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# SOME NEW STEIN AND HARDY TYPE INEQUALITIES

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*We prove a generalization of the pointwise Stein inequality, considering its two truncated versions. More generally than in the Stein inequality, we assume that the kernel is dominated by a radial function almost decreasing after the division by a power function with nonnegative exponent in the case of the truncation to the ball of the radius $x$, and almost increasing after the multiplication by a power function in the case of truncation to the exterior of this ball. We give some applications to a series of inequalities of Hardy type in norms of various function spaces, in particular, in the norm of variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with weights. Bibliography: 40 titles.*

## 1. Introduction

Among the inequalities bearing the names of Hardy and Littlewood, there are the following ones:

$$\int\limits_0^\infty \left| \frac{1}{x^\alpha} \int\limits_0^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}} \right|^p dx \leqslant C_1^p \int\limits_0^\infty |f(x)|^p\,dx \tag{1.1}$$

with $1 < p \leqslant \infty$, $\alpha > 0$, and

$$\int\limits_0^\infty \left| \frac{1}{x^\alpha} \int\limits_x^\infty \frac{f(t)\,dt}{(t-x)^{1-\alpha}} \right|^p dx \leqslant C_2^p \int\limits_0^\infty |f(x)|^p\,dx, \tag{1.2}$$

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with $1 \leqslant p < \infty$, $0 < \alpha < \frac{1}{p}$, where

$$C_1 = B\left(\alpha, \frac{1}{p'}\right), \quad C_2 = \Gamma(\alpha)\Gamma\left(\frac{1}{p} - \alpha\right)$$

are the best constants (cf. [**1**] and also [**2**]). In the case $\alpha = 1$, (1.1) coincides with the Hardy inequality, $C_1 = \frac{p}{p-1}$ in this case (cf. [**3**] and also, for example, [**2**]). The dramatic more than 10 years period of research until Hardy proved (1.1) with $\alpha = 1$ in his paper [**3**] of 1925, was recently described in detail in [**4**]. After that period Hardy inequalities have been generalized and applied in an almost unbelievable way (cf., for example, the monographs [**2**], [**5**]–[**8**] and the references therein). For instance, a characterization of the weights $u = u(x)$ and $v = v(x)$ for which the weighted Hardy inequality

$$\left(\int_0^\infty (Hf(x))^q \, dx\right)^{\frac{1}{q}} \leqslant C_2^p \left(\int_0^\infty |f(x)|^p \, dx\right)^{\frac{1}{p}}, \quad Hf(x) = \int_0^x f(t)\, dt, \tag{1.3}$$

is known for all $1 \leqslant p < \infty$, $0 < q < \infty$. When the Hardy operator $H$ is replaced by a more general operator

$$H_{k,v,w} f(x) := w(x) \int_0^x k(x,t) v(t) f(t)\, dt,$$

much less is known. Here, $k(x,t)$ is a kernel, i.e., $k(x,t)$ is locally integrable in $t$ and $k(x,t) \geqslant 0$. In this case, some characterizations of weights, for which the corresponding Hardy type inequality of the form (1.3) holds, are known, in fact, only for the Oinarov kernels defined by the condition that there exists $C > 1$ such that

$$\frac{1}{C} k(x,t) \leqslant k(x,z) + k(x,t) \leqslant C k(x,t)$$

for all $0 < t \leqslant z \leqslant x < \infty$ (cf., for example, [**8**, Section 2.3] and [**9**] and the references given there).

Moreover, for the multidimensional case there exist very few characterizations of weights for Hardy type inequalities of the form (1.3) to hold (cf., for example, [**8**], the recent review article [**10**] and the references therein).

Note that some Hardy inequalities of the form (1.3) are known for some spaces other than weighted Lebesgue spaces, for example, for Orlicz, Lorentz, rearrangement invariant spaces, Morrey spaces and even general Banach function spaces (cf., for example, [**11, 10**] and the references therein).

In this paper, we prove some new Hardy type inequalities, but we cannot obtain necessary and sufficient conditions of their validity. More exactly, we prove some new generalizations of the inequalities (1.1) and (1.2) by considering the following multidimensional Hardy–Littlewood type operators with a radial kernel defined by

$$k = k(t) \geqslant 0, \quad t \in \mathbb{R}^1_+$$

as follows:

$$\mathbb{H}_\mu f(x) := u(x) \int\limits_{|y|<\mu(x)} k(|x-y|)\frac{f(y)\,dy}{v(y)}, \qquad x \in \mathbb{R}^n, \tag{1.4}$$

$$\mathcal{H}_\nu f(x) := u(x) \int\limits_{|y|>\nu(x)} k(|x-y|)\frac{f(y)\,dy}{v(y)}, \qquad x \in \mathbb{R}^n, \tag{1.5}$$

where $\mu(x)$ and $\nu(x)$ are arbitrary functions with the only restriction that

$$\mu(x) \geqslant 0, \quad \nu(x) > |x| \text{ a.e.}$$

and $u$ and $v$ are weight functions, within the frameworks of Banach Function spaces, in particular, for the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Observe that one may admit $\nu(x) = |x|$ on a set of positive measure by imposing a stronger condition on the kernel (cf. Remark 3.2).

Since

$$\left| \int\limits_{|y|<\mu(x)} \cdots dy \right| \leqslant \int\limits_{|x-y|<|x|+\mu(x)} |\cdots| dy, \tag{1.6}$$

integrals of such a type admit a pointwise estimate by the Hardy–Littlewood maximal function in view of the *Stein pointwise inequality*

$$\left| \frac{1}{\varepsilon^n} \int\limits_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy \right| \leqslant CMf(x), \tag{1.7}$$

where $\varepsilon > 0$ and

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\,dy \tag{1.8}$$

is the maximal operator, under the condition that the least radial decreasing dominant

$$k_0(|x|) := \sup_{|y|>|x|} |k(y)|$$

of the kernel is integrable:

$$\int\limits_{\mathbb{R}^n} |k_0(|x|)| dx < \infty,$$

and then $C = \|k_0\|_1$ (cf. [**12**]).

In the pointwise estimate (1.7), the parameter $\varepsilon$ may depend on $x$. Under the choice

$$k(x) = \begin{cases} |x|^{\alpha-n}, & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases}$$

and $\varepsilon = |x|$, in particular, we have the pointwise estimate of the Hardy–Littlewood operator

$$\left| \frac{1}{|x|^\alpha} \int\limits_{|y|<|x|} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right| \leqslant \frac{|S^{n-1}|}{n\alpha} Mf(x), \tag{1.9}$$

where

$$|S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

is the area of the unit sphere in $\mathbb{R}^n$.

In this paper, we prove that the Stein type pointwise inequality is valid under weaker assumptions on the kernel (cf. Corollary 3.1).

This follows from our main Theorems 3.1 and 3.2 on pointwise Stein type inequalities which, together with the known results for the maximal operator, imply new results concerning multidimensional Hardy inequalities even in the case of variable $L^{p(\cdot)}$-spaces and with the general operators (1.4) and (1.5) involved.

We admit kernels which are

(a) almost decreasing after division by a power function in the case of truncated convolutions (1.4), $0 \leqslant \mu(x) < \infty$ a.e. in this case,

(b) almost increasing after the multiplication by a power function in the case of truncated convolutions (1.5), $\nu(x) > |x|$ a.e. in this case.

We may admit $\nu(x) = |x|$ on a set of positive measure if the kernel $k$ satisfies both above conditions (a) and (b) (cf. Remark 3.2).

The paper is organized as follows. In order not to disturb our discussions later on, some preliminaries are presented in Section 2. The announced Stein type inequalities are proved in Section 3. Some preliminaries for applications in Section 4 can be found in Subsection 4.1.1 and the new Hardy type inequalities are presented and proved in Subsection 4.2.

## 2. Preliminaries

Denote by $L^1_{loc}(\mathbb{R}^1_+)$ the class of functions integrable over every finite interval $(0, a)$, $a < \infty$.

Let $n \in \mathbb{Z}_+$, and let $w(x)$ be a weight function on $\mathbb{R}^n$. As is known [**13**], the weighted maximal operator $wM\frac{1}{w}$ is bounded in the space $L^p(\mathbb{R}^n), 1 < p < \infty$, if and only if $w^p$ is in the Muckenhoupt class $A_p$, i.e.,

$$\sup_B \left( \fint_B w^{\frac{1}{p}}(y)dy \right)^{\frac{1}{p}} \left( \fint_B w^{-\frac{1}{p'}}(y)dy \right)^{\frac{1}{p'}} < \infty, \tag{2.1}$$

where

$$\fint_B \cdots dy = \frac{1}{|B|} \int\limits_B \cdots dy,$$

$B$ is a ball in $\mathbb{R}^n$, and $p' = \frac{p}{p-1}$.

**2.1. Assumptions on the kernel.** A nonnegative function $k(t)$ on $\mathbb{R}^1_+$ is called *almost increasing* if

$$k(\tau) \leqslant Ck(t) \quad \text{for } \tau \leqslant t,$$

and *almost decreasing* if

$$g(t) \leqslant Cg(\tau) \quad \text{for } \tau \leqslant t.$$

These notions go back to S. Bernstein [**14**].

As an example, we just mention that the function

$$k(t) = t^a \ln\left(C + \frac{1}{t}\right), \quad a > 0,\ C > 0$$

is almost increasing (increasing when $C$ is large enough), so that to include functions with the behavior of type

$$k(t) = t^a \ln^b \frac{1}{t}, t^a \ln^b \ln\left(\frac{1}{t}\right)$$

etc. near the origin, the notion of the almost monotonicity becomes more appropriate.

**Definition 2.1.** We denote by $W_\beta = W_\beta(\mathbb{R}^1_+)$ and $W^\gamma = W^\gamma(\mathbb{R}^1_+)$, $\beta \geqslant 0$, $\gamma \geqslant 0$, the set of all nonnegative functions $k(t)$ such that $\frac{k(t)}{t^\beta}$ is almost decreasing on $\mathbb{R}^1_+$, and $t^\gamma k(t)$ is almost increasing on $\mathbb{R}^1_+$ respectively.

Note that functions which become almost monotonic after the multiplication by a power function are well known in the literature. For example, in an implicit form such functions were used already in [**15**] and in the case of the usual monotonicity classes of such functions were first introduced in connection with some problems in the theory of Fourier series and the interpolation theory in [**16**] and [**17**] respectively. In the case of almost monotonicity, we also refer to the study of properties of the classes of $W_\beta = W_\beta(\mathbb{R}^1_+)$ and $W^\gamma = W^\gamma(\mathbb{R}^1_+)$, in particular, relations between $\beta$ and $\gamma$ and Matuszewska–Orlicz indices, in [**18**] and [**19**].

We use the notation

$$\mathcal{C}_\beta(k) := \sup_{0<\tau<t<\infty} \left(\frac{\tau}{t}\right)^\beta \frac{k(t)}{k(\tau)} \quad \text{for } k \in W_\beta,$$

$$\mathcal{D}^\gamma(k) := \sup_{0<\tau<t<\infty} \left(\frac{\tau}{t}\right)^\gamma \frac{k(\tau)}{k(t)} \quad \text{for } k \in W^\gamma.$$

Everywhere in the sequel, we assume that the kernel $k(t)$, $0 < t < \infty$, in (1.4) is nonnegative and satisfies the conditions

$$k(t)t^{n-1} \in L^1_{loc}(\mathbb{R}^1_+), \tag{2.2}$$

$$k \in W_\beta(\mathbb{R}^1_+) \quad \text{for some } \beta \geqslant 0, \tag{2.3}$$

in the case of operators of the form (1.4), and

$$\int_\delta^\infty k(t)t^{n-1} < \infty, \tag{2.4}$$

for every $\delta > 0$, and

$$k \in W^{n+\gamma}(\mathbb{R}^1_+) \quad \text{for some } \gamma \geqslant 0, \tag{2.5}$$

in the case of operators of the form (1.5)

**Lemma 2.1.** *Let $\lambda > 1$ and $\delta < t < \lambda\delta$, where $0 < \delta < \infty$. Then*

$$\frac{1}{\lambda^\beta \mathcal{C}_\beta(k)} k(\lambda\delta) \leqslant k(t) \leqslant \lambda^\beta \mathcal{C}_\beta(k) k(\delta) \quad \textit{for } k \in W_\beta \tag{2.6}$$

*and*

$$\frac{1}{\lambda^\gamma \mathcal{D}^\gamma(k)} k(\delta) \leqslant k(t) \leqslant \lambda^\gamma \mathcal{D}^\gamma(k) k(\lambda\delta) \quad \textit{for } k \in W^\gamma. \tag{2.7}$$

The proof of Lemma 2.1 follows almost immediately by using the definitions above, so we omit the details.

## 3. Some Stein Type Inequalities

Keeping (1.6) in mind, everywhere in the sequel we use the notation

$$\mu_0(x) = \mu(x) + |x|$$

and

$$\nu_0(x) = \nu(x) - |x|.$$

Our first main result reads as follows.

**Theorem 3.1.** *Let the kernel $k$ satisfy the conditions* (2.2) *and* (2.3), *and let $\ell(x)$ be a measurable function on $\mathbb{R}^n$ satisfying the condition*

$$\mathcal{A}(k, \ell) := \sup_x |\ell(x)| \int_0^{\mu_0(x)} k(t) t^{n-1} dt < \infty. \tag{3.1}$$

*Then the following Stein type pointwise inequality holds*:

$$\left| \ell(x) \int_{|y| < \mu(x)} k(|x-y|) f(y)\, dy \right| \leqslant C_0 M f(x), \tag{3.2}$$

*where*

$$C_0 = |S^{n-1}| a(\beta) \mathcal{C}_\beta^2(k) \mathcal{A}(k, \ell)$$

*and*

$$a(\beta) = 2\left(1 + \frac{\beta}{n}\right) \left(\frac{2 + \frac{\beta}{n}}{1 + \frac{\beta}{n}}\right)^{\frac{2\beta}{n}}.$$

**Proof.** We have

$$\left| \int\limits_{|y|<\mu(x)} k(|x-y|)f(y)\,dy \right| = \left| \int\limits_{|z-x|<\mu(x)} k(|z|)f(x-z)\,dz \right|$$
$$\leqslant \int\limits_{|z|<\mu_0(x)} k(|z|)|f(x-z)|\,dz. \tag{3.3}$$

To prove (3.2), we represent the last integral as

$$\int\limits_{|z|<2\mu_0(x)} k(|z|)|f(x-z)|\,dz = \sum_{m=1}^{\infty} \int\limits_{\lambda^{-m}\mu_0(x)<|z|<\lambda^{-m+1}\mu_0(x)} k(|z|)|f(x-z)|\,dz,$$

where $\lambda > 1$. Observe that we use the decomposition with the parameter $\lambda$ instead of the binary decomposition in order to optimize the arising constant; in this relation we follow [**20**].

Making use of (2.6), we get

$$\int\limits_{|z|<\mu_0(x)} k(|z|)|f(x-z)|\,dz \leqslant \lambda^{\beta}\mathcal{D}_{\beta}(k) \sum_{m=1}^{\infty} k\left(\lambda^{-m}\mu_0(x)\right) \int\limits_{|z|<\lambda^{-m+1}\mu_0(x)} |f(x-z)|\,dz. \tag{3.4}$$

Consequently,

$$\int\limits_{|z|<\mu_0(x)} k(|z|)|f(x-z)|\,dz \leqslant A(\mu_0(x))(Mf)(x) \tag{3.5}$$

with

$$A(r) := \lambda^{\beta}\mathcal{D}_{\beta}(k) \sum_{m=1}^{\infty} k\left(\lambda^{-m}r\right)|B(0,\lambda^{-m+1}r)|, \quad r = \mu_0(x).$$

The function $A(r)$, $0 < r < \infty$, may be estimated via the integral which appeared in (3.1). To this end, we proceed as follows:

$$\int\limits_0^r k(t)t^{n-1}\,dt = \sum_{m=1}^{\infty} \int\limits_{\lambda^{-m}r}^{\lambda^{-m+1}r} k(t)t^{n-1}\,dt \geqslant \frac{1}{\lambda^{\beta}\mathcal{C}_{\beta}(k)} \sum_{m=1}^{\infty} k(\lambda^{-m+1}r) \int\limits_{\lambda^{-m}r}^{\lambda^{-m+1}r} t^{n-1}\,dt$$

where we made use of the left-hand side inequality in (2.6). Hence

$$\int_0^r k(t)t^{n-1}\,dt \geqslant \frac{\lambda^n-1}{n}\sum_{m=1}^{\infty} k\left(\lambda^{-m+1}r\right)\lambda^{-mn}$$
$$\geqslant \frac{r^n}{\lambda^\beta \mathcal{C}_\beta(k)}\frac{\lambda^n-1}{n\lambda^n}\sum_{m=1}^{\infty} k\left(\lambda^{-m}r\right)\lambda^{-mn}$$
$$= \frac{\lambda^n-1}{\lambda^{2n+2\beta}|S^{n-1}|\mathcal{C}_\beta^2(k)}A(r),$$

so that

$$A(r) \leqslant \frac{|S^{n-1}|\mathcal{C}_\beta^2(k)\lambda^{2n+2\beta}}{\lambda^n-1}\int_0^r k(t)t^{n-1}\,dt. \tag{3.6}$$

This is minimized at

$$\lambda^n = \frac{2n+2\beta}{n+2\beta},$$

and then from (3.5) we have

$$\int\limits_{|z|<\mu_0(x)} k(|z|)|f(x-z)|\,dz \leqslant |S^{n-1}|a(\beta)\mathcal{C}_\beta^2(k)\int_0^{\mu_0(x)} k(t)t^{n-1}\,dt\ (Mf)(x),$$

which yields (3.2) under the condition (3.1). The proof is complete. □

**Remark 3.1.** Note that for the constant $a(\beta)$ we have

$$a(\beta) \sim \frac{2e^2}{n}\beta \quad \text{as } \beta\to\infty$$

which, in particular, shows that the constant in the estimate tends to infinity as $\beta\to\infty$, when the kernel $k(t)$ loses the property to be almost decreasing after the division by a power function.

The corresponding version of Theorem 3.1 adjusted for the operator $\mathcal{H}_\nu$ reads as follows.

**Theorem 3.2.** *Let $\nu(x) > |x|$ a.e., let the kernel $k$ satisfy the conditions* (2.4) *and* (2.5), *and let $\ell(x)$ be a measurable function on $\mathbb{R}^n$ satisfying the condition*

$$\mathcal{B}(k,\ell) := \left|\sup_x \ell(x)\right|\int_{\nu_0(x)}^{\infty} k(t)t^{n-1}dt < \infty. \tag{3.7}$$

*Then the following pointwise inequality holds*:

$$\left|\ell(x)\int\limits_{|y|>\nu(x)} k(|x-y|)f(y)\,dy\right| \leqslant C_0 Mf(x), \tag{3.8}$$

*where*

$$C_0 = |S^{n-1}| b(\gamma) [\mathcal{D}^{n+\gamma}(k)]^2 \mathcal{B}(k, \ell)$$

*and*

$$b(\gamma) = \left(1 + \frac{\gamma}{n}\right) \left(\frac{2 + \frac{\gamma}{n}}{1 + \frac{\gamma}{n}}\right)^{\frac{2\gamma}{n}}.$$

**Proof.** The proof of Theorem 3.2 is similar to that of Theorem 3.1, so that we omit the details of this proof, but give some inequalities of the main steps. It is obvious that

$$\left| \int\limits_{|y|>\nu(x)} k(|x-y|) f(y)\, dy \right| \leqslant \int\limits_{|z|>\nu_0(x)} k(|z|) |f(x-z)|\, dz. \tag{3.9}$$

Via the corresponding decompositions as in (3.4), we find

$$\left| \int\limits_{|y|>\nu(x)} k(|x-y|) f(y)\, dy \right| \leqslant \mathcal{B}(\nu_0(x)) Mf(x),$$

where

$$\mathcal{B}(r) = |B(0,1)| \lambda^{n+\gamma} \mathcal{D}^{n+\gamma}(k) \sum_{m=0}^{\infty} (\lambda^{m+1} r)^n k(r\lambda^{m+1})$$
$$\leqslant \frac{|S^{n-1}| \lambda^{2n+2\gamma} [\mathcal{D}^{\gamma}(k)]^2}{\lambda^n - 1} \int\limits_r^{\infty} k(t) t^{n-1} dt, \quad r = \nu_0(x),$$

compare with (3.6). By now minimizing the estimate with respect to $\lambda$, we obtain (3.8). □

**Remark 3.2.** In the case $\nu(x) = |x|$ on a set of positive measure, a similar inequality holds if the kernel $k$ satisfies both conditions (2.3) and (2.5). To see this, it suffices to split the integral on the right-hand side of (3.9) as

$$\int\limits_{|y|>\nu_0(x)} \leqslant \int\limits_{|y|<\delta+\nu_0(x)} + \int\limits_{|y|>\delta+\nu_0(x)}, \delta > 0,$$

and use Theorems 3.1 and 3.2. We do not dwell on the estimation of the arising constant in this case.

From Theorems 3.1 and 3.2 we obtain, as a consequence, the following version of the Stein type inequality.

**Corollary 3.1.** *Let*

$$\varkappa(k) := \int\limits_0^{\infty} k(t) t^{n-1} dt < \infty,$$

*and let*

$$k \in W_\beta(\mathbb{R}^1_+) \cap W^{n+\gamma}(\mathbb{R}^1_+), \quad \beta \geqslant 0,\ \gamma \geqslant 0.$$

*Then*

$$\left|\frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} k\left(\frac{|x-y|}{\varepsilon}\right) f(y)\,dy\right| \leqslant C\varkappa(k)\,Mf(x), \tag{3.10}$$

*where $C > 0$ depends on $\beta$ and $\gamma$, but is independent of the function $f$, the kernel $k$, and $\varepsilon > 0$.*

**Proof.** Let first $\varepsilon = 1$. Split the integral

$$\int\limits_{\mathbb{R}^n} k(|z|)f(x-z)\,dz = \int\limits_{|z|<2|x|} k(|z|)f(x-z)\,dz + \int\limits_{|z|>2|x|} k(|z|)f(x-z)\,dz$$

and apply the estimates obtained in the proofs of Theorems 3.1 and 3.2, with $\ell(x) \equiv 1$. The arising constant has the form $C\varkappa(k)$, where $C$ depends only on $\beta$ and $\gamma$. To cover the case of $\varepsilon > 0$, it suffices to denote

$$k_1(x) = \frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right),$$

make the change of variables, and observe that $\varkappa(k_1) = \varkappa(k)$. □

**Remark 3.3.** The corresponding "dilation–invariant" versions of the inequalities (3.2) and (3.8) have the following forms:

$$\left|\ell\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon^n}\int\limits_{|y|<\varepsilon\mu\left(\frac{x}{\varepsilon}\right)} k\left(\frac{|x-y|}{\varepsilon}\right) f(y)\,dy\right| \leqslant C_0 Mf(x), \tag{3.11}$$

$$\left|\ell\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon^n}\int\limits_{|y|>\varepsilon\nu\left(\frac{x}{\varepsilon}\right)} k\left(\frac{|x-y|}{\varepsilon}\right) f(y)\,dy\right| \leqslant C_0 Mf(x) \tag{3.12}$$

under the same conditions (3.1) and (3.7), respectively. To check this, it suffices to observe that

$$Mf_\varepsilon\left(\frac{x}{\varepsilon}\right) \equiv Mf(x),$$

where

$$f_\varepsilon(y) = f(\varepsilon y).$$

From Theorems 3.1 and 3.2 we also get the following corollary, in which we use the notation

$$\mathfrak{A}\left(k, \frac{v}{w}\right) := \sup_{x\in\mathbb{R}^n} \frac{v(x)}{w(x)} \int\limits_0^{\mu_0(x)} k(t)t^{n-1}dt, \tag{3.13}$$

$$\mathfrak{B}\left(k, \frac{v}{w}\right) := \sup_{x\in\mathbb{R}^n} \frac{v(x)}{w(x)} \int\limits_{\nu_0(x)}^{\infty} k(t)t^{n-1}dt, \tag{3.14}$$

where $u$ and $w$ are weight functions, and the Banach function space $X$ is understood in the sense of [**21**].

**Corollary 3.2.** *Let $u$ and $w$ be weight functions, and let $X = X(\mathbb{R}^n)$ be an arbitrary Banach function space, in which the weighted maximal operator $wM\frac{1}{w}$ is bounded. Then*

$$\left\| v(x) \int\limits_{|y|<\mu(x)} k(|x-y|)\frac{f(y)}{w(y)}\,dy \right\|_X \leqslant A\,\|f\|_X \tag{3.15}$$

*if the kernel $k$ satisfies the conditions* (2.2) *and* (2.3), *and* $\mathfrak{A}\left(k,\frac{v}{w}\right) < \infty$, *and*

$$\left\| v(x) \int\limits_{|y|>\nu(x)} k(|x-y|)\frac{f(y)}{w(y)}\,dy \right\|_X \leqslant A\,\|f\|_X \tag{3.16}$$

*if the kernel $k$ satisfies the conditions* (2.4) *and* (2.5) *and* $\mathfrak{B}\left(k,\frac{v}{w}\right) < \infty$.

*Under these conditions, one may take*

$$A = |S^{n-1}|a(\beta)\mathcal{C}_\beta^2(k)\mathfrak{A}\left(k,\frac{v}{w}\right)\|M\|_{X\to X} \quad \textit{in } (3.15)$$

*and*

$$A = |S^{n-1}|b(\gamma)[\mathcal{D}^\gamma(k)]^2\mathfrak{B}\left(k,\frac{v}{w}\right)\|M\|_{X\to X} \quad \textit{in } (3.16).$$

**Proof.** Apply Theorems 3.1 and 3.2. □

## 4. Applications

Corollary 3.2 admits various useful consequences under a concrete choice of the Banach function space $X$. We first give, in Subsection 4.1, some preliminaries on the basics of the spaces we use, after which we formulate the corresponding corollaries in Subsection 4.2.

### 4.1. Preliminaries.

4.1.1. *On variable exponent Lebesgue spaces.* We refer to the paper [**22**] for the basics on the variable exponent Lebesgue spaces and the surveying papers [**23**]–[**26**], but give necessary definitions and some statements.

Let $p$ be a measurable function on $\mathbb{R}^n$ such that $p : \mathbb{R}^n \to (1,\infty)$, $n \geqslant 1$. The generalized Lebesgue space with variable exponent is defined via the modular

$$I^p(f) := \int\limits_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \tag{4.1}$$

by the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I^p\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

We denote by $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$, where $\varrho(x) \geqslant 0$, the weighted Banach space of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^n} \left|\frac{\varrho(x)f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leqslant 1\right\} < \infty. \tag{4.2}$$

We denote

$$p'(x) = \frac{p(x)}{p(x)-1}.$$

In what follows, we assume that $p(x)$ satisfies the conditions

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \leqslant \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty, \tag{4.3}$$

$$|p(x)-p(y)| \leqslant \frac{A}{ln\ \frac{1}{|x-y|}}, \quad x,y\in\mathbb{R}^n, \quad |x-y| \leqslant \frac{1}{2}, \tag{4.4}$$

and

$$|p(x)-p(\infty)| \leqslant \frac{A}{ln\ (2+\mu_0(x))}, \quad x\in\mathbb{R}^n. \tag{4.5}$$

We denote by $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ the class of exponents $p$ satisfying the conditions (4.3), (4.5).

The boundedness of the maximal operator in variable exponent Lebesgue spaces was first proved by Diening [**27, 28**] for bounded domains in $\mathbb{R}^n$ under the conditions (4.3)–(4.4). The result for the whole space $\mathbb{R}^n$ reads as follows (cf. [**29**]).

**Theorem 4.1.** *Let $p \in \mathcal{P}$. Then the maximal operator* (1.8) *is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.*

For power weights the following statement is valid.

**Theorem 4.2.** *Let $p \in \mathcal{P}$, and let $p(x) \equiv p_\infty = const$ outside some large ball. Then the maximal operator $M$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ with the weight*

$$\varrho(x) = (1+\mu(x))^{\beta} \prod_{k=1}^{m} |x-x_k|^{\beta_k}, \quad x_k \in \mathbb{R}^n, \tag{4.6}$$

*if and only if*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \quad k=1,\dots,m, \quad \textit{and} \quad -\frac{n}{p_\infty} < \beta + \sum_{k=1}^{m} \beta_k < \frac{n}{p'_\infty}. \tag{4.7}$$

Theorem 4.2 was proved in [**30**] in the one-dimensional case (on infinite Carleson curves), but the proof remains the same for $\mathbb{R}^n$ (cf. also its proof for bounded sets in the general setting of measure metric spaces in [**31**]).

Recently, in [**32**] an analog of the Muckenhoupt result for the maximal operator was obtained. It uses the Muckenhoupt type class

$$A_{p(\cdot)} := \left\{ w : \sup_B \frac{1}{|B|^{p_B}} \|w\|_{L^1(B)} \left\| \frac{1}{w} \right\|_{L^{\frac{1}{p(\cdot)-1}}(B)} > \infty \right\}, \tag{4.8}$$

where

$$p_B = \left( \fint_B \frac{dx}{p(x)} \right)^{-1}$$

and the supremum is taken with respect to all balls in $\mathbb{R}^n$, and reads as follows.

**Theorem 4.3.** *Let $p \in \mathcal{P}$. Then the weighted maximal operator is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ if and only if $\varrho^{p(\cdot)} \in A_{p(\cdot)}$.*

**Corollary 4.1.** *Weights $\varrho$ of the form* (4.6), (4.7) *satisfy the condition $\varrho^{p(\cdot)} \in A_{p(\cdot)}$.*

**Proof.** The proof is indirect: the inclusion $\varrho^{p(\cdot)} \in A_{p(\cdot)}$ for weight (4.6) is a consequence of the fact that the conditions (4.7) are necessary for the boundedness of the maximal operator in the weighted space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ with such a weight, by Theorem 4.2. □

4.1.2. *On variable exponent Morrey spaces.* For Morrey spaces $\mathcal{L}^{p,\lambda}$ with constant parameters $p$ and $\lambda$, $1 \leqslant p < \infty$, $0 < \lambda < n$, introduced in [**33**] in relation to the study of partial differential equations, we refer, for example, to the book [**7**]. Such spaces with variable parameters $p(x)$ and $\lambda(x)$ were studied in [**34**]–[**37**]. Let $\lambda(\cdot)$ be a measurable function on $\Omega$ with values in $[0, n]$. The variable exponent Morrey space $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is introduced as the set of all locally integrable functions $f$ such that

$$I_{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in\Omega,\, r>0} r^{-\lambda(x)} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy < \infty. \tag{4.9}$$

The norm in the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ is introduced in the form

$$\|f\| := \inf\left\{\eta > 0 :\ I_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \leqslant 1\right\} = \sup_{x\in\Omega,\, r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\, \chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)},$$

the last equality being proved in [**34**].

The boundedness of the maximal operator in Morrey spaces in the case of constant exponents $p$ and $\lambda$ was proved in [**38**]. The corresponding result for variable exponents was proved in [**34**] in the case of bounded domains in $\mathbb{R}^n$. Hästö [**32**] extended this result to the whole space $\mathbb{R}^n$ by means of his "local-to-global" trick. The final result runs as follows.

**Theorem 4.4.** *Let $p \in \mathcal{P}$, and let $0 \leqslant \lambda(x) \leqslant \lambda_+ < n$. Then the maximal operator is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.*

**4.2. Some new Hardy type inequalities via Corollary 3.2.** We give some applications of the Hardy–Littlewood type inequality (3.15) of Corollary 3.2.

Our first application will be to the case of the classical Lebesgue spaces.

**Corollary 4.2.** *Let the kernel $k$ satisfy the conditions* (2.3) *and* (2.2). *Then the inequality* (3.15) *with $X = L^p(\mathbb{R}^n), 1 < p < \infty$, holds under the conditions* (3.13) *and* (2.1).

**Remark 4.1.** A one-dimensional nonweighted version close to the statement of Corollary 4.2 was given in [**39**]. Also, in the one-dimensional case this result does not follow from the well-known characterization in the literature (cf. for example, [**8**, Section 2.3] and [**9**], where, in particular, the kernels in the integral operator satisfy the Oinarov condition).

Of more interest is the next corollary for the variable exponent spaces. We start with the nonweighted case.

**Corollary 4.3.** *Let the kernel $k$ satisfy the conditions* (2.3) *and* (2.2)*. If $p \in \mathcal{P}(\mathbb{R}^n)$, then the inequality*

$$\left\| \ell(x) \int\limits_{|y|<\mu(x)} k(|x-y|) f(y)\, dy \right\|_{p(\cdot)} \leqslant A \, \|f\|_{p(\cdot)} \tag{4.10}$$

*holds for any measurable function $\ell(x)$ satisfying the condition* (3.1)*.*

**Proof.** Apply Corollary 3.2 with

$$X = L^{p(\cdot)}(\mathbb{R}^n), \quad w(x) \equiv 1, \quad u(x) = \ell(x)$$

and Theorem 4.1. □

Note that Corollary 4.4 is a particular case of what is stated in Corollary 4.5, but we single out this case by the reasons given in Remark 4.2.

**Corollary 4.4.** *Let the kernel $k$ satisfy the assumptions* (2.3)*,* (2.2)*, and let $p(x)$ fulfill the conditions* (4.3)*,* (4.4) *and be constant outside some large ball. Then the inequality*

$$\left\| \ell(x)\varrho(x) \int\limits_{|y|<\mu(x)} k(|x-y|) \frac{f(y)}{\varrho(y)}\, dy \right\|_{p(\cdot)} \leqslant A \, \|f\|_{p(\cdot)} \,, \tag{4.11}$$

*holds for any measurable function $\ell(x)$ satisfying the condition* (3.1) *and weight $\varrho$ of the form* (4.6) *with the conditions* (4.7)*.*

**Proof.** Apply Corollary 3.2 with

$$X = L^{p(\cdot)}(\mathbb{R}^n), \quad w(x) \equiv \varrho(x), \quad u(x) = \ell(x)\varrho(x)$$

and Theorem 4.2. □

**Corollary 4.5.** *Let the kernel $k$ satisfy the conditions* (2.3) *and* (2.2)*, and let $p \in \mathcal{P}(\mathbb{R}^n)$. Then the inequality* (4.11) *holds for any measurable function $\ell(x)$ satisfying the condition* (3.1) *and weight $\varrho$ such that $\varrho^{p(\cdot)} \in A_{p(\cdot)}$.*

**Proof.** Apply Corollary 3.2 with

$$X = L^{p(\cdot)}(\mathbb{R}^n), \quad w(x) \equiv \varrho(x), \quad u(x) = \ell(x)\varrho(x)$$

and Theorem 4.3. □

**Remark 4.2.** The statement of Corollary 4.5 covers, in particular, radial weights $\varrho(x) = \varphi(|x-x_0|)$, where $\varphi(t)$ has the property that $t^\alpha\varphi(t)$ is almost increasing and $\frac{\varphi(t)}{t^\beta}$ is almost decreasing for some $\alpha$ and $\beta$. However, note that it seems to be a hard task to check directly that such weights satisfy or not the general Muckenhoupt type condition (4.8), but we can state this indirectly since (4.8) is a necessary condition for the boundedness of the maximal operator, as shown in [**32**], and the boundedness of the maximal operator in case of such radial type weights was proved in [**40**]. Note that for such a weight $\varrho = \varphi(|x-x_0|)$ the conditions on the weight may be given in terms of the Matuszewska–Orlicz indices of the function $\varphi$; we refer to [**40**] for details.

Finally, we treat the case of the variable exponent Morrey spaces.

**Corollary 4.6.** *Let the kernel* $k$ *satisfy the conditions* (2.3), *and* (2.2), *let* $p \in \mathcal{P}(\mathbb{R}^n)$, *and let* $0 \leqslant \lambda(x) \leqslant \lambda_+ < n$. *Then the inequality*

$$\left\| \ell(x) \int\limits_{|y|<\mu(x)} k(|x-y|) f(y)\, dy \right\|_{M^{p(\cdot),\lambda(\cdot)}\mathbb{R}^n} \leqslant A \, \|f\|_{M^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \tag{4.12}$$

*holds for any measurable function* $\ell(x)$ *satisfying the condition* (3.1).

**Remark 4.3.** The obtained inequalities may be extended to the case of metric measure spaces, in the spirit of the approaches, developed for instance in [**5**]. The inequality (3.2), for example, should be then read as follows:

$$\left| \ell(x) \int\limits_{B(x_0,\mu(x)} k(d(x,y)) f(y)\, dy \right| \leqslant CMf(x).$$

We do not dwell on such extensions in this paper.

**Remark 4.4.** Finally, we note that all the inequalities in this section are related to the inequality (3.15). The "dual" inequalities related to (3.16) can be proved in the same way. We leave the details to the interested reader.

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