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Generalized potentials in variable exponent Lebesgue spaces on homogeneous spaces

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We consider generalized potential operators with the kernel $\frac{a([\varrho(x,y)])}{[\varrho(x,y)]^N}$ on bounded quasimetric measure space (X,μ,d) with doubling measure μ satisfying the upper growth condition $\mu B(x,r) \le Kr^N, N \in (0,\infty)$. Under some natural assumptions on $a(r)$ in terms of almost monotonicity we prove that such potential operators are bounded from the variable exponent Lebesgue space $L^{p(\cdot)}(X,\mu)$ into a certain Musielak-Orlicz space $L^\Phi(X,\mu)$ with the N-function $\Phi(x,r)$ defined by the exponent $p(x)$ and the function $a(r)$. A reformulation of the obtained result in terms of the Matuszewska-Orlicz indices of the function $a(r)$ is also given.

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1 Introduction

The Lebesgue spaces $L^{p(\cdot)}$ with variable exponent have been intensively investigated during the last years. The growing interest to this topic is connected with applications of this theory in problems of fluid dynamics, elasticity theory and differential equations with non-standard growth conditions (see e.g. [38], [47]).

The spaces $L^{p(\cdot)}$ with variable exponent probably first appeared in the book of Nakano ([35]) as an example of the modular spaces studied in this book. They are special cases of Orlicz-Musielak spaces, see [32] for these spaces. We refer to [5] where the maximal operator was studied in the context of Orlicz-Musielak spaces.

A significant progress has been made in the study of classical integral operators in the context of the $L^{p(\cdot)}$ spaces. We refer to the papers [46], [27] for the basic properties of these spaces and to the surveying papers [6], [20] and [42] for the development, up to 2005, of the study of maximal, potential and singular operators in such spaces.

The spaces $L^{p(\cdot)}$ on measure quasimetric spaces and maximal and potential operators in such spaces were studied in [15], [16], [17], [19], [25].

In this paper we study - within the variable exponent setting - the generalized Riesz potential operators

$$I_a f(x) := \int\limits_X \mathcal{K}(x,y) f(y) d\mu(y), \quad \mathcal{K}(x,y) = \frac{a\left[\varrho(x,y)\right]}{\left[\varrho\left(x,y\right)\right]^N} \tag{1}$$

over a bounded measure space X with quasimetric ϱ, where N is the upper Ahlfors dimension of X. Under some assumptions on the function $a(\varrho)$ we prove a Sobolev-type theorem on the boundedness of the operator I_a from $L^{p(\cdot)}(X)$ into a certain Orlicz-Musielak space.

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The generalized Riesz potential operators I_a attracted attention last years, we refer in particular to [14], [33], where such potentials were studied in Orlicz spaces in the case $X = \mathbb{R}^n$ and Euclidean metric, and to [34] where homogeneous spaces with constant dimension were admitted. We refer also to [37] for the study of the similar generalized potentials in the Euclidean setting in rearrangement invariant spaces. For non-generalized potentials (that is, for the case $a(r) \equiv r^\alpha, 0 < \alpha < n$) on quasimetric measure spaces we refer to [7], [8], [9], [10], [11], [12],[13], [21], [22].

For the case where X is bounded, we generalize the known results to the case of variable exponent $p(x)$ and admit upper Ahlfors-regular quasimetric measure spaces (see (2)). In the case of bounded space X, the obtained result is valid under more general assumptions on $a(r)$ that known even in the case of constant p and Euclidean space.

The main results are formulated in Section 3 and proved in Section 4. Section 2 contains necessary preliminaries. In Section 5 we reformulate the obtained result in terms of Matuszewska-Orlicz type indices of the function $a(r)$.

2 Orlicz-Musielak space and Lebesgue space with variable exponent

By (X, ϱ, μ) we denote a space with quasimetric ϱ and positive Borel regular measure μ. We recall that the measure μ is called doubling if

$$\mu B(x, 2r) \leq \mu B(x, r)$$

for every open ball $B(x, r) = \{y \in X : \varrho(x, y) < r\}$, with $C > 0$ not depending on x and r.

The space X is called upper Ahlfors N-regular, $N \in (0, \infty)$, if there exists a constant $K > 0$, not depending on $x \in X$ and $r > 0$, such that

$$\mu B(x, r) \leq K r^N \tag{2}$$

Everywhere in the sequel we assume that X is a bounded quasimetric space.

Following [32], p. 82, we use the following definitions.

Definition 2.1 A function $\Phi : X \times [0, \infty) \to [0, +\infty)$ is said to be an N-function, if

1. for every $x \in X$ the function $\Phi(x, t)$ is convex, nondecreasing and continuous in $t \in [0, \infty)$,
2. $\Phi(x, 0) = 0$, and $\Phi(x, t) > 0$ for every $t > 0$,
3. $\Phi(x, t)$ is a μ-measurable function of x for every $t \geq 0$.

The integral

$$\mathcal{M}_\Phi(f) = \int_X \Phi(x, |f(x)|) d\mu(x)$$

is called the modular.

Definition 2.2 Let Φ be an N-function. The Orlicz-Musielak space $L^\Phi(X)$ is defined as the set of all real-valued μ-measurable and μ-almost everywhere finite functions f on X such that $\mathcal{M}_\Phi\left(\frac{f}{\lambda}\right) < \infty$ for some $\lambda > 0$. This is a Banach space with respect to the norm

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \mathcal{M}_\Phi\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Observe that

$$\mathcal{M}_\Phi(f) \leq 1 \Longleftrightarrow \|f\|_\Phi \leq 1. \tag{3}$$

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In the case $\Phi(x,t) = t^{p(x)}$, where $p : X \to [1, +\infty)$ is a μ-measurable function, we obtain the variable exponent Lebesgue space $L^{p(\cdot)}(X)$. We also use the notation

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int\limits_X \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} d\mu(x) \le 1 \right\}.$$

Variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ within the frameworks of quasimetric measure spaces were considered in [15], [16], [17], [19], [25]. We also specially refer to [5], where the spaces $L^{p(\cdot)}(X)$ were studied in the context of Musielak-Orlicz spaces in the Euclidean setting.

Obviously,

$$\|f\|_{p(\cdot)} \le \max \left\{ \left(\int\limits_X |f(x)|^{p(x)} d\mu(x) \right)^{\frac{1}{p_-}}, \left(\int\limits_X |f(x)|^{p(x)} d\mu(x) \right)^{\frac{1}{p_+}} \right\}, \tag{4}$$

where

$$p_- = \operatorname*{ess\,inf}_{x \in X} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x \in X} p(x) < \infty.$$

In the sequel we suppose that $p(x)$ satisfies the assumption

$$1 < p_- \le p(x) \le p_+ < +\infty \tag{5}$$

and the "weak Lipschitz" condition

$$|p(x) - p(y)| \le \frac{\mathbb{A}}{-\ln \varrho(x,y)}, \quad \varrho(x,y) \le \frac{1}{2}, \tag{6}$$

where $\mathbb{A} > 0$ does not depend on x, y. We denote $p'(x) = \dfrac{p(x)}{p(x) - 1}$.

In [16], Theorem 4.3, the following statement on the boundedness of maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)| \, d\mu(y)$$

in the Lebesgue spaces with variable exponent was proved.

Theorem 2.3 *Let X be a bounded quasimetric measure space with doubling measure and the exponent $p : X \to (1, +\infty)$, satisfy conditions* (5) *and (6). Then there exists a constant $C > 1$ such that*

$$\|Mf\|_{p(\cdot)} \le C \|f\|_{p(\cdot)}.$$

3 Main result

In our main result in Theorem 3.1, we suppose that X is a bounded quasimetric measure space with doubling measure, upper Ahlfors N-regular, and denote $d = diam X,\ 0 < d < \infty$. Some of the auxiliary results will be obtained without the assumption that the measure is doubling.

By $W_0 = W_0([0,d])$ we denote the class of continuous non-negative functions $a = a(r)$ on $[0,d]$ such that $a(r)$ is almost increasing and $\lim\limits_{r \to 0} a(r) = 0$. We recall that a non-negative function $a(r)$ defined on $[0,d],\ 0 < d \le \infty$, is called almost increasing (almost decreasing), if there exist a constant $C > 0$ such that

$$a(t_1) \le C a(t_2)$$

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for all $0 < t_1 < t_2 < d$ $(0 < t_2 < t_1$, respectively).

In most of the statements in the sequel it is assumed that

$$a \in W_0([0,d]) \quad \text{and} \quad \int_0^d \frac{a(t)}{t} dt < \infty. \tag{7}$$

In the main result (Theorem 3.1) the condition

$$\frac{a(r)}{r^\lambda} \quad \text{is almost decreasing on} \quad [0,d] \quad \text{for some} \quad 0 < \lambda < \frac{N}{p_+} \tag{8}$$

will be also used.

Our main result is the following theorem, where we make use of the notation

$$A(r) = \int_0^r \frac{a(t)}{t} dt.$$

Observe that a function $a \in W_0$ is equivalent to $A(r)$, that is, $C_1 a(r) \le A(r) \le C_2 a(r)$, under the following conditions:

a) there exists a $\delta > 0$ such that $\frac{a(r)}{r^\delta}$ is almost increasing,

b) there exists a $\lambda > 0$ such that $\frac{a(r)}{r^\lambda}$ is almost decreasing (it is obvious that $\lambda > \delta$).

It is known that for $a \in W_0$ condition a) is also necessary for the validity of the inequality $A(r) \le C_2 a(r)$, see [18], Theorem 3.1.

Theorem 3.1 *Let X be a bounded quasimetric measure space with doubling measure, upper Ahlfors N-regular, let the exponent $p(x)$ satisfy conditions* (5) *and* (6) *and let $a(r)$ satisfy* (7) *and* (8). *Then the operator I_a is bounded from $L^{p(\cdot)}(X)$ into the Orlicz-Musielak space $L^\Phi(X)$, where the N-function is defined by its inverse (for every fixed $x \in X$)*

$$\Phi^{-1}(x,u) = \int_0^u A\left(t^{-\frac{1}{N}}\right) t^{-\frac{1}{p'(x)}} dt \tag{9}$$

and $a(t)$ is assumed to be continued as $a(t) \equiv a(d)$ for $r > d$.

Remark 3.2 The function $\Phi^{-1}(x,u)$ may be equivalently represented as

$$\Phi^{-1}(x,u) = \int_r^\infty \frac{A(t)\,dt}{t^{1+\frac{N}{p(x)}}} \sim \frac{1}{r^{\frac{N}{p(x)}}} \int_0^r \frac{a(t)\,dt}{t} + \int_r^\infty \frac{a(t)\,dt}{t^{1+\frac{N}{p(x)}}}, \quad r = u^{-\frac{1}{N}}$$

which follows from the identity

$$\frac{N}{p(x)} \int_r^\infty \frac{A(t)\,dt}{t^{1+\frac{N}{p(x)}}} = \frac{1}{r^{\frac{N}{p(x)}}} \int_0^r \frac{a(t)\,dt}{t} + \int_r^\infty \frac{a(t)\,dt}{t^{1+\frac{N}{p(x)}}}$$

obtained by the direct interchange of order of integration in the repeated integral.

Note that the statement of Theorem 3.1 is new even in the case of constant p: the corresponding result in [34] was obtained under stronger assumptions on $a(r)$. In the case where $a(r) \equiv r^\alpha$ and p is constant, when the corresponding Musielak-Orlicz space is the Lebesgue space L^q with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$, statements of the type of Theorem 3.1 were earlier obtained in [21] (see also [22] and [7], Theorem 6.1.1) and in [8], [9], [11] (note that in [21] there was also shown that the growth condition (2) is necessary for the validity of such a Sobolev theorem).

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Observe that for the case of non-generalized potentials and constant p the corresponding Sobolev theorem was proved without the assumption that the measure is doubling. In the case of variable exponent $p(x)$ we have to impose this condition, because - up to authors' knowledge - the boundedness of the maximal operator is an open question without the doubling condition.

We give also a reformulation of Theorem 3.1 in terms of the upper Matuszewska-Orlicz type index of the function $a(r)$, see Theorem 5.2.

To prove Theorem 3.1, we use the Hedberg approach of estimation of potentials, adjusted for the case of quasimetric measure spaces and variable exponents. The main difficulty is the estimation of the variable norms of the kernel of the potential truncated to an exterior of balls. This estimation being trivial for the Riesz potential in the case of constant exponents p and Euclidean setting (and standard in the case of constant exponents and homogeneous spaces), becomes a more difficult task for the generalized potentials, even in the case of constant exponents. In the case of variable exponents such an estimation, and consequently realization of the Hedberg approach, turns to be a heavy problem. In the case of the Riesz potentials this problem for variable exponent norms was overcome in the Euclidean setting in [43], another way to realize Hedberg's approach for variable exponents was suggested in [26] for (non-generalized) potentials on Carleson curves. The variable norm estimation of the truncated generalized potential kernels in the following subsection partially follows the approach of [43].

4 Proofs

4.1 Preliminaries

Definition 4.1 By $\mathcal{V}_{N,p(\cdot)}$ we denote the class of functions $a \in W_0([0,d]),\ 0 < d < \infty$, such that $\int\limits_0^d \frac{a(t)}{t} dt < \infty$ and

$$\sup_{x\in X}\sup_{0<r<d}\left(\int\limits_r^d \left[\frac{a(t)}{t^{\frac{N}{p(x)}}}\right]^{p'(x)} \frac{dt}{t}\right)\cdot\left(\frac{1}{r^{\frac{N}{p(x)}}}\int\limits_0^r \frac{a(t)}{t}dt\right)^{-p'(x)} < \infty. \tag{10}$$

The power function $a(r) = r^\alpha$ belongs to the class $\mathcal{V}_{N,p(\cdot)}$, if and only if $0 < \alpha < \frac{N}{p_+}$. The following lemma gives a sufficient condition for a non-negative function $a(t)$ to belong to the class $\mathcal{V}_{N,p(\cdot)}$.

Lemma 4.2 *Let $p(x)$ satisfy condition* (5) *and $a(t)$ a non-negative function satisfying condition* (8). *Then $a(t)$ satisfies condition* (10).

P r o o f. The proof is direct: under the conditions of the lemma on $p(x)$ and $a(t)$, the integral in the first parenthesis in (10) has an upper bound $C\left(\frac{a(r)}{r^{\frac{N}{p(x)}}}\right)^{p'(x)}$ with $C > 0$ not depending on x and r, and integral in the second parenthesis has the same lower bound, the condition $\lambda < \frac{N}{p_+}$ being needed only for the upper bound in the left-hand side parenthesis. □

Lemma 4.3 *Let $a(r)$ be a non-negative continuous almost increasing function on $[0,d], 0 < d \le \infty$ and let the variable exponents $\lambda(x)$ and $\gamma(x)$ satisfy the assumptions*

$$\inf_{x\in X}\lambda(x) > 0, \quad \sup_{x\in X}\lambda(x) < \infty \qquad \textit{and} \qquad \inf_{x\in X}|\gamma(x)| > 0, \quad \sup_{x\in X}|\gamma(x)| < \infty.$$

Then

$$C_1\int\limits_r^{\frac{d}{2}}\left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t} \le \sum_{k=1}^{\left[\log_2\frac{d}{r}\right]}\left[\frac{a\left(2^k r\right)}{(2^k r)^{\gamma(x)}}\right]^{\lambda(x)} \le C_2\int\limits_r^{d}\left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t}, \tag{11}$$

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where it is assumed that $a(t)$ satisfies also the doubling condition $a(2t) \le Ca(t)$ in the case where the right-hand side inequality is considered for $d < \infty$, and $0 < r \le \frac{d}{2}$ in the left-hand side inequality and $0 < r < d$ in the right-hand side; $C_1 > 0$ and $C_2 > 0$ do not depend on r and x.

P r o o f. Since $a(t)$ is almost increasing we have

$$\int\limits_{2^{k-1}r}^{2^k r} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t} \le C \left[a(2^k r)\right]^{\lambda(x)} \int\limits_{2^{k-1}r}^{2^k r} t^{-\lambda(x)\gamma(x)-1} dt \le C \left[\frac{a(2^k r)}{(2^k r)^{\gamma(x)}}\right]^{\lambda(x)}.$$

Hence

$$\sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \left[\frac{a(2^k r)}{(2^k r)^{\gamma(x)}}\right]^{\lambda(x)} \ge C \sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \int\limits_{2^{k-1}r}^{2^k r} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t} = C \int\limits_{r}^{d\cdot 2^{-\theta}} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t}$$

where

$$\theta = \theta(r) = \log_2 \frac{d}{r} - \left[\log_2 \frac{d}{r}\right] \in [0, 1) \tag{12}$$

and then

$$\sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \left[\frac{a(2^k r)}{(2^k r)^{\gamma(x)}}\right]^{\lambda(x)} \ge C \int\limits_{r}^{\frac{d}{2}} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t}. \tag{13}$$

To prove the inverse inequality, we again use the almost monotonicity of $a(t)$ and have

$$\int\limits_{2^{k-1}r}^{2^k r} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t} \ge C \left[a(2^{k-1} r)\right]^{\lambda(x)} \int\limits_{2^{k-1}r}^{2^k r} t^{-\lambda(x)\gamma(x)-1} dt \ge C \left[a(2^{k-1} r)\right]^{\lambda(x)} (2^k r)^{-\lambda(x)\gamma(x)}.$$

Therefore,

$$\sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \left[\frac{a(2^{k-1} r)}{(2^k r)^{\gamma(x)}}\right]^{\lambda(x)} \le C \sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \int\limits_{2^{k-1}r}^{2^k r} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t} \le C \int\limits_{r}^{d} \left[\frac{a(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t}.$$

Since $a(2^{k-1}r) \ge Ca(2^k r)$, we obtain the right-hand side inequality in (11). □

Lemma 4.4 *Let $a(r)$ be a non-negative continuous almost increasing function on $[0, d], 0 < d \le \infty$, let it satisfy also the doubling condition $a(2t) \le Ca(t)$ in the case $d < \infty$. If $p : X \to [0, \infty)$ satisfy condition* (5)*, then*

$$\int\limits_{X \setminus B(x,r)} \left(\frac{a\left[\varrho(x,y)\right]}{\varrho(x,y)^N}\right)^{p(x)} d\mu(y) \le C \int\limits_{r}^{d} \left[\frac{a(t)}{t^{\frac{N}{p'(x)}}}\right]^{p(x)} \frac{dt}{t} + C[a(d)]^{p(x)}, \quad 0 < r < d, \tag{14}$$

where $C > 0$ does not depend on $x \in X$ and $r \in (0, d)$, which may be also written in the form

$$\int\limits_{X \setminus B(x,r)} \left(\frac{a\left[\varrho(x,y)\right]}{\varrho(x,y)^N}\right)^{p(x)} d\mu(y) \le C \int\limits_{r}^{d} \left[\frac{a(t)}{t^{\frac{N}{p'(x)}}}\right]^{p(x)} \frac{dt}{t}, \tag{15}$$

when $0 < r < \frac{d}{2}$.

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P r o o f. We have

$$\int\limits_{X\setminus B(x,r)} \left(\frac{a\,[\varrho(x,y)]}{\varrho(x,y)^N}\right)^{p(x)} d\mu(y) = \sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \int\limits_{2^{k-1}r<\varrho(x,y)<2^k r} \left(\frac{a\,[\varrho(x,y)]}{\varrho(x,y)^N}\right)^{p(x)} d\mu(y)$$

$$+ \int\limits_{2^{-\theta}d<\varrho(x,z)<d} \left(\frac{a\,[\varrho(x,y)]}{\varrho(x,y)^N}\right)^{p(x)} d\mu(y) =: F_1(x,r) + F_2(x,r),$$

where $F_2(x,r) \equiv 0$ in the case $d = \infty$ and $\theta = \theta(r)$ is the same as in (12). For $F_1(x,r)$ by the almost monotonicity of $a(x)$ we obtain

$$F_1(x,r) \le C \sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \left[\frac{a\left(2^k r\right)}{\left(2^{k-1}r\right)^N}\right]^{p(x)} \left(2^k r\right)^N = C2^{Np(x)} \sum_{k=1}^{\left[\log_2 \frac{d}{r}\right]} \left[\frac{a\left(2^k r\right)}{\left(2^k r\right)^{\frac{N}{p'(x)}}}\right]^{p(x)}.$$

Then

$$F_1(x,r) \le C \int\limits_r^d \left[\frac{a(t)}{t^{\frac{N}{p'(x)}}}\right]^{p(x)} \frac{dt}{t}$$

by Lemma 4.3.

For $F_2(x,r)$ we make use of the fact that $\frac{d}{2} \le 2^{-\theta} d$ and obtain

$$F_2(x,r) \le \int\limits_{\frac{d}{2}<\varrho(x,y)<d} \left(\frac{a\,[\varrho(x,y)]}{\varrho(x,y)^N}\right)^{p(x)} d\mu(y) \le C[a(d)]^{p(x)}$$

which yields (14). The passage to (15) in the case $0 < r < \frac{d}{2}$ is obvious. □

4.2 Estimation of the variable exponent norm of the truncated generalized potentials

We are interested in the estimation of the norm

$$\beta_p = \beta_p(x,r) := \left\|\mathcal{K}(x,\cdot)\chi_{X\setminus B(x,r)}(\cdot)\right\|_{p(\cdot)} \quad \text{as} \quad r \to 0, \tag{16}$$

of the kernel $\mathcal{K}(x,y) := \dfrac{a[\varrho(x,y)]}{[\varrho(x,y)]^N}$, truncated to the exterior of the ball $B(x,r)$. We will need this estimate with $p(x)$ replaced by the conjugate exponent $p'(x)$ when making use of Hedberg's approach.

We first prove a "rough" estimate in Lemma 4.5, which will be used in Theorem 4.6 to get a more precise estimate which will suit well for our purposes.

Lemma 4.5 *Let X be a bounded upper Ahlfors N-regular measure quasimetric space and p satisfy condition (5), let $a(r) : (0,d) \to (0,+\infty)$ and let $\dfrac{a(r)}{r^N}$ be almost decreasing. Then there exists a constant $C > 0$, not depending on $x \in X$ and $r \in (0,d)$, such that*

$$\beta_p(x,r) \le Cr^{-N}a(r). \tag{17}$$

P r o o f. By the definition of the norm we have

$$\int\limits_{X\setminus B(x,r)} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu(y) = 1. \tag{18}$$

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Hence, taking into account that $(\cdots)^{p(y)} \leq (\cdots)^{p_-} + (\cdots)^{p_+}$ and that $\dfrac{a(r)}{r^N}$ is almost decreasing, we get

$$1 \leq \int\limits_{X\setminus B(x,r)} \left[\left(\frac{a[\varrho(x,y)]}{[\varrho(x,y)]^N\beta_p}\right)^{p_-} + \left(\frac{a[\varrho(x,y)]}{\varrho(x,y)^N\beta_p}\right)^{p_+}\right] d\mu\,(y)$$

$$\leq \left[\left(\frac{a(r)}{r^N\beta_p}\right)^{p_-} + \left(\frac{a\,(r)}{r^N\beta_p}\right)^{p_+}\right]\mu(X)$$

If $\frac{a(r)}{r^N\beta_p} \geq 1$, there is nothing to prove. When $\frac{a(r)}{r^N\beta_p} \leq 1$, we obtain $1 \leq 2\mu(X)\left(\frac{a(r)}{r^N\beta_p}\right)^{p_-}$, which proves the estimate. □

Theorem 4.6 *Let X be a bounded upper Ahlfors N-regular measure quasimetric space and p satisfy conditions* (5) *and (6). Also suppose that a non-negative continuous function $a(r)$ is almost increasing and $\dfrac{a(r)}{r^N}$ is almost decreasing on $(0,d]$, $d = diam(X) < \infty$. Then there exists a constant $C > 0$, not depending on $x \in X$ and $r \in (0,d)$, such that*

$$\beta_p(x,r) \leq C\left(\int\limits_r^d \left[\frac{a(t)}{t^{\frac{N}{p'(x)}}}\right]^{p(x)} \frac{dt}{t}\right)^{\frac{1}{p(x)}} + C\chi_{\left[\frac{d}{2},d\right]}(r). \tag{19}$$

Proof. Since $1 < \inf_X p'(x)$ and $\sup_X p'(x) < \infty$, the right-hand side of (19) is obviously bounded from below by a uniform constant. Therefore, it suffices to estimate the norm $\beta_p(x,r)$ when $\beta_p(x,r) \geq 1$ and $0 < r < 1$ (the latter in the case $d > 1$). From (18) we have

$$1 = \int\limits_{\substack{r<\varrho(x,y)<1 \\ \mathcal{K}(x,y)>\beta_p}} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu\,(y) + \int\limits_{\substack{r<\varrho(x,y)<1 \\ \mathcal{K}(x,y)\leq\beta_p}} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu(y)$$

$$+ \int\limits_{\varrho(x,y)>1} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu(y) = I_1 + I_2 + I_3.$$

We need to estimate I_1, I_2, I_3 from above. For I_1 we have

$$I_1 = \int\limits_{\substack{r<\varrho(x,y)<1 \\ \mathcal{K}(x,y)>\beta_p}} g_r(x,y)\left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(x)} d\mu\,(y)\,,$$

where $g_r(x,y) = \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(x)-p(y)}$. By (6) we get

$$|\ln g_r(x,y)| \leq \mathbb{A}\left|\frac{\ln\left[\mathcal{K}(x,y)\beta_p^{-1}\right]}{\ln\varrho\,(x,y)}\right| =$$

$$= \mathbb{A}\frac{\ln[\mathcal{K}(x,y)] - \ln\beta_p}{\ln\dfrac{1}{\varrho\,(x,y)}} \leq \mathbb{A}\frac{\ln\left(\dfrac{a[\varrho(x,y)]}{[\varrho(x,y)]^N}\right)}{\ln\dfrac{1}{\varrho\,(x,y)}} \leq C$$

where in the last inequality we made use of the boundedness of $a(r)$. Then

$$I_1 \leq \frac{C}{\beta_p^{p(x)}} \int\limits_{X\setminus B(x,r)} \left(\frac{a\,[\varrho(x,y)]}{\varrho\,(x,y)^N}\right)^{p(x)} d\mu\,(y)\,.$$

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Then by (14) and (15) we get

$$I_1 \leq \frac{C}{\beta_p^{p(x)}} \left(\int_r^d \left[\frac{a(t)}{t^{\frac{N}{p'(x)}}} \right]^{p(x)} \frac{dt}{t} + \chi_{\left[\frac{d}{2},d\right]}(r) \right). \tag{20}$$

For I_2 we obtain

$$I_2 \leq \int_{r<\varrho(x,y)<1} \left(\frac{\mathcal{K}(x,y)}{\beta_p} \right)^{p_-} d\mu(y) = \frac{C}{\beta_p^{p_-}} \int_{r<\varrho(x,y)<1} \left(\frac{a[\varrho(x,y)]}{[\varrho(x,y)]^N} \right)^{p_-} d\mu(y)$$

and the application of Lemma 4.4 gives

$$I_2 \leq \frac{C}{\beta_p^{p_-}} \left(\int_r^d \left[\frac{a(t)}{t^{\frac{N}{(p_-)'}}} \right]^{p_-} \frac{dt}{t} + \chi_{\left[\frac{d}{2},d\right]}(r) \right). \tag{21}$$

The estimation of I_3 is easy:

$$I_3 \leq \frac{C}{\beta_p^{p_-}} \tag{22}$$

because

$$I_3 \leq \frac{C}{\beta_p^{p_-}} \int_{\varrho(x,y)>1} \left(\frac{\frac{a[\varrho(x,y)]}{\sup_{t\in(0,d)} a(t)}}{[\varrho(x,y)]^N} \right)^{p(y)} d\mu(y) \leq \frac{C}{\beta_p^{p_-}} \int_{\varrho(x,y)>1} \left(\frac{\frac{a[\varrho(x,y)]}{\sup_{t\in(0,d)} a(t)}}{\varrho^N} \right)^{p_-} d\mu(y)$$

where the last integral is convergent and uniformly bounded with respect to x by Lemma 4.4.

Therefore, by (20), (21), (22) we have

$$1 \leq C \left[\frac{1}{\beta_p^{p(x)}} \left(\int_r^d t^{N-1} \left(\frac{a(t)}{t^N} \right)^{p(x)} dt + \chi_{\left[\frac{d}{2},d\right]}(r) \right) \right. \tag{23}$$

$$\left. + \frac{1}{\beta_p^{p_-}} \left(\int_r^d t^{N-1} \left(\frac{a(t)}{t^N} \right)^{p_-} + \chi_{\left[\frac{d}{2},d\right]}(r) \right) + \frac{1}{\beta_p^{p_-}} \right].$$

We may suppose that $\beta_p(x,r)$ only for those x, r for which $\beta_p(x,r)$ is sufficiently large: $\beta_p(x,r) \geq (2C)^{\frac{1}{p_-}}$, where C is the constant from (23) (otherwise, there is nothing to prove). For such x, r we have $\frac{C}{\beta_p^{p_-}} \leq \frac{1}{2}$ and then from (23) we obtain

$$\frac{1}{2} \leq C \int_r^d t^{N-1} \left[\left(\frac{a(t)}{\beta_p t^N} \right)^{p(x)} + \left(\frac{a(t)}{\beta_p t^N} \right)^{p_-} \right] dt + C \frac{\chi_{\left[\frac{d}{2},d\right]}(r)}{\beta_p^{p(x)}}, \tag{24}$$

where we have used the fact that $\frac{\chi_{\left[\frac{d}{2},d\right]}(r)}{\beta_p^{p_-}} \leq C \frac{\chi_{\left[\frac{d}{2},d\right]}(r)}{\beta_p^{p(x)}}$. By (17) we have

$$\left(\frac{a(t)}{\beta_p t^N} \right)^{p_-} \leq C \left(\frac{a(t)}{\beta_p t^N} \right)^{p(x)}.$$

Therefore, from (24) we get

$$1 \leq C \int_r^d t^{N-1} \left(\frac{a(t)}{\beta_p t^N} \right)^{p(x)} dt + C \frac{\chi_{\left[\frac{d}{2},d\right]}(r)}{\beta_p^{p(x)}},$$

whence (19) follows. □

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Corollary 4.7 *Let $p(x)$ and $a(r)$ satisfy the assumptions of Theorem 4.6. If $a \in \mathcal{V}_{N,p(\cdot)}$, then*

$$\beta_{p'}(x,r) \le C\frac{A(r)}{r^{\frac{N}{p(x)}}}. \tag{25}$$

4.3 An appropriate N-function

The N-function $\Phi(x,u)$ defining the Musielak-Orlicz space $L^{\Phi}(X)$ into which the generalized potential maps the variable space $L^{p(\cdot)}(X)$ is defined by the relation

$$\Phi^{-1}(x,u) = \int_r^\infty \frac{A(t)\,dt}{t^{1+\frac{N}{p(x)}}}, \quad r = u^{-\frac{1}{N}}, \tag{26}$$

where Φ^{-1} stands for the inverse function with respect to r. We always, whenever necessary, continue the function $a(t)$ as $a(d)$ for $t > d$, so that $A(t) \equiv c + a(d)\ln\frac{t}{d}$ for large $t (> d)$.

In the following two lemmas we check that $\Phi(x,r)$ is indeed an N-function and it is equivalent to $\frac{A(r)}{r^{\frac{N}{p(x)}}}$.

Lemma 4.8 *Let $p(x)$ satisfy condition* (5) *and $a(r)$ be a non-negative continuous on $[0,d], 0 < d < \infty$ function such that*

$$\int_0^d \frac{a(t)dt}{t} < \infty, \quad \int_0^d \frac{a(t)dt}{t^{1+\frac{N}{p_+}}} = \infty. \tag{27}$$

Then the function $\Phi(x,r)$ defined by its inverse (26)*, satisfies the requirements of Definition 2.1.*

P r o o f. The condition $\lim\limits_{r\to 0}\Phi(x,r) = 0 \longleftrightarrow \lim\limits_{r\to 0}\Phi^{-1}(x,r) = 0$ is obvious because of the convergence at infinity of the integral in (26) for every x. So we have only to check that $\Phi(x,r)$ is a convex function of r or equivalently $\Phi^{-1}(x,r)$ is a concave function. To this end, it suffices to check that $\frac{\partial^2}{\partial r^2}\Phi^{-1}(x,r) \le 0$. Since

$$\Phi^{-1}(x,r) = \frac{1}{N}\int_0^r A\left(t^{-\frac{1}{N}}\right) t^{\frac{1}{p(x)}-1}\,dt,$$

the inequality $\frac{\partial^2}{\partial r^2}\Phi^{-1}(x,r) \le 0$ is checked by direct verification:

$$\frac{\partial^2}{\partial r^2}\Phi^{-1}(x,r) = -\frac{1}{N^2} r^{\frac{1}{p(x)}-2}\left[A'\left(r^{-\frac{1}{N}}\right)r^{-\frac{1}{N}} + N\left(1-\frac{1}{p(x)}\right)A\left(r^{-\frac{1}{N}}\right)\right] \le 0$$

taking into account that $A(r) \ge 0, A'(r) \ge 0$. □

Lemma 4.9 *Let $p(r)$ satisfy condition* (5) *and $a(r)$ be a non-negative almost increasing continuous on $[0,d], 0 < d < \infty$ function such that the function $\frac{a(t)}{t^{\frac{N}{p_+}-\varepsilon}}$ is almost decreasing for some $\varepsilon > 0$. Then there exist constants $C_1 > 0, C_2 > 0$ not depending on x and r such that*

$$C_1\frac{A(r)}{r^{\frac{N}{p(x)}}} \le \Phi^{-1}\left(x, \frac{1}{r^N}\right) \le C_2\frac{A(r)}{r^{\frac{N}{p(x)}}}. \tag{28}$$

P r o o f. We have to prove that

$$C_1\frac{A(r)}{r^{\frac{N}{p(x)}}} \le \int_r^\infty \frac{A(t)\,dt}{t^{1+\frac{N}{p(x)}}} \le C_2\frac{A(r)}{r^{\frac{N}{p(x)}}}. \tag{29}$$

It is easily proved that the function $A(t)$ also satisfies the properties that it is almost increasing and $\frac{A(t)}{t^{\frac{N}{p_+}-\varepsilon}}$ is almost decreasing. Then the left-hand side inequality in (29) immediately follows from the fact that $A(r)$ is almost increasing and it is easily checked that from the another property of $A(t)$ there follows the right-hand side inequality. □

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4.4 The proof of Theorem 3.1

First we note that the function $\Phi(x,r)$ defined by (9) is indeed an N-function under the conditions of Theorem 3.1 on $p(x)$ and $a(r)$. This follows from Lemma 4.8, the first of the conditions in (27) being satisfied by the assumption, the second easily following from the assumption that $\frac{a(r)}{r^\lambda}$ is almost decreasing for some $\lambda < \frac{N}{p_+}$.

By the linearity of the operator I_a, it suffices to prove that $\|I_a f\|_\Phi \le C < \infty$ for $\|f\|_{p(\cdot)} \le 1$. We split $I_a f\,(x)$ in the standard way

$$I_a f\,(x) = \int\limits_{B(x,r)} \frac{a\,[\varrho(x,y)]}{\varrho\,(x,y)^N} f\,(y)\, d\mu\,(y) + \int\limits_{X\setminus B(x,r)} \frac{a\,(\varrho(x,y))}{\varrho\,(x,y)^N} f(y) d\mu(y) = \mathcal{A}_r(x) + \mathcal{B}_r(x)$$

and suppose that $f(x) \ge 0$. Since $\frac{a(t)}{t^N}$ is almost decreasing, for $\mathcal{A}_r(x)$ we have

$$\mathcal{A}_r(x) = \sum_{k=0}^{\infty} \int\limits_{2^{-k-1}r \le \varrho(x,y) < 2^{-k}r} \frac{a\,[\varrho(x,y)]}{\varrho\,(x,y)^N} f\,(y)\, d\mu\,(y) \le$$

$$\le C \sum_{k=0}^{\infty} \frac{a\left(2^{-k-1}r\right)}{\left(2^{-k-1}r\right)^N} \int\limits_{2^{-k-1}r \le \varrho(x,y) < 2^{-k}r} f\,(y)\, d\mu\,(y) \le CMf\,(x) \sum_{k=0}^{\infty} a\left(2^{-k-1}r\right) \le$$

$$\le CMf\,(x) \sum_{k=0}^{\infty} \int\limits_{2^{-k-1}r}^{2^{-k}r} \frac{a(t)}{t} dt.$$

Therefore,

$$\mathcal{A}_r(x) \le CA(r) Mf(x), \quad A(r) = \int\limits_0^r \frac{a(t)}{t}\, dt. \tag{30}$$

For $\mathcal{B}_r(x)$ by Hölder inequality for variable exponents and the condition $\|f\|_{p(\cdot)} \le 1$, we obtain

$$\mathcal{B}_r(x) \le C\, \|f\|_{p(\cdot)} \left\|\mathcal{K}(x,\cdot)\chi_{X\setminus B(x,r)}(\cdot)\right\|_{p'(\cdot)} \le C \left\|\mathcal{K}(x,\cdot)\chi_{X\setminus B(x,r)}(\cdot)\right\|_{p'(\cdot)} = C\beta_{p'}(x,r).$$

Then by Theorem 4.6

$$\mathcal{B}_r(x) \le C \left(\int\limits_r^d \left[\frac{a(t)}{t^{\frac{N}{p(x)}}}\right]^{p'(x)} \frac{dt}{t} \right)^{\frac{1}{p'(x)}} + C \frac{\chi_{[\frac{d}{2},d]}(r)}{\beta_p^{p(x)}}.$$

By Lemma 4.2, inequality (10) is applicable and we get

$$\mathcal{B}_r(x) \le C r^{-\frac{N}{p(x)}} A\,(r)\,.$$

Therefore,

$$I_a f(x) \le C \left(Mf\,(x) + r^{-\frac{N}{p(x)}} \right) A\,(r)\,.$$

Then

$$I_a f(x) \le C \left[Mf(x) r^{\frac{N}{p(x)}} + 1 \right] \Phi^{-1}\left(x, \frac{1}{r^N}\right) \tag{31}$$

by (28). Now we choose $r = [Mf(x)]^{-\frac{p(x)}{N}}$. Then the last inequality turns into

$$I_a f(x) \le C\Phi^{-1}\left(x, [Mf(x)]^{p(x)}\right)$$

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and consequently ,

$$\int_X \Phi\left(x, \frac{I_a f(x)}{C}\right) d\mu(x) \leq \int_X [Mf(x)]^{p(x)} d\mu(x) \leq 1,$$

where we have used (3) and the fact that $\|f\|_{p(\cdot)} \leq 1$. Hence

$$\|I_a f\|_\Phi \leq C, \tag{32}$$

which completes the proof.

5 Reformulation of assumptions on the function $a(r)$ in terms of its upper Matuszewska-Orlicz index

In Theorem 3.1, the main assumptions on the non-negative function $a(r)$ were that it was almost increasing and the function $\frac{a(r)}{r^\lambda}$ is almost decreasing for some $\lambda < \frac{N}{p_+}$. It is known that the property of a function to be almost decreasing is closely related to the notion of the so called Matuszewska-Orlicz indices. We refer to [18], [28], [29] (p.20), [30], [31], [39], [41] for the properties of the indices of such a type. For a function $a \in W_0$, the numbers

$$m(a) = \sup_{0<x<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{a(hx)}{a(h)}\right)}{\ln x} = \lim_{x\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{a(hx)}{a(h)}\right)}{\ln x}$$

and

$$M(a) = \sup_{x>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{a(hx)}{a(h)}\right)}{\ln x} = \lim_{x\to \infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{a(hx)}{a(h)}\right)}{\ln x}$$

are known as *the Matuszewska-Orlicz type lower and upper indices* of the function $a(r)$. Note that in this definition $a(r)$ need not to be an N-function: only its behaviour at the origin is of importance. Observe that $0 \leq m(a) \leq M(a) \leq \infty$ for $a \in W_0$.

The following statement is known.

Lemma 5.1 *The index $M(a)$ of a function $a \in W_0$ is finite if and only if there exists a number $\gamma > 0$ and a constant $C > 0$ such that*

$$\int_r^\ell \frac{a(t)\, dt}{t^{1+\gamma}} \leq C \frac{a(r)}{r^\gamma} \tag{33}$$

and in this case $M(a) < \gamma$.

Lemma 5.1 follows from the known facts, see [40], p. 100, or [18], Theorem 3.2. We will make use of the following important property of the upper index $M(a)$: in the case where $M(a)$ is finite, it coincides with the infinum of all those δ for which the function $\frac{a(r)}{r^\delta}$ is almost decreasing:

$$M(a) = \inf\left\{\delta > 0 : \ \frac{a(x)}{x^\delta} \quad \text{is almost decreasing}\right\}, \tag{34}$$

see [40], p. 100, or [18], p. 448. (In [40], [18] property (34) is formulated for functions in W_0 which satisfy Zygmund type condition (33) with some $\gamma > 0$, which always holds when $M(a)$ is finite, according to Lemma 5.1).

In view of (34), the main result given in Theorem 3.1, may be reformulated as follows

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Theorem 5.2 *Let X be a bounded quasimetric measure space with doubling measure, upper Ahlfors N-regular, let the exponent $p(x)$ satisfy conditions* (5) *and* (6). *The operator I_a is bounded from $L^{p(\cdot)}(X)$ into the Orlicz-Musielak space $L^{\Phi}(X)$, if the function $a(r)$ satisfies condition* (7) *and its upper Matuszewska-Orlicz index is less than $\frac{N}{p_+}$:*

$$M(a) < \frac{N}{p_+}.$$

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