasion of his 70th birthday

Abstract. We give a survey of a selection of recent results on weighted and non-weighted estimations of classical operators of Harmonic Analysis in variable exponent Lebesgue, Morrey and Hölder spaces, based on the talk presented at International Conference *Analysis, PDEs and Applications* on the occasion of the 70th birthday of Vladimir Maz'ya, Rome, June 30–July 3, 2008. We touch both the Euclidean case and the general setting within the frameworks of quasimetric measure spaces. Some of the presented results are new.

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1. Introduction

Last decade there was a strong increase of interest to studies of various operators and function spaces in the "variable setting", when parameters defining the operator or the space (which usually are constant), may vary from point to point. A number of mathematical problems leading, for instance, to Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, or Sobolev spaces $W^{m,p(\cdot)}$ arise in applications to PDE, variational problems and continuum mechanics (in particular, in the theory of the so-called electrorheological fluids), see [76]; see also a recent paper [9] on applications in the problems of image restoration. These applications stipulated a significant interest to the spaces $L^{p(\cdot)}$ in the last decade. The study of classical operators of harmonic analysis (maximal, singular operators and potential type operators) in the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, weighted or non-weighted, undertaken last decade, continues to attract a strong interest of researchers, influenced in particular by possible applications. We refer in particular to the surveying articles [22], [51], [82]. A progress in the study of these spaces raised a natural interest to other spaces whose parameters may be variable, for instance, Morrey spaces or Hölder (Lipschitz) spaces. The development of Harmonic Analysis and Operator Theory in the spaces $L^{p(\cdot)}$ led also to an interest to variability of parameters defining an operator.

The area which is now called variable exponent analysis, last decade became a rather branched field with many interesting results obtained in Harmonic Analysis, Approximation Theory, Operator Theory, Pseudo-Differential Operators. We present a survey of a certain selection of results on estimation of the classical operators of harmonic analysis, mainly obtained after surveys [22], [51], [82] had appeared, and present some new results on such estimations, mainly in variable Morrey and Hölder spaces. The survey is far from being complete and reflects a part of results obtained last several years. For earlier results in the topic related to Lebesgue and Sobolev spaces $L^{p(\cdot)}, W^{m,p(\cdot)}$ of variable order we refer to the above-mentioned surveys.

We start with typical examples of operators of variable order and spaces with variable exponents.

1.1. Typical examples of operators with variable orders

1^0. *The Riesz fractional integration operator* of functions on $\mathbb{R}^n$ may be considered in the case of variable order:

$$I^{\alpha(\cdot)}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha(x)}}, \quad \alpha(x) > 0. \tag{1.1}$$

(We omit the usual normalizing constant $\frac{1}{\gamma_n(\alpha)}$; in the case where α is constant, it is for the validity of the semigroup property $I^\alpha I^\beta = I^{\alpha+\beta}$.) In general, $\alpha(x)$ may be allowed to approach singular value $\alpha(x) = 0$ at some points, and then we have to study mapping properties of $I^{\alpha(\cdot)}$ in these or other function spaces, taking into account the degeneracy of the order $\alpha(x)$.

2^0. *Hypersingular integrals.* In the case of constant α, the operator (left)-inverse to the Riesz potential operator is the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ and it may be realized as a hypersingular integral, see [81]. The corresponding variable order construction (written for the case $0 < \alpha(x) < 1$) is:

$$\mathbb{D}^{\alpha(\cdot)}f(x) = \int\limits_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+\alpha(x)}} dy.$$

3^0. *One-dimensional Riemann-Liouville fractional integration*:

$$I^{\alpha(\cdot)}f(x) = \frac{1}{\Gamma[\alpha(x))} \int_a^x f(y)(x-y)^{\alpha(x)-1}\, dy, \quad \alpha(x) > 0,$$

as well as the corresponding fractional differentiation. Such operators have applications in Physics, see, for instance, [50].

4^0. *Fractional operators over quasimetric measure spaces.* More generally, fractional operators of variable order may be considered on arbitrary domains in $\mathbb{R}^n$, surfaces, manifolds, fractal sets, and in general, in the setting of quasimetric measure spaces (X, d, μ) with a quasimetric d and positive Borel measure μ. It is known, see, for instance, [25], [51], [52], that they are defined in different forms, not equivalent in general,

$$\begin{aligned}\mathfrak{I}^{\alpha(\cdot)} f(x) &= \int\limits_X \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x, d(x,y))} f(y)\, d\mu(y),\\ \mathfrak{I}^{\alpha} f(x) &= \int_\Omega \frac{f(y)\, d\mu(y)}{\mu B(x, d(x,y))^{1-\alpha(x)}},\end{aligned} \tag{1.2}$$

and

$$I^{\alpha(\cdot)} f(x) = \int\limits_X \frac{f(y)\, d\mu(y)}{[d(x,y)]^{N-\alpha(x)}}, \tag{1.3}$$

where $\alpha(x) > 0$ and N should be thought as a kind of dimension of X. However, in general, X may have no "dimension", but may have the so-called lower and upper dimensions, which in their turn may depend on the point x. In the case where the measure satisfies the growth condition $\mu B(x, r) \le Cr^N$ with some $N > 0$, this exponent N (not necessarily an integer), may be used to define $I^{\alpha(\cdot)} f(x)$.

5^0. *Fractional maximal function.* Another example of an operator of variable order is the fractional maximal function

$$M^{\alpha(\cdot)} f = \sup_{r>0} r^{-\alpha(x)} \int_{|y-x|<r} f(y)\, dy$$

and its corresponding version for an arbitrary quasimetric measure space.

6^0. *Fractional powers of operators of variable order.* In general, one may also consider fractional powers $A^{\alpha(x)}$ of this or other operator A; however, different definitions of such powers, which coincide in the case $\alpha = \text{const}$, now may lead do quite different objects. We do not touch this topic here.

1.2. Typical examples of spaces with variable exponents

1. *Generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent* (see [85], [62] and surveys [22], [51], [82]) defined by the condition

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

2. More generally, *Musielak-Orlicz spaces $L^{\Phi(\cdot)}(\Omega)$ with the Young function also varying from point to point* (see [68], [21]):

$$\int_\Omega \Phi[x, f(x)]\, dx < \infty.$$

3. *Variable exponent Morrey spaces* $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ ([3], [53], [54]) defined by

$$\sup_{x\in\Omega,\, r>0} r^{-\lambda(x)} \int\limits_{B(x,r)\cap\Omega} |f(y)|^{p(y)}dy < \infty. \tag{1.4}$$

4. *Hölder spaces* $H^{\lambda(\cdot)}(\Omega)$ *of variable order* ([47], [48], [74]), defined by the condition

$$\sup_{|h|<t} |f(x+h)-f(x)| \le Ct^{\lambda(x)}, \ x\in\Omega.$$

5. More generally, *generalized Hölder spaces with variable characteristic* $\omega(h) = \omega(x,h)$ depending on x ([88]):

$$\sup_{|h|<t} |f(x+h)-f(x)| \le C\omega(x,t),$$

that is, the spaces of continuous functions with a given dominant of their continuity modulus, which may vary from point to point.

Notation

(X,d,μ) is a measure space with quasimetric d and a non-negative measure μ;
$B(x,r) = B_X(x,r) = \{y\in X : d(x,y) < r\}$;
$p'(x) = \frac{p(x)}{p(x)-1}, \quad 1 < p(x) < \infty, \quad \frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$;
$p_- = p_-(X) = \inf_{x\in X} p(x)$, $p^+ = p^+(X) = \sup_{x\in X} p(x)$;
$p'_- = \inf_{x\in X} p'(x) = \frac{p^+}{p^+-1}, \ \ (p')^+ = \sup_{x\in X} p'(x) = \frac{p_-}{p_--1}$;
$\mathbb{P}(X)$, see (2.2)–(2.3);
a.i. = almost increasing $\Longleftrightarrow u(x) \le Cu(y)$ for $x \le y, C > 0$.
$\mathbb{R}^n$ is the n-dimensional Euclidean space;
Ω is a non-empty open set in $\mathbb{R}^n$ or Ω;
d_Ω denotes the diameter of Ω;
χ_E is a characteristic function of a measurable set $E\subset\mathbb{R}^n$;
$|E|$ is the Lebesgue measure of E;
by c and C we denote various absolute positive constants, which may have different values even in the same line.

2. Some basics for variable exponent Lebesgue spaces

In the sequel (X,d,μ) is a homogeneous type space, i.e., a measure space with a quasimetric d and a non-negative measure μ satisfying the doubling condition; we refer to [11], [25], [42], for the basic notions of function spaces on quasimetric measure spaces. The space (X,d,μ) is assumed to satisfy the conditions:

1) all the balls $B(x,r)$ are measurable,
2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(\mu)$.

The doubling condition means that $\mu B(x,2r) \le C\mu B(x,r)$.

By $L^{p(\cdot)}(X,\varrho)$, where $\varrho(x) \geq 0$, we denote the weighted Banach space of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(X,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_X \left|\frac{\varrho(x)f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leq 1\right\} < \infty. \tag{2.1}$$

We write $L^{p(\cdot)}(X,1) = L^{p(\cdot)}(X)$ and $\|f\|_{L^{p(\cdot)}(X)} = \|f\|_{p(\cdot)}$ in the case $\varrho(x) \equiv 1$.

The generalized Lebesgue spaces $L^{p(\cdot)}(X)$ with variable exponent on quasimetric measure spaces have been considered in [31], [35], [44], [45], [46], [49], [66], the Euclidean case being studied in [26], [29], [62], [85], see also references therein.

By $\mathbb{P}(X)$ we denote the set of bounded measurable functions $p(x)$ defined on X which satisfy the condition

$$1 < p_- \leq p(x) \leq p^+ < \infty, \quad x \in X \tag{2.2}$$

and by $WL(X)$ we denote the set of functions $p(x)$ such that

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{d(x,y)}}, \quad d(x,y) \leq \frac{1}{2}, \quad x, y \in X. \tag{2.3}$$

3. On some recent results on boundedness of classical operators in spaces $L^{p(\cdot)}(\Omega,\varrho)$

The boundedness of various classical operators in $L^{p(\cdot)}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, in the non-weighted case was proved in [12] by the extrapolation method. An extension to the case of weighted estimates, including the setting of quasimetric measure spaces, was given in [58], [57]. In relation to the extrapolation method, we refer to [75], [15], [16], [17].

We touch the cases not covered in [12], [58], [57] for the following operators

1) *Convolution operators*

$$Af(x) = \int\limits_{\mathbb{R}^n} k(y)f(x-y)dy \tag{3.1}$$

with rather "nice" kernels for which the local log-condition is not needed,

2) *Hardy-Littlewood maximal operator*

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y), \quad x \in X \tag{3.2}$$

where X in general is a quasimetric measure space, being either an open set in $\mathbb{R}^n$ or a Carleson curve on the complex plane in this section; we pay a special attention to this special case of metric measure spaces with constant dimension – Carleson curves – because of important application in operator theory;

3) *the Cauchy singular integral operator*

$$S_\Gamma f(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{3.3}$$

along a Carleson curve Γ on complex plane, where ν is the arc-length measure;

4) *potential type operators.*

3.1. On convolution operators

We single out a result on convolution operators, obtained without local log-condition. As is known, the Young theorem in its natural form is not valid in the case of variable exponent, whatsoever smooth exponent $p(x)$ is. As observed by L. Diening, the Young theorem is valid under the log-condition on $p(x)$ if the kernel is dominated by a radial integrable non-increasing function. However, a natural expectation was that the Young theorem may be valid in the case of rather "nice" kernels without the local log-condition, which was proved in [23], see Theorem 3.1.

Let $\mathcal{P}_\infty(\mathbb{R}^n)$ be the set of measurable bounded functions on $\mathbb{R}^n$ such that $1 \le p_- \le p(x) \le p_+ < \infty, \quad x \in \mathbb{R}^n$, there exists $p(\infty) = \lim\limits_{x\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \le \frac{A}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n. \tag{3.4}$$

Theorem 3.1. *Let $|k(y)| \le C(1+|y|)^{-\lambda}, y \in \mathbb{R}^n$ for some $\lambda > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$. Then the operator* (3.1) *is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ under the only assumption that $p, q \in \mathcal{P}_\infty(\mathbb{R}^n)$ and $q(\infty) \ge p(\infty)$.*

3.2. On the maximal operator

Let

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y) \tag{3.5}$$

be the Hardy-Littlewood maximal operator. In the case of constant $p \in (1,\infty)$ the boundedness of the maximal operator on bounded quasimetric measure spaces is well known, due to A.P. Calderón [8] and R. Macías and C. Segovia [65], for weights in the Muckenhoupt class $A_p = A_p(X)$. For variable exponents, the non-weighted boundedness of the maximal operator was first proved in the Euclidean case in [18], [19], for bounded domains or for $\mathbb{R}^n$ with $p(x) \equiv$ const outside some large ball. For further results in non-weighted case see [13], [14], [21], [63], [69], [70]. Extensions to the case of quasimetric measure spaces were considered in [45] and [49].

Let, by definition, $A_{p(\cdot)}(X)$:= Muckenhoupt class be the class of weights for which the maximal operator is bounded in the space $L^{p(\cdot)}(X, \varrho)$. By $\widetilde{A}_{p(\cdot)}(X)$ we denote the class of weights, which satisfy the "Muckenhoupt-like looking" condition

$$\sup_{x\in X, r>0}\left(\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}|\varrho(y)|^{p(y)}d\mu(y)\right)\left(\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}\frac{d\mu(y)}{|\varrho(y)|^{\frac{p(y)}{p_--1}}}\right)^{p_--1}<\infty, \tag{3.6}$$

where $p_- = \inf_{x\in X} p(x)$. The class $\widetilde{A}_{p(\cdot)}(X)$ coincides with $A_{p(\cdot)}(X)$ in the case where p is constant. The next theorem ([61] in the case X is a Carleson curve and [60], [59] in the general case) states that $\widetilde{A}_{p(\cdot)}(X) \subset A_{p(\cdot)}(X)$ under natural conditions.

Theorem 3.2. *Let X be a bounded doubling measure quasimetric space. Under conditions* (2.2)*,* (2.3) *and* (3.6)*, $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X,\varrho)$.*

In the case of power weights or radial type weights, the boundedness of the maximal operator was obtained under conditions weaker than derived from (3.6). We refer for details to [58], [57], but mention that for a radial weight $w(|x-a|)$, $a \in \Omega$, with w in the so-called Bary-Stechkin type class, the condition on the weight, in the Euclidean case, reduces to

$$-\frac{n}{p(a)} < m(w) \leq M(w) < \frac{n}{p'(a)} \tag{3.7}$$

in terms of the Matuszewska-Orlicz indices $m(w)$ and $M(w)$ of the function $w(r)$; see a version of (3.7) for quasimetric measure spaces in [58], [57]. The sufficiency of the above condition in terms of the numbers $m(w)$ and $M(w)$ seems to be a new result even in the case of constant p. In relation with (3.7), note that in applications, the verification of the Muckenhoupt condition for a concrete weight may be an uneasy task, even in the case of constant p. Therefore, it is always of importance to have easier sufficient conditions for weight functions, as, for instance, in (3.7).

3.3. On the Cauchy singular operator

We specially dwell on the case of the Cauchy singular operator along Carleson curves because of of its importance in application to singular integral equations.

Theorem 3.3 ([55])**.** *Let Γ be a simple Carleson curve of finite or infinite length, let $p \in \mathbb{P}(\Gamma)\cap WL(\Gamma)$ and the following condition at infinity*

$$|p(t)-p(\tau)| \leq \frac{A_\infty}{\ln\frac{1}{\left|\frac{1}{t}-\frac{1}{\tau}\right|}}, \qquad \left|\frac{1}{t}-\frac{1}{\tau}\right| \leq \frac{1}{2},$$

for $|t| \geq L$, $|\tau| \geq L$ with some $L > 0$, in the case Γ is infinite. Then the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma,\varrho)$ with weight $\varrho(t) = (1+|t|)^\beta \prod\limits_{k=1}^{m}|t-t_k|^{\beta_k}$, $t_k \in \Gamma$, *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, \dots, m, \qquad \text{and} \qquad -\frac{1}{p(\infty)} < \beta + \sum_{k=1}^{m} \beta_k < \frac{1}{p'(\infty)}, \tag{3.8}$$

the latter condition appearing in the case Γ *is infinite.*

An extension of Theorem 3.3 to the case of radial type oscillating weights from the Zygmund-Bary-Stechkin class Φ_δ^β may be found in [56]. The following is an extension of the Guy David theorem to the case of variable exponent $p(x)$.

Theorem 3.4. *Let* Γ *be a finite rectifiable curve and* $p : \Gamma \to [1, \infty)$ *a continuous function. If the operator* S_Γ *is bounded in* $L^{p(\cdot)}(\Gamma)$*, then the curve* Γ *has the property*

$$\sup_{\substack{t\in\Gamma \\ r>0}} \frac{\nu(\Gamma \cap B(t, r))}{r^{1-\varepsilon}} < \infty \tag{3.9}$$

for every $\varepsilon > 0$*. If* $p(t)$ *satisfies the log-condition* (2.3)*, then* (3.9) *holds with* $\varepsilon = 0$*, i.e.,* Γ *is a Carleson curve.*

3.4. On potential operators

For non-weighted results on potentials and Sobolev embeddings we refer to [12], [20], [24], [27], [28], [34], [67], [80].

a) Weighted $p(\cdot) \to q(\cdot)$**-boundedness.** The known generalization of Sobolev theorem by Stein-Weiss for the case of power weights was extended in [83], [84] to the variable exponent setting as follows.

Theorem 3.5. *Let* $p \in \mathbb{P}(\mathbb{R}^n) \cap WL(\mathbb{R}^n)$*,* $\sup\limits_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha}$*,* $\varrho(x) = |x|^{\gamma_0}(1+|x|)^{\gamma_\infty - \gamma_0}$ *and*

$$|p_*(x) - p_*(y)| \le \frac{A_\infty}{\ln \frac{1}{|x-y|}}, \qquad |x-y| \le \frac{1}{2}, \;\; x, y \in \mathbb{R}^n, \quad p_*(x) = p\left(\frac{x}{|x|^2}\right). \tag{3.10}$$

Then operator (1.1) *with* $\alpha(x) = \alpha = \mathrm{const}$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ *to* $L^{q(\cdot)}(\mathbb{R}^n, \varrho)$*, if*

$$\alpha - \frac{n}{p(0)} < \gamma_0 < \frac{n}{p'(0)}, \qquad \alpha - \frac{n}{p(\infty)} < \gamma_\infty < \frac{n}{p'(\infty)}. \tag{3.11}$$

We refer to [78] for a generalization of Theorem 3.5 to the case of more general radial type weights. In connection with estimation of operators over unbounded domains, we refer also to a certain general approach suggested in [40].

b) Characterization of the range of potential operators. The inversion of the Riesz potentials with densities in $L^{p(\cdot)}(\mathbb{R}^n)$ by means of hypersingular integrals

$$\mathbb{D}^\alpha f = \frac{1}{d_{n,\ell}(\alpha)} \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}}\, dy, \tag{3.12}$$

known also as Riesz fractional derivatives of order α, was obtained in [2] (we refer to [81] for the case of constant p and hypersingular integrals in general). This gave a possibility to obtain in [6] a characterization of the range $I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ in terms of convergence of $\mathbb{D}^\alpha f$ in $L^{p(\cdot)}(\mathbb{R}^n)$ as follows.

Theorem 3.6. *Let $p \in WL(\mathbb{R}^n)$, $1 < p_-(\mathbb{R}^n) \leq p^+(\mathbb{R}^n) < \frac{n}{\alpha}$ and f a locally integrable function. Then $f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$, if and only if $f \in L^{q(\cdot)}(\mathbb{R}^n)$ with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and $\mathbb{D}^\alpha f \in L^{p(\cdot)}$ (treated in the sense of convergence in $L^{p(\cdot)}$).*

A study of the range $I^\alpha[L^{p(\cdot)}(\Omega)]$ for domains $\Omega \subset \mathbb{R}^n$ is an open question; in the form given in Theorem 3.6 it is open even in the case of constant p, one of the reasons being in the absence of the corresponding apparatus of hypersingular integrals adjusted to domains in $\mathbb{R}^n$; some their analogue reflecting the influence of the boundary was recently suggested in [72] for the case $0 < \alpha < 1$. In the one-dimensional case for $\Omega = (a, b), -\infty < a < b \leq \infty$, when the range of the potential coincides with that of the Riemann-Liouville fractional integral operators (in the case $1 < p^+ < \frac{1}{\alpha}$), the characterization for variable $p(x)$ was obtained in [73], where for $-\infty < a < b < \infty$ there was also shown its coincidence with the space of restrictions of Bessel potentials.

The result of Theorem 3.6 was used in [6] to obtain a characterization of the *Bessel potential space*

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f : \ f = \mathcal{B}^\alpha\varphi, \quad \varphi \in L^{p(\cdot)}(\mathbb{R}^n)\}, \quad \alpha \geq 0,$$

where $\mathcal{B}^\alpha\varphi = F^{-1}(1 + |\xi|^2)^{-\alpha/2}F\varphi$. It runs as follows.

Theorem 3.7. *Under the conditions of Theorem 3.6*

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = L^{p(\cdot)}(\mathbb{R}^n) \bigcap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f \in L^{p(\cdot)}(\mathbb{R}^n) : \ \mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)\} \tag{3.13}$$

and $\mathcal{B}^m[L^{p(\cdot)}(\mathbb{R}^n)] = W^{m,p(\cdot)}(\mathbb{R}^n)$ for any integer $m \in \mathbb{N}_0$, where $W^{m,p(\cdot)}(\mathbb{R}^n)$ is the Sobolev space with the variable exponent $p(x)$.

Statement (3.13) has the following generalization, see [73], Theorem 4.10.

Theorem 3.8. *Let $\mathrm{Y} = \mathrm{Y}(\mathbb{R}^n)$ be a Banach function space, satisfying the assumptions*

i) *C_0^∞ is dense in* Y*;*
ii) *the maximal operator $\mathcal{M}$ is bounded in* Y*;*
iii) *$I^\alpha f(x)$ converges absolutely for almost all x for every $f \in \mathrm{Y}$ and $(1 + |x|)^{-n-\alpha} I^\alpha f(x) \in L^1(\mathbb{R}^n)$.*

Then

$$\mathcal{B}^\alpha(\mathrm{Y}) = \mathrm{Y} \bigcap I^\alpha(\mathrm{Y}) = \{f \in \mathrm{Y} : \mathbb{D}^\alpha f = \lim_{\substack{\varepsilon \to 0 \\ (\mathrm{Y})}} \mathbb{D}^\alpha_\varepsilon f \in \mathrm{Y}\}. \tag{3.14}$$

From Theorem 3.8 there follows, in particular, the characterization of the ranges of potential operators over weighted Lebesgue spaces with variable exponent obtained by means of results of Subsection 3.2 for the maximal operator.

Certain results related to imbedding of the range of the Riesz potential operator into Hölder spaces (of variable order) in the case $p(x) \geq n$ were obtained in [7]. The results proved in [7] run as follows. In Theorem 3.9 we use the notation $\Pi_{p,\Omega} := \{x \in \Omega : p(x) > n\}$ and $H^{\alpha(\cdot)}(\Omega)$ in Theorem 3.15 is the Hölder-type space with a finite seminorm $[f]_{\alpha(\cdot),\Omega} := \sup_{x,x+h\in\Omega, 0<|h|\leq 1} |h|^{-\alpha(x)}|f(x) - f(x+h)|$.

Theorem 3.9. *Let Ω be a bounded open set with Lipschitz boundary, let $p \in WL(\Omega), p_+(\Omega) < \infty$ and let the set $\Pi_{p,\Omega}$ be non-empty. If $f \in W^{1,p(\cdot)}(\Omega)$, then*

$$|f(x) - f(y)| \leq C(x,y)\, \||\nabla f|\|_{p(\cdot),\Omega}\, |x-y|^{1-\frac{n}{\min[p(x),p(y)]}} \tag{3.15}$$

for all $x, y \in \Pi_{p,\Omega}$ such that $|x-y| \leq 1$, where $C(x,y) = \frac{c}{\min[p(x),p(y)]-n}$ with $c > 0$ not depending on f, x and y.

Theorem 3.10. *Let Ω be a bounded open set with Lipschitz boundary, $p \in WL(\Omega)$ and $p_+(\Omega) < \infty$. If $\inf\limits_{x\in\Omega} p(x) > n$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow H^{1-\frac{n}{p(\cdot)}}(\Omega)$.*

Theorem 3.10 is an improved version of the result earlier obtained in [27], [30]. The papers [32], [33] are also relevant to the topic. We refer also to [43] where the capacity approach was used to get embeddings into the space of continuous functions or into $L^\infty(\Omega)$. In [7] there were also obtained $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}$-estimates of hypersingular integrals (fractional differentiation operators)

$$\mathcal{D}^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x) - f(y)}{|x-y|^{n+\alpha(x)}}\, dy, \quad x \in \Omega. \tag{3.16}$$

We dwell briefly also on extensions to the case of Hajłasz-Sobolev spaces on quasimetric measure spaces. In [4], by means of the estimate

$$|f(x) - f(y)| \leq C(\mu, \alpha, \beta) \left[d(x,y)^{\alpha(x)} \mathcal{M}^\sharp_{\alpha(\cdot)} f(x) + d(x,y)^{\beta(y)} \mathcal{M}^\sharp_{\beta(\cdot)} f(y) \right]$$

where $\mathcal{M}^\sharp_{\alpha(\cdot)} f(x) = \sup\limits_{r>0} \frac{r^{-\alpha(x)}}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, d\mu(y)$, generalizing an estimate from [41], there was given an extension of (3.15) with n replaced by the exponent from the growth condition and ∇f replaced by the generalized gradient of f. This led to the following result for the Hajłasz-Sobolev space $M^{1,p(\cdot)}(X)$.

Theorem 3.11. *Let the set X be bounded and the measure μ be doubling. If $p(\cdot)$ is* log*-Hölder continuous and $p_- > N$, then $M^{1,p(\cdot)}(X) \hookrightarrow H^{1-\frac{N}{p(\cdot)}}(X)$.*

We refer also to [5] with regards to Sobolev-type estimations with variable $p(\cdot)$ of potentials $\mathfrak{I}^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ on metric measure spaces.

4. Maximal and potential operators in variable exponent Morrey spaces

In this section we present results obtained for maximal and potential operators in variable exponent Morrey spaces.

$$I^{\alpha(\cdot)}f(x) = \int_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}$$

of variable order $\alpha(x)$. We prove the boundedness of the maximal operator in Morrey spaces under the log-condition on $p(\cdot)$. For potential operators, under the same log-condition and the assumptions $\inf_{x\in\Omega}\alpha(x) > 0$, $\sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)] < n$, we present a Sobolev type $L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}$-theorem. In the case of constant α, we also give a result on the boundedness theorem in the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha}$, when the potential operator I^α acts from $L^{p(\cdot),\lambda(\cdot)}$ into BMO.

Let $p(\cdot)$ and $\lambda(\cdot)$ be measurable functions on $\Omega \subseteq \mathbb{R}^n$ with values in $[0,n]$. We define the variable Morrey space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ by condition (1.4). Equipped with the norm

$$\begin{aligned}\|f\| &= \inf\left\{\eta > 0 : \sup_{x\in\Omega,\ r>0} r^{-\lambda(x)} \int\limits_{B(x,r)\cap\Omega} \left(\frac{|f(y)|}{\eta}\right)^{p(y)} dy \le 1\right\}\\ &= \sup_{x\in\Omega,\ r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)},\end{aligned}$$

this is a Banach space. In the case where $|\Omega| < \infty$ and $\lambda(\cdot)$ is log-continuous, this norm is equivalent to $\sup_{x\in\Omega,\,r>0}\left\| r^{-\frac{\lambda(\cdot)}{p(\cdot)}} f\,\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)}$. There holds the embedding $L^{q(\cdot),\mu(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(\Omega)$, when $\frac{n-\lambda(x)}{p(x)} \ge \frac{n-\mu(x)}{q(x)}$.

In the sequel we suppose that $0 \le \lambda(x) \le \lambda_+ < n, \quad x \in \Omega$. For constant exponents $p(x) \equiv p$ and $\lambda(x) \equiv \lambda$ the following two theorems were proved in [10], [1], respectively.

Theorem 4.1. *Let Ω be bounded and $p \in \mathbb{P}(\Omega)\cap WL(\Omega)$. Then the maximal operator M is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

Theorem 4.2. *Let Ω be bounded, $p \in \mathbb{P}(\Omega) \cap WL(\Omega)$ and $\alpha \in WL(\Omega)$. Under the conditions $\inf\limits_{x\in\Omega}\alpha(x) > 0$, $\quad \sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)] < n$, the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\lambda(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}$.*

For the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha(x)}$, we have the following statement, proved in [71], Theorem 5.4, for constant exponents.

Theorem 4.3. *Let $0 < \alpha < n$, $\lambda(x) \ge 0$, $\sup_{x\in\Omega}\lambda(x) < n-\alpha$, $\lambda \in WL(\Omega)$ and let $p(x) = \frac{n-\lambda(x)}{\alpha}$. Then the operator I^α is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $BMO(\Omega)$.*

Theorem 4.3 is derived – via the pointwise estimate $M^\sharp(I^\alpha f)(x) \le c\,M^\alpha f(x)$, ([1], Proposition 3.3) – from the following statement.

Theorem 4.4. *Let Ω be bounded, $p \in \mathbb{P}(\Omega) \cap WL(\Omega)$ and $\inf_{x\in\Omega} \alpha(x) > 0$. In the case $p(x) = \frac{n-\lambda(x)}{\alpha}$, the fractional maximal operator*

$$M^{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy$$

is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^\infty(\Omega)$.

5. Fractional integrals and hypersingular integrals in variable order Hölder spaces on homogeneous spaces

The results we present here are new, they were obtained in [77]. For a version of such results when X is a sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, we refer to [79]. We consider the mapping properties of potential type operators in Hölder spaces $H^{\lambda(\cdot)}$ of variable order, which in case of domains Ω in $\mathbb{R}^n$ are defined by the condition $\sup_{|h|<t} |f(x+h)-f(x)| \leq Ct^{\lambda(x)}, \ x \in \Omega$. It is done in the general setting of quasimetric measure spaces (X, d, μ) which satisfy the growth condition

$$\mu B(x,r) \leq K r^N \quad \text{as} \quad r \to 0, \quad K > 0, \tag{5.1}$$

where $N > 0$ need not be an integer, for the potentials of form (1.3), where we admit variable exponent $\alpha(x)$, $0 \leq \alpha(x) < 1$, and Ω is an open bounded set in a quasimetric measure space X. We will also study the corresponding hypersingular operators

$$(D^\alpha f)(x) = \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:\varrho(x,y)>\varepsilon} \frac{f(y)-f(x)}{\varrho(x,y)^{N+\alpha(x)}} d\mu(y), \quad x \in \Omega, \tag{5.2}$$

within the frameworks of the Hölder spaces $H^{\lambda(\cdot)}(\Omega)$ with a variable exponent. In the case of constant α such a study in the general setting of quasimetric measure spaces (X, ϱ, μ) with growth condition, is known, see [36], [37], [38], [39].

The estimate we present here reveal the mapping properties of the operators I^α and D^α in dependence of local values of $\alpha(x)$ and $\lambda(x)$. Note that estimations with variable $\lambda(x)$ and $\alpha(x)$ were known in the special case $X = \mathbb{S}^{n-1}$ for spherical potential operators and related hypersingular integrals, and even in a more general setting of generalized Hölder spaces defined by a given (variable) dominant $w(x,h)$ of continuity modulus, see, for instance, [86], [87], [89].

The estimates we present here are related to a general quasimetric measure spaces and admit the situation when $\alpha(x)$ may be degenerate on Ω (on a set of measure zero). We denote

$$\Pi_\alpha = \{x \in \Omega : \alpha(x) = 0\}$$

and suppose that $\mu(\Pi_\alpha) = 0$.

To obtain results stating that the range of the potential operator over this or that Hölder space is imbedded into a better space of a similar nature, we prove Zygmund type estimates for the continuity modulus. In the case we study, these estimates are local, depending on points x. By means of such Zygmund type estimates of such a kind, we prove theorems on the mapping properties $I^{\alpha(\cdot)}$: $H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$, and similar results for the operator $D^{\alpha(\cdot)}$, $0 < \alpha(x) < 1$.

Let (X, d, μ) be a homogeneous quasimetric measure space. As shown in [64], it admits an equivalent quasimetric d_1 for which there exists an exponent $\theta \in (0, 1]$ such that the property

$$|d_1(x,z) - d_1(y,z)| \le M d_1^\theta(x,y) \left\{d_1(x,z) + d_1(y,z)\right\}^{1-\theta} \tag{5.3}$$

holds. When d is a quasimetric, then d automatically satisfies (5.3) with $\theta = 1$ and $M = 1$. For brevity, we will say that the quasimetric d is *regular of order* $\theta \in (0, 1]$, if it satisfies property (5.3).

In the sequel we suppose that all the balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are measurable and $\mu S(x, r) = 0$ for all the spheres $S(x, r) = \{y \in X : d(x, y) = r\}$, $\quad x \in X \quad r \ge 0$.

For fixed $x \in \Omega$ we consider the local continuity modulus

$$\omega(f, x, h) = \sup_{\substack{z\in\Omega:\\ d(x,z)\le h}} |f(x) - f(z)| \tag{5.4}$$

of a function f at the point x. Everywhere below we assume that $|h| < 1$. The function $\omega(f, x, h)$ is non-decreasing in h and tends to zero as $h \to +0$ for any continuous function on Ω and fixed x.

Lemma 5.1. *For all $x, y \in \Omega$ such that $d(x, y) \le h$, the inequality*

$$\frac{1}{C}\omega(f, x, h) \le \omega(f, y, h) \le C\omega(x, y, h) \tag{5.5}$$

holds, where $C = [2k] + 2$ and k is the constant from the triangle university. If $a(x) \in WL(\Omega)$, then

$$\frac{1}{C}h^{a(x)} \le h^{a(y)} \le Ch^{a(x)} \tag{5.6}$$

for all x, y such that $d(x, y) < h$, where $C \ge 1$ depends on the function a, but does not depend on x, y and h.

For a function $\lambda(x)$ defined on Ω we suppose that

$$\lambda_- := \inf_{x\in X} \lambda(x) > 0 \qquad \text{and} \quad \lambda_+ := \sup_{x\in X} \lambda(x) < 1.$$

Definition 5.2. *By $H^{\lambda(\cdot)}(\Omega)$ we denote the space of functions $f \in C(\Omega)$ such that $\omega(f, x, h) \le Ch^{\lambda(x)}$, where $C > 0$ does not depend on $x, y \in \Omega$. Equipped with the norm*

$$\|f\|_{H^{\lambda(\cdot)}(\Omega)} = \|f\|_{C(\Omega)} + \sup_{x\in\Omega} \sup_{h\in(0,1)} \frac{\omega(f, x, h)}{h^{\lambda(x)}},$$

this is a Banach space.

In Hölder norm estimations of functions $I^\alpha f$, the case $f \equiv \text{const}$ plays an important role, in the case where

$$\mathfrak{I}_\alpha(x) := I^\alpha(1)(x) = \int\limits_\Omega \frac{d\mu(z)}{d(x,z)^{N-\alpha(x)}} \tag{5.7}$$

is well defined. Observe that in the Euclidean case $\Omega = X = \mathbb{R}^N$, this integral although not well directly defined, may be treated as a constant in the case $\alpha(x) = \alpha = \text{const}$ in the sense that the cancellation property

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|z-x|^{N-\alpha}} - \frac{1}{|z-y|^{N-\alpha}} \right] dz \equiv 0, \quad 0 < \alpha < 1, \quad x, y \in \mathbb{R}^N$$

holds. For constant α, the function $\mathfrak{I}_\alpha(x)$ is also constant in the case $\Omega = X = \mathbb{S}^{N-1}$, which fails when $\alpha = \alpha(x)$ and the cancellation property of the type

$$\int\limits_\Omega \left[\frac{1}{|z-x|^{N-\alpha(x)}} - \frac{1}{|z-y|^{N-\alpha(y)}} \right] d\mu(z) \equiv 0,$$

no more holds even for $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{S}^{N-1}$; see, for instance, [36] on the importance of the cancellation property $I^\alpha(1) \equiv \text{const}$ for the validity of mapping properties of potentials within Hölder spaces on quasimetric measure spaces. When we consider Hölder type spaces $H^{\lambda(\cdot)}(\Omega)$ which contain constants, the condition

$$\mathfrak{I}_\alpha(1) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

is necessary for the mapping $I^\alpha : H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$ to hold.

Remark 5.3. Let $\inf\limits_{x\in\Omega} \alpha(x) \geq 0$ and $x, y \notin \Pi_\alpha$. Then

$$|\mathfrak{I}_\alpha(x) - \mathfrak{I}_\alpha(y)| \leq C \frac{|\alpha(x)-\alpha(y)|}{\min(\alpha(x),\alpha(y))} + \left| \int\limits_\Omega \left[d(x,z)^{\alpha(x)-N} - d(y,z)^{\alpha(x)-N} \right] d\mu(z) \right| \tag{5.8}$$

and

$$\begin{aligned} &|\alpha(x)\mathfrak{I}_\alpha(x) - \alpha(y)\mathfrak{I}_\alpha(y)| \qquad (5.9) \\ &\quad \leq C\,|\alpha(x)-\alpha(y)| + \min(\alpha(x),\alpha(y)) \left| \int\limits_\Omega \left[d(x,z)^{\alpha(x)-N} - d(y,z)^{\alpha(x)-N} \right] d\mu(z) \right| \end{aligned}$$

where $C > 0$ does not depend on $x, y \in \Omega$.

Remark 5.4. The meaning of estimates (5.8)–(5.9) is in the fact that the second term on the right-hand sides may be subject to the cancellation property: at the least it disappears when $\Omega = X = \mathbb{R}^N$ or $\Omega = X = \mathbb{S}^{N-1}$.

The estimate (5.10) provided by the following theorem clearly shows the worsening of the behaviour of the local continuity modulus $\omega(I^\alpha f, x, h)$ when x approaches the points where $\alpha(x)$ vanishes. We also give a weighted estimate exactly with the weight $\alpha(x)$.

We use the notation

$$\alpha_h(x) = \min_{d(x,y)<h} \alpha(y).$$

Theorem 5.5. *Let Ω be a bounded open set in X, let $\alpha \in WL(\Omega)$ and $0 \le \inf_{x\in\Omega} \alpha(x) \le \sup_{x\in\Omega} \alpha(x) < \min(1, N)$. Then for all the points $x \in \Omega\backslash\Pi_\alpha$ such that $\alpha_h(x) \neq 0, 0 < h < \frac{d}{2}$, the following Zygmund type estimate is valid*

$$\omega(I^\alpha f, x, h) \le \frac{C}{\alpha_h(x)} h^{\alpha(x)} \omega(f, x, h) + Ch^\theta \int_h^d \frac{\omega(f, x, t)dt}{t^{1+\theta-\alpha(x)}} \tag{5.10}$$

$$+ C\omega(\alpha, x, h) \int_h^d \frac{\omega(f, x, t)dt}{t^{2-\alpha(x)}} + C\omega(\Im_\alpha, x, h)\|f\|_{C(\Omega)},$$

where the constant $C > 0$ does not depend on f, x and h.

Also, for all the points $x \in \Omega\backslash\Pi_\alpha$ the weighted estimate holds

$$\omega(\alpha I^\alpha f, x, h) \le Ch^{\alpha(x)} \omega(f, x, h) + Ch^\theta \int_h^d \frac{\omega(f, x, t)dt}{t^{1+\theta-\alpha(x)}} \tag{5.11}$$

$$+ C\omega(\alpha, x, h) \int_h^d \frac{\omega(f, x, t)dt}{t^{2-\alpha(x)}} + C\omega(\alpha\Im_\alpha, x, h)\|f\|_{C(\Omega)},$$

5.1. Zygmund type estimates of hypersingular integrals

Theorem 5.6. *Let $\alpha \in C(\Omega), \alpha \in WL(\Omega)$ and $0 \le \inf_{x\in\Omega} \alpha(x) \le \max_{x\in\Omega} \alpha(x) < 1$. If $f \in C(\Omega)$, then for all $x, y \in \Omega$ with $d(x, y) < h$ such that $\alpha(x) \neq 0$ and $\alpha(y) \neq 0$, the following estimate is valid*

$$|(D^\alpha f)(x) - (D^\alpha f)(y)| \le \frac{C}{\min(\alpha(x), \alpha(y))} \int_0^h \left[\frac{\omega(f, x, t)}{t^{1+\alpha(x)}} + \frac{\omega(f, y, t)}{t^{1+\alpha(y)}}\right] dt \tag{5.12}$$

$$+ C \int_h^2 \left[\omega(\alpha, x, h) + h^\theta t^{1-\theta}\right] \frac{\omega(f, x, t)dt}{t^{2+\alpha(x)}},$$

where $C > 0$ does not depend on x, y and h.

5.2. Theorems on mapping properties

Recall that for the potential operator $I^{\alpha(\cdot)}$ we allow the variable order $\alpha(x)$ to be degenerate on a set Π_α (of measure zero). We consider the weighted space

$$H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha) = \{f : \alpha(x)f(x) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega\}.$$

Theorem 5.7. *Let* $\alpha(x) \geq 0, \quad \max_{x\in\Omega} \alpha(x) < \min(\theta, N), \quad \alpha(x) \in \mathrm{Lip}(\Omega),$ *and*

$$\sup_{x\in\Omega}[\lambda(x) + \alpha(x)] < \theta. \tag{5.13}$$

If

$$\alpha \mathfrak{I}_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}, \tag{5.14}$$

then the operator $I^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the weighted space* $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha)$*.*

A "non-degeneracy" version of Theorem 5.7 obtained from (5.10), runs as follows.

Theorem 5.8. *Let*

$$0 < \min_{x\in\Omega} \alpha(x) \leq \max_{x\in\Omega} \alpha(x) < \min(\theta, N) \quad \text{and} \quad \alpha \in \mathrm{Lip}(\Omega). \tag{5.15}$$

Under conditions (5.13) *and* (5.14)*, the operator* $I^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the space* $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$*.*

The corresponding mapping theorem for the hypersingular operator runs as follows.

Theorem 5.9. *Under conditions* (5.14)*,* (5.15)*, the operator* $D^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the space* $H^{\lambda(\cdot)-\alpha(\cdot)}(\Omega)$*, if*

$$0 < \inf_{x\in\Omega}\{\lambda(x) - \alpha(x)\}, \quad \sup_{x\in\Omega} \lambda(x) < 1.$$

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References

[1] D.R. Adams. A note on Riesz potentials. *Duke Math. J.*, 42.

[2] A. Almeida. Inversion of the Riesz Potential Operator on Lebesgue Spaces with Variable Exponent. *Frac. Calc. Appl. Anal.*, 6(3):311–327, 2003.

[3] A. Almeida, J. Hasanov and S. Samko. Maximal and potential operators in variable exponent Morrey spaces. *Georgian Math. J.*, 15(2):195–208, 2008.

[4] A. Almeida and S. Samko. Embeddings of variable Hajlasz-Sobolev spaces into Hölder spaces of variable order. *Manuscript*.

[5] A. Almeida and S. Samko. Fractional and hypersingular operators in variable exponent spaces on metric measure spaces. *Meditter. J. Math.*, DOI 10.1007/s00009-003-0000.

[6] A. Almeida and S. Samko. Characterization of Riesz and Bessel potentials on variable Lebesgue spaces. *J. Function Spaces and Applic.*, 4(2):113–144, 2006.

[7] A. Almeida and S. Samko. Pointwise inequalities in variable Sobolev spaces and applications. *Zeit. Anal. und ihre Anwend.*, 26(2):179–193, 2007.

[8] A.-P. Calderón. Inequalities for the maximal function relative to a metric. *Studia Math.*, 57(3):297–306, 1976.

[9] Y. Chen, S. Levine, and Rao M. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66:1383–1406, 2006.

[10] F. Chiarenza and M. Frasca. Morrey spaces and Hardy-Littlewood maximal function. *Rend. Math.*, 7:273–279, 1987.

[11] R.R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certaines espaces homogènes*, volume 242. Lecture Notes Math., 1971. 160 pages.

[12] D. Cruz-Uribe, A. Fiorenza, J.M. Martell, and C Perez. The boundedness of classical operators on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 31(1):239–264, 2006.

[13] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer. The maximal function on variable L^p-spaces. *Ann. Acad. Scient. Fennicae, Math.*, 28:223–238, 2003.

[14] D. Cruz-Uribe, A. Fiorenza, and C.J. Neugebauer. Corrections to "The maximal function on variable L^p-spaces". *Ann. Acad. Scient. Fennicae, Math.*, 29:247–249, 2004.

[15] D. Cruz-Uribe, J.M. Martell, and C. Pérez. Extrapolation from A_∞ weights and applications. *J. Funct. Anal.*, 213(2):412–439, 2004.

[16] D. Cruz-Uribe, J.M. Martell, and C. Pérez. Extensions of Rubio de Francia's extrapolation theorem. *Collect. Math.*, (Vol. Extra):195–231, 2006.

[17] G.P. Curbera, J. García-Cuerva, J.M. Martell, and C. Pérez. Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals. *Adv. Math.*, 203(1):256–318, 2006.

[18] L. Diening. Theoretical and numerical results for electrorheological fluids. Ph.D Thesis, University of Freiburg, Germany, 2002.

[19] L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.

[20] L. Diening. Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Mathem. Nachrichten*, 268:31–43, 2004.

[21] L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.

[22] L. Diening, P. Hästö, and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May* 28*–June* 2*,* 2004. Math. Inst. Acad. Sci. Czech Republick, Praha.

[23] L. Diening and S. Samko. Hardy inequality in variable exponent Lebesgue spaces. Albert-Ludwigs-Universität Freiburg, 2006. Preprint Nr. 1/2006-17.03.2006, 15p.

[24] D.E. Edmunds. Sobolev embeddings with variable exponent, II. *Math. Nachr.*, 246/247:53–67, 2002.

[25] D.E. Edmunds, V. Kokilashvili, and A. Meskhi. Two-weight estimates for singular integrals defined on spaces of homogeneous type. *Canad. J. Math.*, 52(3):468–502, 2000.

[26] D.E. Edmunds, J. Lang, and A. Nekvinda. On $L^{p(x)}$ norms. *Proc. R. Soc. Lond.*, A 455:219–225, 1999.

[27] D.E. Edmunds and J. Rákosník. Sobolev embeddings with variable exponent. *Studia Math.*, 143(3):267–293, 2000.

[28] X. Fan, J. Shen, and D. Zhao. Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 262(2):749–760, 2001.

[29] X. Fan and D. Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 263(2):424–446, 2001.

[30] A. Fiorenza. A mean continuity type result for certain Sobolev spaces with variable exponent. *Commun. Contemp. Math.*, 4(3):587–605, 2002.

[31] T. Futamura, P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura. Variable exponent spaces on metric measure spaces. *Proceedings of the* 5*th ISAAC Congress, Catania,* 2005. To appear; available at
http://www.helsinki.fi/∼pharjule/varsob/publications.shtml.

[32] T. Futamura and Y. Mizuta. Continuity of weakly monotone Sobolev functions of variable exponent. *Adv. Math. Sci. Appl., Maruzen Co. Ltd*, 15:571–585, 2005.

[33] T. Futamura and Y. Mizuta. Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent. *Math. Inequal. Appl*, 8(1):619–631, 2005.

[34] T. Futamura, Y. Mizuta, and T. Shimomura. Sobolev embeddings for Riesz potential space of variable exponent. *Math. Nachr.*, 31(279):495–522, 2006.

[35] T. Futamura, Y. Mizuta, and T. Shimomura. Sobolev embeddings for variable exponent Riesz potentials on metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 31(2):495–522, 2006.

[36] A.E. Gatto. On fractional calculus associated to doubling and non-doubling measures. In J. Marshall (ed.) et al. Ash, editor, *Harmonic analysis. Calderon-Zygmund and beyond. A conference in honor of Stephen Vagi's retirement, Chicago, IL, USA, December* 6–8*,* 2002, volume 411, pages 15–37. Providence, RI: American Mathematical Society, 2006. Contemporary Mathematics.

[37] A.E. Gatto and J. Garcia-Cuerva. Boundedness properties of fractional integral operators associated to non-doubling measures. *Studia Math.*, 162(3):245–261, 2004.

[38] A.E. Gatto, C. Segovia, and S. Vagi. On fractional differentiation on spaces of homogeneous type. *Revista Mat. Iberoamer*, 12(1), 1996. 1–35.

[39] A.E. Gatto and S. Vagi. Fractional integrals on spaces of homogeneous type. In Cora Sadosky (ed.), editor, *Analysis and Partial Differential Equations*, volume 122, pages 171–216. Marcel Dekker, New York and Basel, 1990. Lecture Notes in Pure and Appl. Math.

[40] P. Hästö. Local-to-global results in variable exponent spaces. *Math. Res. Letters. To appear.*

[41] P. Hajłasz and J. Kinnunen. Hölder quasicontinuity of Sobolev functions on metric spaces. *Rev. Mat. Iberoamericana*, 14(3):601–622, 1998.

[42] P. Hajłasz and P. Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[43] P. Harjulehto and P. Hästö. A capacity approach to the Poincaré inequality in the Sobolev imbeddings in variable exponent Sobolev spaces. *Rev. Mat. Complut.*, 17(1):129–146, 2004.

[44] P. Harjulehto, P. Hästö, and V. Latvala. Sobolev embeddings in metric measure spaces with variable dimension. *Math. Z.*, 254(3):591–609, 2006.

[45] P. Harjulehto, P. Hästö, and M. Pere. Variable Exponent Lebesgue Spaces on Metric Spaces: The Hardy-Littlewood Maximal Operator. *Real Anal. Exchange*, 30(1):87–104, 2004.

[46] P. Harjulehto, P. Hästö, and M. Pere. Variable exponent Sobolev spaces on metric measure spaces. *Funct. Approx. Comment. Math.*, 36:79–94, 2006.

[47] N.K. Karapetyants and A.I. Ginzburg. Fractional integrals and singular integrals in the Hölder classes of variable order. *Integral Transform. Spec. Funct.*, 2(2):91–106, 1994.

[48] N.K. Karapetyants and A.I. Ginzburg. Fractional integro-differentiation in Hölder classes of variable order. *Dokl. Akad. Nauk*, 339(4):439–441, 1994.

[49] M. Khabazi. The maximal operator in spaces of homogenous type. *Proc. A. Razmadze Math. Inst.*, 138:17–25, 2005.

[50] Ya.L. Kobelev, L.Ya. Kobelev, and Yu.L. Klimontovich. Statistical physics of dynamic systems with variable memory. *Doklady Physics*, 48(6):285–289, 2003.

[51] V. Kokilashvili. On a progress in the theory of integral operators in weighted Banach function spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May* 28*–June* 2*,* 2004. Math. Inst. Acad. Sci. Czech Republick, Praha.

[52] V. Kokilashvili and A. Meskhi. On some weighted inequalities for fractional integrals on nonhomogeneous spaces. *Zeitschr. Anal. Anwend.*, 24(4):871–885, 2005.

[53] V. Kokilashvili and A. Meskhi. Boundedness of maximal and singular operators in Morrey spaces with variable exponent. *Govern. College Univ., Lahore*, 72:1–11, 2008.

[54] V. Kokilashvili and A. Meskhi. Boundedness of maximal and singular operators in Morrey spaces with variable exponent. *Armenian J. Math.*, 1(1):18–28, 2008.

[55] V. Kokilashvili, V. Paatashvili, and Samko S. Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operators on Carleson curves. In Ya. Erusalimsky, I. Gohberg, S. Grudsky, V. Rabinovich, and N. Vasilevski, editors, *"Operator Theory: Advances and Applications", dedicated to* 70*th birthday of Prof. I.B. Simonenko*, volume 170, pages 167–186. Birkhäuser Verlag, Basel, 2006.

[56] V. Kokilashvili, N. Samko, and S. Samko. Singular operators in variable spaces $L^{p(\cdot)}(\omega, \rho)$ with oscillating weights. *Math. Nachrichten*, 280(9-10):1145–1156, 2007.

[57] V. Kokilashvili and S. Samko. Operators of harmonic analysis in weighted spaces with non-standard growth. *J. Math. Anal. Appl.*
To appear doi:10.1016/j.jmaa.2008.06.056

[58] V. Kokilashvili and S. Samko. Singular operators and Fourier multipliers in weighted Lebesgue spaces with variable exponents. *Vestnik Saint-Petersburg University, ser.* 1, issue 2, 56–68, Engl. Transl. in "Vestnik St. Petersburg University. Mathematics", 2008, Vol. 41, No. 2, pp. 134–144

[59] V. Kokilashvili and S. Samko. Boundedness in Lebesgue spaces with variable exponent of the cauchy singular operator. *Izvestija VUZov. Severo-Kavkazskii region. Estestvennie nauki, Special issue "Pseudodifferential equations and some problems of mathematical physics", dedicated to* 70*th birthday of Prof. I.B. Simonenko*, (Special Issue):152–158, 2005.

[60] V. Kokilashvili and S. Samko. The maximal operator in weighted variable spaces on metric measure spaces. *Proc. A. Razmadze Math. Inst.*, 144:137–144, 2007.

[61] V. Kokilashvili and S. Samko. Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent. *Acta Mathematica Sinica*, 24(11):1775–1800, 2008.

[62] O. Kovácǐk and J. Rákosnǐk. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116):592–618, 1991.

[63] A. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Zeit.*, 251(3):509–521, 2005.

[64] R.A. Macías and C. Segovia. Lipshitz functions on spaces of homogeneous type. *Adv. Math.*, 33:257–270, 1979.

[65] R.A. Macías and C. Segovia. A Well Behaved Quasidistance for Spaces of Homogeneous Type. *Trab. Mat. Inst. Argentina Mat.*, 32:1–18, 1981.

[66] Y. Mizuta and T. Shimomura. Continuity of Sobolev functions of variable exponent on metric spaces. *Proc. Japan Acad.*, 80, Ser. A:96–99, 2004.

[67] Y. Mizuta and T. Shimomura. Sobolev's inequality for Riesz potentials with variable exponent satisfying a log-Hölder condition at infinity. *J. Math. Anal. Appl.*, 311:268–288, 2005.

[68] J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[69] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Math. Inequal. and Appl.*, 7(2):255–265, 2004.

[70] A. Nekvinda. A note on maximal operator on ℓ^{p_n} and $L^{p(x)}(\mathbb{R}^1)$. *J. Funct. Spaces and Appl.*, 5:49–88, 2007.

[71] J. Peetre. On the theory of $\mathcal{L}_{p,\lambda}$ spaces. *Function. Analysis*, 4:71–87, 1969.

[72] H. Rafeiro and S. Samko. On multidimensional analogue of Marchaud formula for fractional Riesz-type derivatives in domains in R^n. *Fract. Calc. and Appl. Anal.*, 8(4):393–401, 2005.

[73] H. Rafeiro and S. Samko. Characterization of the range of one-dimensional fractional integration in the space with variable exponent. In *Operator Algebras, Operator Theory and Applications* (*WOAT Conference, Lisbon,* 2006), Operator Theory. Birkhäuser, Basel, 2007 (to appear).

[74] B. Ross and S.G. Samko. Fractional integration operator of variable order in the Hölder spaces $H^{\lambda(x)}$. *Internat. J. Math. Sci.*, 18(4):777–788, 1995.

[75] J.L. Rubio de Francia. Factorization theory and A_p weights. *Amer. J. Math.*, 106(3):533–547, 1984.

[76] M. Ružička. *Electrorheological Fluids: Modeling and Mathematical Theory.* Springer, Lecture Notes in Math., 2000, vol. 1748, 176 pages.

[77] N. Samko, S. Samko, and Vakulov B. Fractional integrals and hypersingular integrals in variable order Hölder spaces on homogeneous spaces (*manuscript*).

[78] N. Samko, S. Samko, and Vakulov B. Weighted Sobolev theorem in Lebesgue spaces with variable exponent. *J. Math. Anal. and Applic.*, 335:560–583, 2007.

[79] N. Samko, and Vakulov B. Spherical fractional and hypersingular integrals in generalized Hölder spaces with variable characteristic *Math. Nachrichten*, to appear.

[80] S. Samko. Convolution and potential type operators in $L^{p(x)}$. *Integr. Transf. and Special Funct.*, 7(3-4):261–284, 1998.

[81] S. Samko. *Hypersingular Integrals and their Applications.* London-New York: Taylor & Francis, Series *"Analytical Methods and Special Functions"*, vol. 5, 2002. 358 + xvii pages.

[82] S. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integr. Transf. and Spec. Funct*, 16(5-6):461–482, 2005.

[83] S.G. Samko, E. Shargorodsky, and B. Vakulov. Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, II. *J. Math, Anal. Appl.*, 325(1):745–751, 2007.

[84] S.G. Samko and B.G. Vakulov. Weighted Sobolev theorem with variable exponent. *J. Math, Anal. Appl.*, 310:229–246, 2005.

[85] I.I. Sharapudinov. The topology of the space $\mathcal{L}^{p(t)}([0, 1])$ (Russian). *Mat. Zametki*, 26(4):613–632, 1979.

[86] B.G. Vakulov. Spherical potentials of complex order in the variable order Hölder spaces. *Integral Transforms Spec. Funct.*, 16(5-6):489–497, 2005.

[87] B.G. Vakulov. Spherical convolution operators in Hölder spaces of variable order. *Matem. Zametki*, 80(5):683–695, 2006. Transl. in Math. Notes, 80:5, 2006, 645–657.

[88] B.G. Vakulov. Spherical potentials of complex order in the spaces with generalized variable Hölder condition (in Russian). *Dokl. Akad. Nauk*, 407(1):12–15, 2006. Transl. in Doklady Mathematics, 2006, 73(2): 165–168.

[89] B.G. Vakulov, N.K. Karapetiants, and L.D. Shankishvili. Spherical hypersingular operators of imaginary order and their multipliers. *Fract. Calc. Appl. Anal.*, 4(1):101–112, 2001.

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