Published in:
Revista Matematica Complutense,
Volume 24, Number 2 2011 , 357-373
DOI: 10.1007/s13163-010-0043-6

On potentials in generalized Hölder spaces over uniform domains in $\mathbb{R}^n$

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Abstract

We show that Riesz-type potential operators of order α over uniform domains Ω in $\mathbb{R}^n$ map the subspace $H_0^\lambda(\Omega)$ of functions in Hölder space $H^\lambda(\Omega)$ vanishing on $\partial\Omega$, into the space $H^{\lambda+\alpha}(\Omega)$, if $\lambda + \alpha \leq 1$. This is proved in a more general setting of generalized Hölder spaces with a given dominant of continuity modulus. Statements of such a kind are known for instance for the whole space $\mathbb{R}^n$ or more generally for metric measure spaces with cancellation property. In the case of domains in $\mathbb{R}^n$ when the cancellation property fails, our proofs are based on a special treatment of potential of a constant function.

Keywords: potential operators, Hölder space, generalized Hölder space, uniform domains

Mathematics Subject Classification (2000):Primary 46E15; Secondary 42B35

1 Introduction

Mapping properties of potential operators within the frameworks of Hölder spaces are well studied in the case of the whole space $\mathbb{R}^n$, see for instance [15], Theorem 25.5, in the case of spherical potentials, see [13], Theorems 6.37 and 6.38, for more general setting of generalized Hölder spaces on sphere we refer to [17], [18], [19], [20]. Such mapping properties are also known in the general setting of metric measure

[1]supported by Russian Federal Targeted Programme "Scientific and Research-Educational Personnel of Innovative Russia" for 2009-2013, project N 02.740.11.5024

spaces X, see [2], [3], [4], [5] under the assumptions that X satisfies the so called "cancellation" property. In cases where the potential of a constant function on X is well defined, this cancellation property means that the potential of a constant is constant. This property was also used in the recent paper [12], where there were admitted potentials of variable order $\alpha(x)$ with possible degeneration: $\alpha(x) = 0$ on a set of measure zero.

The cancellation property holds in the cases $X = \mathbb{R}^n$ and $X = \mathbb{S}^{n-1}$, but it is very restrictive in applications, never being valid for domains Ω in $\mathbb{R}^n$. As shown in [9], the Riesz potential of order $\alpha \in (2, n)$ of a constant on a convex bounded domain Ω is constant *even only on the boundary* $\partial\Omega$ if and only if Ω is a ball; the same being also valid for $\alpha = 2$ without the assumption on convexity, see [1]. To this we have to add that the potential of a constant on a ball is constant on the boundary, but is not constant in the ball, see Subsection 2.1.

In general, statements of the type

$$I_\Omega^\alpha : H^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega)$$

for the potential operator

$$I_\Omega^\alpha f(x) := \int\limits_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha}}$$

may not be valid for domains, since potential of a constant behaves like $[\delta(x)]^\alpha$ near the boundary, where $\delta(x)$ is the distance to the boundary, and consequently is a Hölderian function only of order α. However, one should expect that there should be valid a statement

$$I_\Omega^\alpha : H_0^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega) \tag{1.1}$$

for the space $H_0^\lambda(\Omega)$ of functions vanishing at the boundary. Such mapping is known in the one-dimensional case and goes back to Hardy and Littlewood, see for instance [15], Corollary 1 on p. 56.

To our surprise, we did not find any multi-dimensional result of such a kind in the literature. The goal of this paper is to fill in the gap. We show that a mapping of type (1.1) (and more generally, for spaces of the type $H^\omega(\Omega)$) holds at the least for the so called uniform domains (known also as satisfying the banana condition). The obtained results are based on the properties, near the boundary $\partial\Omega$ of Ω, of the potential of a constant function, that is,

$$J_{\Omega,\alpha}(x) = \int\limits_\Omega \frac{dy}{|x-y|^{n-\alpha}}, \qquad x \in \Omega. \tag{1.2}$$

To show a typical behaviour of $J_{\Omega,\alpha}(x)$ near the boundary, in Section 2 we start with the study of $J_{\Omega,\alpha}(x)$ for balls, half-space and a quarter-plane.

2 Potentials of constant functions for special domains

For a bounded, measurable set $\Omega \subset \mathbb{R}^n$ and $\alpha > 0$ we define the potential $J_{\Omega,\alpha}$ by

$$J_{\Omega,\alpha}(x) = \int_{\mathbb{R}^n} \frac{\chi_\Omega(y)}{|x-y|^{n-\alpha}}\,dy = \int_\Omega \frac{dy}{|x-y|^{n-\alpha}}, \qquad x \in \mathbb{R}^n. \tag{2.1}$$

Since Ω is bounded and $|\cdot|^{n-\alpha} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $\alpha > 0$, this is well defined.

Further, for a measurable set $\Omega \subset \mathbb{R}^n$ with $\alpha \in (0,1)$ (or $\alpha > 0$ if Ω is bounded) we define the *difference of the potential* by

$$J_{\Omega,\alpha}(x,y) := \int_\Omega \left(\frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right) dz, \quad x \in \mathbb{R}^n. \tag{2.2}$$

If Ω is bounded, then $J_{\Omega,\alpha}(x,y) = J_{\Omega,\alpha}(x) - J_{\Omega,\alpha}(y)$. However, if Ω is not bounded, then $J_{\Omega,\alpha}(x)$ may be not well defined. Nevertheless, $J_{\Omega,\alpha}(x,y)$ is well defined (as an L^1-integral) also for unbounded Ω, since $z \mapsto \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}}$ is in $L^1(\mathbb{R}^n)$ for $\alpha \in (0,1)$ due to the estimate

$$\left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| \le c\, \frac{|x-y|}{|x-z|^{n+1-\alpha}} \qquad \text{for } |x-y| \le \frac{1}{2}|x-z|. \tag{2.3}$$

If $\alpha \in (0,1)$, then $J_{\Omega,\alpha}(x,x) = 0$, $J_{\Omega,\alpha}(x,y) = -J_{\Omega,\alpha}(y,x)$, $J_{\mathbb{R}^n,\alpha}(x,y) = 0$, and $J_{\Omega,\alpha}(x,y) = -J_{\mathbb{R}^n\setminus\Omega,\alpha}(x,y)$, where the first two equalities also hold for $\alpha > 0$ if Ω is bounded. Moreover, $J_{\Omega,\alpha}(x,y) = J_{\Omega,\alpha}(x,a) + J_{\Omega,\alpha}(a,y)$ for all $a, x, y \in \mathbb{R}^n$.

If Ω is bounded and $\alpha > 0$, then $J_{\Omega,\alpha}(x)$ is continuous with respect to $x \in \mathbb{R}^n$ and in general $J_{\Omega,\alpha}(x,y)$ is continuous with respect to $x, y \in \mathbb{R}^n$ when $0 < \alpha < 1$, which may be easily shown by splitting $J_{\Omega,\alpha}(x+h) - J_{\Omega,\alpha}(x)$ into the integrals $\int_{\Omega\setminus B(x,r)}$ with small $r > 0$ and $\int_{\Omega\cap B(x,r)}$. As a consequence $y \mapsto J_{\Omega,\alpha}(x,y)$ is continuous at $y = x$. A simple modification of the argument shows that this continuity also holds for Ω unbounded if $\alpha \in (0,1)$. Using $J_{\Omega,\alpha}(x, y+h) - J_{\Omega,\alpha}(x,y) = J_{\Omega,\alpha}(y, y+h)$ and the antisymmetry $J_{\Omega,\alpha}(x,y) = -J_{\Omega,\alpha}(y,x)$ it follows easily that $J_{\Omega,\alpha}(x,y)$ is continuous with respect to $x, y \in \mathbb{R}^n$. However, $J_{\Omega,\alpha}(x,y)$ is better than just continuous, see Lemma 3.1 in Section 3.

2.1 The case of balls

In the lemma below, we give an explicit formula for $J_{\Omega,\alpha}$ in the case of the ball $\Omega = B(0,a) := \{y : |y| < a\}$ in terms of the Gauss hypergeometric function F, by means of which we show that

$$J_{B(0,a),\alpha}(x) = c_0 + c_1(a - |x|)^\alpha + O\left((a-|x|)^{\alpha+1}\right) \qquad \text{for } x \in B(0,a) \tag{2.4}$$

near the boundary $\partial B(0,a)$.

Lemma 2.1. *Let $0 < \alpha < n$. Then*

$$J_{B(0,a),\alpha}(x) = \frac{2\pi^{\frac{n}{2}} a^\alpha}{\alpha \Gamma\left(\frac{n}{2}\right)} F\left(-\frac{\alpha}{2}, \frac{n-\alpha}{2}; \frac{n}{2}; \frac{|x|^2}{a^2}\right) \qquad \text{for } x \in B(0,a), \tag{2.5}$$

and (2.4) *holds with* $c_0 = \frac{2^{\alpha-1}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+\alpha}{2}\right)}$ *and* $c_1 = \frac{2^{\alpha-1}\Gamma\left(\frac{n}{2}\right)}{a^\alpha\sqrt{\pi}\Gamma\left(\frac{n-\alpha}{2}\right)}$. *Moreover,*

$$J_{B(0,a),\alpha}(x) = c_1(a-|x|)^\alpha + g(x) \qquad \text{for } x \in B(0,a) \tag{2.6}$$

where $g \in \mathrm{Lip}(\overline{B(0,a)})$.

Proof. There is known a general formula for the calculation of potentials over balls for radial functions ([10]; it may be found in [15], p. 589), which gives

$$J_{B(0,a),\alpha}(x) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\left[r^{\alpha-n}\int_0^r t^{n-1} F\left(\frac{n-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{n}{2}; \frac{t^2}{r^2}\right) dt \right.$$
$$\left. + \int_r^a t^{\alpha-1} F\left(\frac{n-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{n}{2}; \frac{r^2}{t^2}\right) dt\right],$$

where $r = |x|$, but instead of calculating the integrals arising in this formula, it is easier to make use of another representation of the potential of radial functions, namely,

$$\frac{1}{\gamma_n(\alpha)}\int_{|y|<a} \frac{f(|y|)dy}{|x-y|^{n-\alpha}} = 2^{-\alpha} r^{2-n}\left(I_{0+}^{\frac{\alpha}{2}}\left[s^{\frac{n-\alpha}{2}-1}\left(I_{a^2-}^{\frac{\alpha}{2}} f^*\right)(s)\right]\right)(r^2), \tag{2.7}$$

where $I_{0+}^{\frac{\alpha}{2}}$ and $I_{a^2-}^{\frac{\alpha}{2}}$ are left-hand sided and right-hand sided Riemann-Liouville fractional integration operators of order $\frac{\alpha}{2}$ (see [15], formulas (2.17) and (2.18)) and $f^*(t) = f(\sqrt{t})$ ([10], see also [15], p. 590). Thus for $f(r) \equiv 1$ we have

$$J_{B(0,a),\alpha}(x) = \frac{\pi^{\frac{n}{2}} r^{2-n}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)} \int_0^{r^2} \frac{s^{\frac{n-\alpha}{2}-1}}{(r^2-s)^{1-\frac{\alpha}{2}}}\, ds \int_s^{a^2} \frac{dt}{(t-s)^{1-\frac{\alpha}{2}}}$$
$$= \frac{\pi^{\frac{n}{2}} r^{2-n}}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)} \int_0^{r^2} \frac{s^{\frac{n-\alpha}{2}-1}(a^2-s)^{\frac{\alpha}{2}}}{(r^2-s)^{1-\frac{\alpha}{2}}}\, ds.$$

With the change of variables $s = r^2\sigma$ we have

$$J_{B(0,a),\alpha}(x) = \frac{\pi^{\frac{n}{2}} a^\alpha}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)} \int_0^1 \sigma^{\frac{n-\alpha}{2}-1}(1-\sigma)^{\frac{\alpha}{2}-1}\left(1-\frac{r^2}{a^2}\sigma\right)^{\frac{\alpha}{2}} d\sigma,$$

which yields (2.5). To single out the behaviour of $J_{B(0,a),\alpha}(x)$ as $|x| \to a$, we make use of one of the transformation formulas for the Gauss function, see [6], formula 9.134.2, and obtain

$$J_{B(0,a),\alpha}(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}} \left\{ \frac{2^{\alpha}}{\Gamma\left(\frac{n+\alpha}{2}\right)} F\left(-\frac{\alpha}{2}, \frac{n-\alpha}{2}; 1-\alpha; 1-\frac{|x|^2}{a^2}\right) \right.$$
$$\left. + \frac{2^{-\alpha}}{\Gamma\left(\frac{n-\alpha}{2}\right)} \left(1-\frac{|x|^2}{a^2}\right)^{\alpha} F\left(\frac{\alpha}{2}, \frac{n+\alpha}{2}; 1+\alpha; 1-\frac{|x|^2}{a^2}\right) \right\}. \tag{2.8}$$

Hence

$$J_{B(0,a),\alpha}(x) = \frac{2^{\alpha-1}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}} \left\{ \frac{1}{\Gamma\left(\frac{n+\alpha}{2}\right)} + \frac{1}{\Gamma\left(\frac{n-\alpha}{2}\right)} \frac{(a^2-|x|^2)^{\alpha}}{(2a^2)^{\alpha}} \right\} + O\left((a-|x|)^{\alpha+1}\right)$$

which proves (2.4). Relation (2.8) also provides (2.6) by analyticity of the Gauss function. □

Observe that when α is variable, we can write

$$J_{B(0,a),\alpha}(x) = \frac{2\pi^{\frac{n}{2}} a^{\alpha(x)}}{\alpha(x)\Gamma\left(\frac{n}{2}\right)} F\left(-\frac{\alpha(x)}{2}, \frac{n-\alpha(x)}{2}; \frac{n}{2}; \frac{|x|^2}{a^2}\right), \quad 0 \le |x| \le a,$$

so that

$$J_{B(0,a),\alpha}(x) \sim \frac{|S^{n-1}|}{\alpha(x)} \quad \text{as} \quad x \to \Pi_{\alpha} := \{x \in \overline{B(0,a)} : \alpha(x) = 0\}.$$

2.2 The case of the half-space

Let $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_n > 0\}$ be the half space. In this case the potential $J_{\mathbb{R}^n_+,\alpha}$ is not well defined, since $\mathbb{R}^n_+$ is unbounded, but the difference $J_{\mathbb{R}^n_+,\alpha}(x,y)$, see (2.2) is well defined for $\alpha \in (0,1)$.

Lemma 2.2. *In the case of the half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_n > 0\}$ with $0 < \alpha < 1$ the formula*

$$J_{\mathbb{R}^n_+,\alpha}(x,y) = c_n(\alpha)(x_n^{\alpha} - y_n^{\alpha}) \qquad \textit{for all } x, y \in \mathbb{R}^n_+ \tag{2.9}$$

is valid, where $c_n(\alpha) = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{\alpha\Gamma\left(\frac{n-\alpha}{2}\right)}$.

Proof. Let $x = (x', x_n)$. We have

$$J_{\mathbb{R}^n_+,\alpha}(x,y) = \int_0^{\infty} [A(x,t) - A(y,t)] \, dt, \tag{2.10}$$

where

$$A(x,t) = \int\limits_{\mathbb{R}^{n-1}} \frac{d\xi}{(|x' - \xi|^2 + (x_n - t)^2)^{\frac{n-\alpha}{2}}}.$$

We have

$$A(x,t) = \int\limits_{\mathbb{R}^{n-1}} \frac{d\xi}{(|\xi|^2 + (x_n - t)^2)^{\frac{n-\alpha}{2}}} = C|x_n - t|^{\alpha-1},$$

where

$$C = \int\limits_{\mathbb{R}^{n-1}} \frac{d\xi}{(|\xi|^2 + 1)^{\frac{n-\alpha}{2}}} = |S_{n-2}| \int\limits_0^\infty \frac{r^{n-2}dr}{(r^2+1)^{\frac{n-\alpha}{2}}}.$$

The last integral converges for $\alpha < 1$ and is easily calculated:

$$\int\limits_0^\infty \frac{r^{n-2}dr}{(r^2+1)^{\frac{n-\alpha}{2}}} = \frac{1}{2}\int\limits_0^\infty \frac{s^{\frac{n-3}{2}}ds}{(s+1)^{\frac{n-\alpha}{2}}} = \frac{1}{2}\int\limits_0^1 t^{-\frac{1+\alpha}{2}}(1-t)^{\frac{n-3}{2}}\,dt = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}{2\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.11}$$

Hence

$$C = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}.$$

By (2.10) we obtain

$$J_{\mathbb{R}^n_+,\alpha}(x,y) = C\lim_{N\to\infty}\int\limits_0^N (|x_n - t|^\alpha - |y_n - t|^\alpha)dt$$

and an easy calculation yields (2.9). □

Remark 2.3. It follows from (2.9), $J_{\mathbb{R}^n_+,\alpha}(x,y) = J_{\mathbb{R}^n_+,\alpha}(x,0) + J_{\mathbb{R}^n_+,\alpha}(0,y)$ and $J_{\mathbb{R}^n_+,\alpha}(x,y) = -J_{\mathbb{R}^n_-,\alpha}(x,y) = -J_{\mathbb{R}^n_+,\alpha}(\hat{x},\hat{y})$, where $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : \ x_n < 0\}$ and $\hat{x} = (x_1, \ldots, x_{n-1}, -x_n)$ that

$$J_{\mathbb{R}^n_+,\alpha}(x,y) = c_n(\alpha)(\mathrm{sgn}(x_n)|x_n|^\alpha - \mathrm{sgn}(y_n)|y_n|^\alpha), \qquad 0 < \alpha < 1 \tag{2.12}$$

for all $x, y \in \mathbb{R}^n$.

2.3 The case of the quarter-plane

Let now $\Omega = \mathbb{R}^2_{++} = \{(x,y) \in \mathbb{R}^2_+ : x > 0, y > 0\}$ be the quarter-plane. Again $\mathbb{R}^2_{++}$ is unbounded, so we have to study the difference $J_{\mathbb{R}^2_{++},\alpha}(x,y)$ for $\alpha \in (0,1)$, see (2.2). We use the notation

$$\mathcal{G}_\alpha(x) = \mathcal{G}_\alpha(x_1, x_2) := \int_0^{x_1}\int_0^{x_2} \frac{d\xi d\eta}{(\xi^2+\eta^2)^{\frac{2-\alpha}{2}}} = \int_0^{x_1}\int_0^{x_2} \frac{d\xi d\eta}{|(\xi,\eta)|^{2-\alpha}},$$

$$\mathcal{H}_\alpha(t) := \mathcal{G}_\alpha((t,\infty)) = \mathcal{G}_\alpha((\infty,t)) = \int_0^{t}\int_0^{\infty} \frac{d\xi d\eta}{|(\xi,\eta)|^{2-\alpha}}.$$

Then

$$\mathcal{H}_\alpha(t) = t^\alpha \int_0^{1}\int_0^{\infty} \frac{d\xi d\eta}{(\xi^2+\eta^2)^{\frac{2-\alpha}{2}}} = t^\alpha \int_0^1 \xi^{\alpha-1} d\xi \int_0^\infty \frac{d\eta}{(1+\eta^2)^{\frac{2-\alpha}{2}}} = \frac{\sqrt{\pi}\Gamma\left(\frac{1-\alpha}{2}\right)}{2\alpha\Gamma\left(\frac{2-\alpha}{2}\right)} t^\alpha. \tag{2.13}$$

by (2.11).

Lemma 2.4. *Let $\Omega = \mathbb{R}^2_{++}$ and $0 < \alpha < 1$. The following representation holds*

$$J_{\mathbb{R}^2_{++},\alpha}(x,y) = \mathcal{G}_\alpha(x) - \mathcal{G}_\alpha(y) + \frac{\sqrt{\pi}\Gamma\left(\frac{1-\alpha}{2}\right)}{2\alpha\Gamma\left(\frac{2-\alpha}{2}\right)}(x_1^\alpha + x_2^\alpha - y_1^\alpha - y_2^\alpha) \tag{2.14}$$

for $x, y \in \mathbb{R}^2_{++}$.

Proof. We have for $x \in \mathbb{R}^2_{++}$

$$\begin{aligned}\mathcal{J}_{\mathbb{R}^2_{++},\alpha}(x,(0,0)) &= \int_{-x_1}^{0}\int_{-x_2}^{0} \frac{d\xi d\eta}{|(\xi,\eta)|^{2-\alpha}} + \int_{0}^{\infty}\int_{-x_2}^{0} \frac{d\xi d\eta}{|(\xi,\eta)|^{2-\alpha}} + \int_{-x_1}^{0}\int_{0}^{\infty} \frac{d\xi d\eta}{|(\xi,\eta)|^{2-\alpha}} \\ &= \mathcal{G}_\alpha(x) + \mathcal{H}_\alpha(x_1) + \mathcal{H}_\alpha(x_2).\end{aligned}$$

and as a consequence

$$\begin{aligned}\mathcal{J}_{\mathbb{R}^2_{++}}(x,y) &= \mathcal{J}_{\mathbb{R}^2_{++}}(x,(0,0)) + \mathcal{J}_{\mathbb{R}^2_{++}}((0,0),y) \\ &= \mathcal{G}_\alpha(x) - \mathcal{G}_\alpha(y) + \mathcal{H}_\alpha(x_1) + \mathcal{H}_\alpha(x_2) - \mathcal{H}_\alpha(y_1) - \mathcal{H}_\alpha(y_2).\end{aligned}$$

The claim follows with (2.13). □

To see the behaviour of $\mathcal{G}_\alpha(x)$ near the boundary, we will use an explicit expression for the $\mathcal{G}_\alpha(x, x_2)$ calculated in [14] in terms of the Gauss hypergeometric function. It has the form

$$\mathcal{G}_\alpha(x) = \frac{|x|^\alpha}{\alpha}\left[\frac{x_2}{x_1}F\left(\frac{1+\alpha}{2}, 1; \frac{3}{2}; -\frac{x_2^2}{x_1^2}\right) + \frac{x_1}{x_2}F\left(\frac{1+\alpha}{2}, 1; \frac{3}{2}; -\frac{x_1^2}{x_2^2}\right)\right], \tag{2.15}$$

valid for $0 < \alpha < 2$, see [14], formulas (4.60) and (4.63). Note that in the case $\alpha = 1$ we have ([14], formula (4.64))

$$\mathcal{G}_1(x) = |x|\left\{x_1 \ln\left(1 + \frac{|x|}{x_1}\right) + x_2 \ln\left(1 + \frac{|x|}{x_2}\right)\right\}. \tag{2.16}$$

In the case $0 < \alpha < 1$, formula (2.15) is also reduced to

$$\mathcal{G}_\alpha(x) = \frac{|x|^\alpha}{\alpha}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{1-\alpha}{2}\right)}{2\Gamma\left(\frac{2-\alpha}{2}\right)}\frac{x_1^\alpha}{|x|^\alpha} + \frac{x_1}{x_2}\left[F\left(\tfrac{1+\alpha}{2}, 1; \tfrac{3}{2}; -\tfrac{x_1^2}{x_2^2}\right) + \tfrac{\alpha}{\alpha-1}F\left(1, \tfrac{1}{2}; \tfrac{3-\alpha}{2}; -\tfrac{x_1^2}{x_2^2}\right)\right]\right), \tag{2.17}$$

see [14], formula (4.66).

Lemma 2.5. *The function $\mathcal{G}_\alpha(x)$ has the following structure*

$$\mathcal{G}_\alpha(x) = \frac{c}{\alpha}[\delta(x)]^\alpha + U(x), \tag{2.18}$$

where $\delta(x) = \min(x_1, x_2)$ is the distance to the boundary of the quarter-plane,

$$U(x) = |x|^\alpha t\mathcal{A}(t), \tag{2.19}$$

where

$$t = \frac{\delta(x)}{\max\{x_1, x_2\}} = \frac{\delta(x)}{\sqrt{|x|^2 - \delta^2(x)}} \quad (\text{so that } \ t = \frac{x_1}{x_2} \ \text{ in the sector } \ 0 < x_1 < x_2)$$

and

$$\mathcal{A}(t) = \frac{1}{\alpha}F\left(\frac{1+\alpha}{2}, 1; \frac{3}{2}; -t^2\right) + \frac{1}{\alpha-1}F\left(1, \frac{1}{2}; \frac{3-\alpha}{2}; -t^2\right) \tag{2.20}$$

is an analytic function (for all t).

Proof. The function $\mathcal{G}_\alpha(x)$ is symmetric: $\mathcal{G}_\alpha(x_1, x_2) = \mathcal{G}_\alpha(x_2, x_1)$. Therefore, it suffices to prove (2.18) for $x_1 \le x_2$. For x with $x_1 \le x_2$, formula (2.18) takes the form

$$\mathcal{G}_\alpha(x) = \frac{c}{\alpha}x_1^\alpha + U(x), \qquad U(x) = |x|^\alpha\frac{x_1}{x_2}\mathcal{A}\left(\frac{x_1}{x_2}\right), \tag{2.21}$$

which is nothing else but (2.17). □

Lemma 2.6. *Let $0 < \alpha < 1$. Then $U \in H^\alpha(\mathbb{R}^2_{++})$, but $U \notin H^{\alpha+\varepsilon}(\mathbb{R}^2_{++})$ for every $\varepsilon > 0$.*

Proof. Since $U(x)$ is continuous in $\mathbb{R}^2_{++}$ and $U(x_1, x_2) = U(x_2, x_1)$, it suffices to check that the function $U(x) = |x|^\alpha\frac{x_1}{x_2}\mathcal{A}\left(\frac{x_1}{x_2}\right)$ satisfies the α-Hölder condition in the sector $\{(x_1, x_2) : 0 < x_1 \le x_2 < \infty\}$. We have

$$|U(x) - U(y)| \le \big||x|^\alpha - |y|^\alpha\big| \left|\mathcal{A}\left(\frac{x_1}{x_2}\right)\right| + |y|^\alpha\left|\frac{x_1}{x_2}\mathcal{A}\left(\frac{x_1}{x}\right) - \frac{y_1}{y_2}\mathcal{A}\left(\frac{y_1}{y_2}\right)\right|.$$

Since the function $t\mathcal{A}(t), 0 < t \le 1$, is bounded and differentiable, we obtain

$$|U(x) - U(y)| \le C\big||x|^\alpha - |y|^\alpha\big| + C|y|^\alpha\left|\frac{x_1}{x_2} - \frac{y_1}{y_2}\right|. \tag{2.22}$$

If $|y| \leq |x-y|$, the statement $U \in H^\alpha$ follows. Let $|y| \geq |x-y|$. From (2.22) we obtain

$$|U(x) - U(y)| \leq C|x-y|^\alpha + C|y|^\alpha \left| \frac{y_2(x_1 - y_1) - y_1(x_2 - y_2)}{x_2 y_2} \right|$$
$$\leq C|x-y|^\alpha + C|y|^\alpha \frac{|x-y|\,|y|}{x_2 y_2}.$$

Since $y_2 \geq \frac{|y|}{\sqrt{2}}$, we obtain

$$|U(x) - U(y)| \leq C|x-y|^\alpha + C\frac{|x-y|}{|y|^{1-\alpha}} \leq C|x-y|^\alpha.$$

It remains to note that the inclusion $U \in H^{\alpha+\varepsilon}(\mathbb{R}^2_{++})$ with $\varepsilon > 0$, that is $|U(x) - U(y)| \leq C|x-y|^{\alpha+\varepsilon}$ is easily checked to be impossible by taking x and y on the same line: $x_2 = kx_1$ and $y_2 = ky_1$. □

Corollary 2.7. *In the case of quarter-plane with $\alpha \in (0,1)$ instead of (2.9) we have*

$$J_{\mathbb{R}^2_{++},\alpha}(x,y) = \frac{c}{\alpha}\left([\delta(x)]^\alpha - [\delta(y)]^\alpha + x_1^\alpha - y_1^\alpha + x_2^\alpha - y_2^\alpha\right) + U(x) - U(y) \qquad (2.23)$$

for all $x, y \in \mathbb{R}^2_{++}$, where $U \in H^\alpha(\mathbb{R}^2_{++})$.

Proof. Equality (2.23) follows immediately from (2.14), (2.18) and Lemma 2.5. □

3 On the α-property of domains

It is known that the potential of a bounded function on a bounded domain is α-Hölder continuous in Ω (which is a particular case of a Sobolev theorem stating that $I^\alpha_\Omega : L^p(\Omega) \to H^{\alpha - \frac{n}{p}}(\Omega)$, $1 < p \leq \infty$ when $\frac{n}{p} < \alpha < \frac{n}{p} + 1$, see [16], p. 256; in the case $p = \infty$ this is Lemma 1.41 in Mikhlin's book [8], Section 41), thus $J_{\Omega,\alpha} \in H^\alpha(\Omega)$ when Ω is bounded. For completeness of presentation, in the following lemma we give the proof for the general case, where Ω may be unbounded and include all $x, y \in \mathbb{R}^n$ into the Hölder condition, not only $x, y \in \Omega$.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be measurable and $\alpha \in (0,1)$. Then*

$$|\mathcal{J}_{\Omega,\alpha}(x,y)| \leq c\,|x-y|^\alpha, \qquad (3.1)$$

where c depends on n and α.

Proof. Let $x, y \in \mathbb{R}^n$, $a := \frac{1}{2}(x+y)$, and $r := 2|x-y|$. Then

$$\begin{aligned} J_{\Omega,\alpha}(x,y) &= \int\limits_{\Omega\setminus B(a,r)} |x-z|^{\alpha-n} - |y-z|^{\alpha-n}\, dz \\ &\quad + \int\limits_{\Omega\cap B(a,r)} |x-z|^{\alpha-n}\, dz - \int\limits_{\Omega\cap B(a,r)} |y-z|^{\alpha-n}\, dz \\ &=: (I) + (II) - (III). \end{aligned}$$

We estimate

$$|(II)| \le \int\limits_{B(x,2r)} |x-z|^{\alpha-n}\, dz = \int\limits_{B(0,2r)} |w|^{\alpha-n}\, dw \le \frac{c}{\alpha} r^\alpha \le \frac{2c}{\alpha}|x-y|^\alpha.$$

where c depends on n. By symmetry we conclude

$$|(II)| + |(III)| \le \frac{4c}{\alpha}|x-y|^\alpha.$$

If $z \in \mathbb{R}^n \setminus B(a,r)$, then $|x-y| = \frac{1}{2}r \le \frac{1}{2}|z-a|$. So it follows with (2.3).

$$|(I)| \le \int\limits_{\mathbb{R}^n\setminus(B(x,r)\cup B(y,r))} c\,|x-y||x-z|^{\alpha-n-1}\, dz \le \frac{c}{1-\alpha}\,|x-y| r^{\alpha-1}.$$

We combine the above estimates to conclude the claim of the lemma. □

Remark 3.2. When Ω is bounded, it is also known that in the case $\alpha = 1$ estimate (3.1) holds at the least in the form

$$|\mathcal{J}_{\Omega,1}(x,y)| \le c\,|x-y| \ln \frac{D}{|x-y|},\ \ D > \operatorname{diam}\Omega, \tag{3.2}$$

see [16], p. 256.

So the difference of the potential $J_{\Omega,\alpha}(x,y)$ is always Hölder continuous on $\mathbb{R}^n \times \mathbb{R}^n$. However, in the case of the ball, the half-space and the quarter-plane, it follows from our calculations above that $J_{\Omega,\alpha}(x,y)$ is Lipschitz off the diagonal $\{(x,x)\}$. We explain this below in more detail after the following definition.

Definition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $\alpha \in (0,1)$ (for bounded Ω we can take $\alpha \in (0,1]$). We say that Ω has the α-property, if there exists $c > 0$ such that difference of the potential satisfies*

$$|J_{\Omega,\alpha}(x,y)| \le c\,\frac{|x-y|}{(\max\{\delta(x),\delta(y)\})^{1-\alpha}} \quad \text{if } |x-y| \le \frac{1}{2}\max\{\delta(x),\delta(y)\}, \tag{3.3}$$

for all $x, y \in \Omega$, where $\delta(x)$ denotes the distance of x to the boundary $\partial\Omega$.

Remark 3.4. In the case $\Omega = \mathbb{R}^n$ use the convention $\delta(x) = \infty$ and $\frac{|x-y|}{\infty} = 0$. Since $J_{\mathbb{R}^n,\alpha}(x, y) = 0$, the set $\mathbb{R}^n$ has the α-property.

The following lemma provides a sufficient condition for a domain Ω to possess the α-property, this condition being inspired by calculation of $J_{\Omega,\alpha}(x)$ for balls, see (2.6), so that balls satisfy this property.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be measurable and bounded, and let $\alpha \in (0, 1]$. If $J_{\Omega,\alpha}(x)$ has the structure*

$$J_{\Omega,\alpha}(x) = c[\delta(x)]^\alpha + g(x), \qquad x \in \Omega, \tag{3.4}$$

where c is a constant and $g \in \mathrm{Lip}(\overline{\Omega})$, then the domain Ω possesses the α-property.

P r o o f. Since g is Lipschitz, we have

$$|g(x) - g(y)| \le c\,|x - y| \le c\,|x - y| \left(\frac{\operatorname{diam} \Omega}{\max\{\delta(x), \delta(y)\}} \right)^{1-\alpha}.$$

If $\alpha = 1$, then $|\delta(x) - \delta(y)| \le |x - y|$. On the other hand if $\alpha \in (0, 1)$, then

$$|a^\alpha - b^\alpha| \le \int_a^b \alpha t^{\alpha-1}\, dt \le \alpha |b - a| \, [\min\{a, b\}]^{\alpha-1}.$$

If $|x-y| \le \frac{1}{2} \max\{\delta(x), \delta(y)\}$, then $\min\{\delta(x), \delta(y)\} \ge \frac{1}{2} \max\{\delta(x), \delta(y)\}$. Therefore with the previous estimate

$$\big|[\delta(x)]^\alpha - [\delta(y)]^\alpha\big| \le \alpha 2^{1-\alpha} \frac{|\delta(x) - \delta(y)|}{[\max\{\delta(x), \delta(y)\}]^{1-\alpha}} \le 2 \frac{|\delta(x) - \delta(y)|}{[\max\{\delta(x), \delta(y)\}]^{1-\alpha}}$$

for all $x, y \in \Omega$. □

In Lemma 3.8 we show that any bounded uniform domain in $\mathbb{R}^n$ satisfies the α-property, but first in Lemma 3.6 below we show the same for the case of some simple unbounded domains.

Lemma 3.6. *The half-space $\mathbb{R}^n_+$ and the quarter-plane $\mathbb{R}^2_{++}$ satisfy the α-property, $0 < \alpha < 1$.*

P r o o f. The statement for the half-space is seen from Lemma 2.2 proceeding similarly as in the proof of Lemma 3.5. In the case of a quarter-plane the term $U(x) - U(y)$ needs a more careful treating, since it is not Lipschitz, see Lemma 2.6.

By symmetry it suffices to check (3.3) in the case

$$|x - y| \le \frac{1}{2} \max\{x_2, y_2\}. \tag{3.5}$$

After easy evaluations, in the sector $0 < x_1 < x_2$ we have

$$\left|\frac{\partial U}{\partial x_j}\right| \leq C\frac{|x|^\alpha}{x_2} \leq \frac{C\sqrt{2}}{|x|^{1-\alpha}}, \quad j = 1, 2. \tag{3.6}$$

Then by the mean value theorem

$$U(x) - U(y) = (x_1 - y_1)\int_0^1 (D_1U)(tx + (1-t)y)\, dt$$
$$+ (x_2 - y_2)\int_0^1 (D_2U)(tx + (1-t)y)\, dt$$

where D_1 and D_2 stand for the first order partial derivatives, we get

$$|U(x) - U(y)| \leq C|x - y| \int_0^1 \frac{dt}{|y + t(x-y)|^{1-\alpha}}.$$

Let $x_2 \leq y_2$ for definiteness. Then by means of (3.5) we obtain $|y + t(x - y)| \geq |y| - |x - y| \geq |y| - \frac{1}{2}\max\{x_2, y_2\} = |y| - \frac{y_2}{2} \geq \frac{y_2}{2}$, so that

$$|U(x) - U(y)| \leq C\frac{|x-y|}{y_2^{1-\alpha}}$$

which completes the proof. □

Definition 3.7. (see [7]) A domain D is called a *uniform domain* or *Jones domain*, if there exists a constant $c > 0$ such that each pair of points $x_1, x_2 \in D$ can be connected by a rectifiable curve Γ in D for which $\ell(\Gamma) \leq c|x_1 - x_2|$ and $\min\{\ell(\Gamma_{x_1,y}), \ell(\Gamma_{x_2,y})\} \leq c\delta(y)$ for all $y \in \Gamma$, where $\ell(\Gamma)$ and $\ell(\Gamma_{x_j,y})$ denote the length of Γ and the length of the subarc $\Gamma_{x_j,y}$ of Γ connecting x_j and y, respectively. The smallest constant is called the *Jones constant* of Ω.

The notion of uniform domains was introduced by Jones in [7] in the context of extensions of Sobolev functions from a domain D to $\mathbb{R}^n$. Jones showed, that for every uniform domain there exists a continuous, linear extension operator from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$. The operator is independent of $k \in \mathbb{N}$ and $p \in [1, \infty]$ as long k is bounded by some k_0.

A ball, the half space $\mathbb{R}^n_+$ and the quarter plane $\mathbb{R}^2_{++}$ are uniform domains, where the Jones constant does not depend on the radius of the ball. Also any bounded Lipschitz domain is uniform. However, uniform domains do not need to have smooth boundary. In fact the interior of the Koch's snow flake is a uniform domain. Note that the boundary of a uniform domain always has measure zero by [7, Lemma 2.3].

Lemma 3.8. *Any uniform domain $\Omega \subset \mathbb{R}^n$ has the α-property.*

P r o o f. Let $x,y \in \Omega$. Then exists a rectifiable path γ from x to y with

$$\begin{aligned} l(\gamma) &\le c_1\,|x-y|, \\ \min\{l(\gamma_{x,z}), l(\gamma_{y,z})\} &\le c_2\,\delta(z) \qquad \text{for all } z \in \gamma, \end{aligned} \tag{3.7}$$

where $l(\gamma)$ and $l(\gamma_{x,z})$ denotes the length of the path γ and the subpath $\gamma_{x,z}$ connecting x and z.

Let γ be parametrized by its arc length with $\gamma(0) = x$ and $\gamma(l(\gamma)) = y$. Let $\beta \in (0,1)$ such that $(1-\beta)c_2 \le \frac{1}{2}$. We define

$$w_j := \begin{cases} \gamma\big(\frac{1}{2}\beta^{|j|} l(\gamma)\big) & \text{for } j \le 0, \\ \gamma\big((1-\frac{1}{2}\beta^{|j|}) l(\gamma)\big) & \text{for } j \ge 0. \end{cases}$$

Certainly, we have $x = \lim_{j\to-\infty} w_j$ and $z = \lim_{j\to-\infty} w_j$. Moreover, we claim that for suitable β (close to 1) we have

$$|w_j - w_{j+1}| \le \frac{1}{2}\max\{\delta(w_j), \delta(w_{j+1})\} \tag{3.8}$$

for all $j \in \mathbb{Z}$. By symmetry $j \leftrightarrow -j$ it suffices to consider the case $j \ge 0$. By (3.7) (for $z = w_j$) and the definition of w_j we get

$$\begin{aligned} l(\gamma) &\le c_1|x-y|, \\ \frac{1}{2} l(\gamma)\beta^{|j|} &\le c_2\delta(w_j), \\ |w_{j+1} - w_j| &\le \frac{1}{2} l(\gamma)(1-\beta)\beta^{|j|} \end{aligned} \tag{3.9}$$

for all $j \ge 0$. Therefore, with $(1-\beta)c_2 \le \frac{1}{2}$ we obtain

$$|w_{j+1} - w_j| \le (1-\beta)c_2\delta(w_j) \le \frac{1}{2}\max\{\delta(w_j), \delta(w_{j+1})\}.$$

So we can apply (3.1) from Lemma 3.1 to the points w_{j+1} and w_j. Since $J_{\Omega,\alpha}$ is continuous on $\mathbb{R}^n$, we have for every $x,y \in \Omega$ by telescoping sum

$$\big|J_{\Omega,\alpha}(x,y)\big| = \left|\sum_{j\in\mathbb{Z}} J_{\Omega,\alpha}(w_{j+1}, w_j)\right| \le \sum_{j\in\mathbb{Z}} |J_{\Omega,\alpha}(w_{j+1}, w_j)|.$$

So with (3.1) and (3.9) we have

$$\begin{aligned} \big|J_{\Omega,\alpha}(x,y)\big| &\le c \sum_{j\in\mathbb{Z}} \frac{|w_{j+1} - w_j|}{[\max\{\delta(w_{j-1}), \delta(w_j)\}]^{1-\alpha}} \\ &\le c \sum_{j\in\mathbb{Z}} \frac{\frac{1}{2} l(\gamma)\beta^{|j|}}{(\frac{1}{2c_2} l(\gamma)\beta^{|j|})^{1-\alpha}} \\ &\le c\, c_2^{1-\alpha} 2^{-\alpha} l(\gamma)^\alpha \sum_{j\in\mathbb{Z}} \beta^{|j|} \\ &\le \frac{c\, c_2^{1-\alpha} 2^{1-\alpha} c_1^\alpha}{1-\beta} |x-y|^\alpha \end{aligned}$$

for all $x, y \in \Omega$. $\square$

Remark 3.9. We already mentioned that a ball B, the half space $\mathbb{R}^n_+$ and the quarter plane $\mathbb{R}^2_{++}$ are uniform domains. In fact also the complements of those domains (removing the boundary) $\mathbb{R}^n \backslash \overline{B}$, $\mathbb{R}^n \backslash \overline{\mathbb{R}^n_+}$ and $\mathbb{R}^2 \backslash \overline{\mathbb{R}^2_{++}}$ are uniform domains, where the Jones constant does not depend on the radius of the ball. Therefore, also the complements satisfy the α-property. As a consequence (3.3) holds for all $x, y \in \mathbb{R}^n$. At this we use that $J_{\Omega,\alpha}(x, y) = -J_{\mathbb{R}^n \backslash \Omega}(x, y) = -J_{\mathbb{R}^n \backslash \overline{\Omega}}(x, y)$, since the boundary of Ω has measure zero, where Ω is B, $\mathbb{R}^n_+$, or $\mathbb{R}^2_{++}$. Note that condition $|x - y| \leq \frac{1}{2} \max\{\delta(x), \delta(y)\}$ ensures that either $x, y \in \Omega$ or $x, y \in \mathbb{R}^n \setminus \overline{\Omega}$. The same conclusion holds for any uniform domain Ω if $\mathbb{R}^n \setminus \overline{\Omega}$ is also a uniform domain.

4 On mapping properties of the potential operator in Hölder type spaces

Definition 4.1. Let $\omega(f, h) = \sup_{\substack{x,y\in\Omega: \\ |x-y|<h}} |f(x) - f(y)|$ be the modulus of continuity of a function $f \in C(\Omega)$. Given a continuous semi-additive function ω on $[0, \operatorname{diam}\Omega]$, positive for $h > 0$, with $\omega(0) = 0$, by $H^\omega(\Omega)$ we denote the space of functions $f \in C(\overline{\Omega})$ with the finite norm

$$\|f\|_{H^\omega} = \|f\|_{C(\bar{\Omega})} + \sup_{0<h< \operatorname{diam}\Omega} \frac{\omega(f, h)}{\omega(h)}.$$

By $H^\omega_0(\Omega)$ we denote the subspace in $H^\omega(\Omega)$ of functions f which vanish on the boundary $\partial\Omega$ of Ω.

Definition 4.2. A non-negative function $\omega(t)$ is called almost increasing (almost decreasing) on $[0, d], 0 < d \leq \infty$, if $\omega(t) \leq C\omega(\tau)$ for all $t \leq \tau$ ($t \geq \tau$, respectively).

Lemma 4.3. *Let $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ have the α-property. Let $f \in H^\omega_0(\Omega)$, where the function $\omega(h)$ has the property that*

$$\frac{\omega(h)}{h^{1-\alpha}} \quad \textit{is almost decreasing.} \tag{4.1}$$

Then

$$\sup_{x,y\in\Omega:|x-y|<h} \left|f(x)[J_{\Omega,\alpha}(x, y)]\right| \leq C\omega_\alpha(h)\|f\|_{H^\omega(\Omega)}, \tag{4.2}$$

where $\omega_\alpha(h) = h^\alpha\omega(h)$. In particular,

$$\sup_{x,y\in\Omega:|x-y|<h} \left|f(x)[J_{\Omega,\alpha}(x, y)]\right| \leq Ch^{\alpha+\lambda}\|f\|_{H^\lambda(\Omega)}, \tag{4.3}$$

when $f \in H^\lambda_0(\Omega)$ and $\lambda + \alpha \leq 1$.

Proof. Given $x \in \Omega$, let $\widetilde{x}$ be a point of the boundary, such that $|x-\widetilde{x}| = \delta(x)$. Then we have

$$|f(x)| = |f(x) - f(\widetilde{x})| \leq C\omega(\delta(x))\|f\|_{H^\omega} \tag{4.4}$$

and

$$\big|f(x)[J_{\Omega,\alpha}(x,y)]\big| \leq C\omega(\delta(x))\|f\|_{H^\omega}|J_{\Omega,\alpha}(x,y)|. \tag{4.5}$$

We distinguish the cases $\frac{1}{2}\max\{\delta(x),\delta(y)\} \leq |x-y|$ and $|x-y| \leq \frac{1}{2}\max\{\delta(x),\delta(y)\}$. In the first case we have by Lemma 3.1 $|J_{\Omega,\alpha}(x,y)| \leq C|x-y|^\alpha$ and therefore

$$\big|f(x)[J_{\Omega,\alpha}(x,y)]\big| \leq C\omega(2|x-y|)|x-y|^\alpha\|f\|_{H^\omega} \leq C\omega_\alpha(h)\|f\|_{H^\omega} \tag{4.6}$$

for all x, y such that $|x-y| < h$. In the second case we have $|J_{\Omega,\alpha}(x) - J_{\Omega,\alpha}(y)| \leq C\frac{|x-y|}{(\max\{\delta(x),\delta(y)\})^{1-\alpha}}$. Then (4.6), (4.5), and (4.1) yield

$$\begin{aligned}\big|f(x)[J_{\Omega,\alpha}(x,y)]\big| &\leq C\|f\|_{H^\omega}\frac{\omega(\delta(x))}{(\max\{\delta(x),\delta(y)\})^{1-\alpha}}|x-y| \\ &\leq C\|f\|_{H^\omega}\frac{\omega(|x-y|)}{|x-y|^{1-\alpha}}|x-y| \leq C\|f\|_{H^\omega}\omega_\alpha(h),\end{aligned}$$

which completes the proof. □

Theorem 4.5 stated below was obtained in [11], see also an electronic pre-publication at [12]. In Theorem 3.9 in [11], [12] there was proved a statement more general than given below: we give its version for the Euclidean setting and constant α; note that Theorem 3.9 in [11], [12] was stated for bounded domains, however this restriction was introduced there only because of variable order $\alpha(x)$ and may be omitted when α is constant).

Definition 4.4. We say that a continuous non-negative function ω belongs to a Zygmund class Φ_β, $\beta > 0$, if it is almost increasing and

$$\int_h^d \left(\frac{h}{t}\right)^\beta \frac{w(t)}{t}dt \leq cw(h), \tag{4.7}$$

where $c > 0$ does not depend on $h \in \left(0, \frac{d}{2}\right]$.

Theorem 4.5. *Let Ω be a domain in $\mathbb{R}^n$, $0 < \alpha < 1$. If $\omega \in \Phi_{1-\alpha}$ and*

$$J_{\Omega,\alpha} \in H^{\omega_\alpha(\cdot)}(\Omega), \quad \textit{where} \quad \omega_\alpha(h) = h^\alpha\omega(h); \tag{4.8}$$

then the operator I^α is bounded from the space $H^{\omega(\cdot)}(\Omega)$ into the space $H^{\omega_\alpha(\cdot)}(\Omega)$.

In [12] this theorem was proved for an open set Ω in a metric measure space X when the set Ω possesses the cancellation property or condition (4.8) is satisfied. Domains in $\mathbb{R}^n$ do not satisfy this condition. As the above examples of simple domains (balls, half-spaces and quarter-plane) show, condition (4.8) is too restrictive, being satisfied in fact only under the cancellation property of Ω, that is, only when

$\Omega = \mathbb{R}^n$. Making use of the above arguments, we may avoid condition (4.8) in the case of domains with α-property.

Theorem 4.6. *Let $0 < \alpha < 1$ and Ω be a domain in $\mathbb{R}^n$ with α-property. If $\omega \in \Phi_{1-\alpha}$, then the potential operator I^α_Ω is bounded from $H^\omega_0(\Omega)$ to $H^{\omega_\alpha}(\Omega)$, where $\omega_\alpha(h) = h^\alpha \omega(h)$. In particular, it is bounded from $H^\lambda_0(\Omega)$ to $H^{\lambda+\alpha}(\Omega)$ if $\lambda + \alpha < 1$.*

Proof. The proof in [12] was based on the direct estimation of the continuity modulus of the potential, see Theorem 3.4 in [12], via the representation

$$\begin{aligned}
&(I^\alpha f)(x) - (I^\alpha f)(y) = \\
&= \int\limits_{\varrho(x,z)<2h} [f(z) - f(x)]\varrho(x,z)^{\alpha-n} d\mu(z) - \int\limits_{\varrho(x,z)<2h} [f(z) - f(x)]\varrho(y,z)^{\alpha-n} d\mu(z) \\
&\quad + \int\limits_{\varrho(x,z)>2h} [f(z) - f(x)] \left\{\varrho(x,z)^{\alpha-n} - \varrho(y,z)^{\alpha-n}\right\} d\mu(z) \\
&\quad + f(x) \int\limits_{\Omega} \left\{\varrho(x,z)^{\alpha-n} - \varrho(y,z)^{\alpha-n}\right\} d\mu(z) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

As can be seen from the proof of Theorem 3.4 in [12], condition (4.8) was used only in the term

$$I_4 = f(x)[J_{\Omega,\alpha}(x) - J_{\Omega,\alpha}(y)],$$

where $|x - y| < h$. This term is now estimated by means of Lemma 4.3. Note that condition (4.1) assumed in that lemma follows from the assumption $\omega \in \Phi_{1-\alpha}$. This completes the proof. □

Corollary 4.7. *The statement of Theorem 4.6 holds, in particular, for uniform domains.*

Proof. Refer to Lemma 3.8. □

Remark 4.8. For simplicity, we dealt with the case where $\omega(h)$ does not depend on x. Since in [12] the general case of $\omega(x,h)$ was treated, Theorem 4.6 is extended in the same way to this case. The only changes in the formulation of Theorem 4.6 are that the condition $\omega \in \Phi_{1-\alpha}$ now should be interpreted as belongness of $\omega(x,h)$ to $\Phi_{1-\alpha}$ in variable h uniformly in x, and we have to write $\omega_\alpha(x,h) = h^\alpha \omega(x,h)$.

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