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An inverse problem of Newtonian aerodynamics

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Abstract. We consider a rarefied medium in $\mathbb{R}^d$, $d \geq 2$ consisting of non-interacting point masses moving at unit velocity in all directions. Given the density of velocity distribution, one easily calculates the pressure created by the medium in any direction. We then consider the inverse problem: given the pressure distribution $f : S^{d-1} \to \mathbb{R}_+$, determine the density $\rho : S^{d-1} \to \mathbb{R}_+$. Assuming that the reflection of medium particles by obstacles is elastic, we show that the solution for the inverse problem is generally non-unique, derive exact inversion formulas, and state necessary and sufficient conditions for existence of a solution. We also present arguments indicating that the inversion is typically unique in the case of non-elastic reflection, and derive exact inversion formulas in a special case of such reflection.

1 Introduction

Consider a homogeneous medium in Euclidean space $\mathbb{R}^d,\ d \geq 2$, rarefied to such extent that mutual interaction of the medium particles can be neglected. The particles chaotically move, besides the density of the velocity distribution in the medium equals $\sigma(v)$. The latter means that at each moment of time and for arbitrary domains $\mathcal{X}$ and $\mathcal{V}$ in $\mathbb{R}^d$, the total mass of particles contained in $\mathcal{X}$ and with the velocity contained in $\mathcal{V}$ equals $|\mathcal{X}| \cdot \int_{\mathcal{V}} \sigma(v)\, dv$. (Here $|\cdot|$ means d-dimensional Lebesgue measure.) We assume σ to be a nonnegative distribution (generalized function), with

$$\int_{\mathbb{R}^d} \sigma(v)\, |v|^2\, dv < \infty.$$

This inequality means that the medium temperature is finite. The medium density is equal to the constant value $\int_{\mathbb{R}^d} \sigma(v)\, dv$. The medium is generally non-isotropic, that is, the function σ is not necessarily radially symmetric.

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Suppose that there is a solid body placed in the medium and that particles hitting the body are reflected according to the law of elastic reflection. Then one can easily calculate the medium pressure $f(n)$ at any point of the body's boundary,

$$f(n) = 2\int_{\mathbb{R}^d} (v \cdot n)^2\, \theta_-(v \cdot n)\, \sigma(n)\, dv, \tag{1.1}$$

where n is the outer unit normal to the boundary at the given point, $v \cdot n$ means the scalar product of v and n, and $\theta_-(t) := (1 - \operatorname{sgn}(t))/2$. Here we are interested in the inverse problem: given the pressure distribution $f(n),\ n \in S^{d-1}$, extract information about the velocity distribution $\sigma(v)$. One can easily see, however, that σ cannot be uniquely determined from f: for example, two different distributions $\sigma(v)$ and $8\sigma(2v)$ produce the same pressure distribution.

It is natural to replace the original distribution σ with the reduced velocity distribution on the unit sphere

$$\frac{1}{2}\rho(v) = \int_0^\infty r^2\, \sigma(rv)\, r^{d-1}\, dr, \quad v \in S^{d-1},$$

which produces the same pressure. Equation (1.1) then takes the form

$$f(n) = \int_{S^{d-1}} (v \cdot n)^2\, \theta_-(v \cdot n)\, \rho(n)\, dv. \tag{1.2}$$

The reduced inverse problem now reads as follows: given $f : S^{d-1} \to \mathbb{R}_+$, find the function $\rho : S^{d-1} \to \mathbb{R}_+$ satisfying (1.2). The main attention in this paper is focused on solving this problem.

Formula (1.2) defines a convolution-type operator. When applied to spherical harmonics, this operator reduces to multiplication by a number. Using this representation, explicit inversion formulas are derived. We also determine necessary and sufficient conditions on f guaranteeing existence of a non-negative solution ρ and explore the relationship between smoothness of ρ an smoothness of f.

We will see below that the operator is injective on the subspace of odd functions and maps the subspace of even functions onto a two-dimensional subspace. This implies that the inversion is not unique: the most part of the even component of ρ vanishes and therefore cannot be uniquely reproduced. In practical terms it means that the medium structure cannot be uniquely reproduced from observations of the pressure created by the medium.

In view of this (somewhat discouraging) conclusion, we decided to extend our consideration to the more general case of (generally) non-elastic reflections of medium particles from the body surface. In this case the normal pressure can be written as

$$f(n) = \int_{S^{d-1}} \frac{2 - \alpha(v \cdot n)}{2}\, (v \cdot n)^2\, \theta_-(v \cdot n)\, \rho(n)\, dv. \tag{1.3}$$

The function $\alpha : [-1, 0] \to [0, 1]$ is called the *accommodation coefficient*. The limiting case $\alpha \equiv 0$ corresponds to *specular* (perfectly elastic) reflection, while the other limiting case $\alpha \equiv 1$ corresponds to *diffuse* (completely non-elastic) reflection where the normal component of incident velocity vanishes. In general α is not constant; that is, the accommodation coefficient depends on the angle of incidence. We consider the simplest case where α is a linear function and show that typically the inversion problem is uniquely solved.

Notice that a medium of non-interacting particles and problems related to interaction of such a medium with a solid body were first considered by Newton in [5]. Nowadays the growing interest to these problems is stimulated by possible applications in aerodynamics of stratospheric and space flights. It is well known that sparse atmosphere at high velocities is extremely changeable; see, e.g.,

[1]. Our approach allows one to describe inhomogeneity of atmosphere at a given location and moment, basing on pressure measurements made in various directions.

The paper is organized as follows. The inverse problem in the two-dimensional case is considered in section 2, and in the case of higher dimensions, in section 3. In the 2D and 3D cases, which are important for applications, explicit inversion formulas are provided. Section 4 is devoted to generalization of the problem to the non-elastic case. We show that the solution for the generalized problem is typically unique and derive the exact inversion formula in the 2D case.

2 The two-dimensional case

The functions ρ and f can be represented as functions on S^1 with the coordinate $\varphi \mod 2\pi$. The formula (1.2) takes the form

$$f(\varphi) = \int_0^{2\pi} k(\varphi - \theta)\, \rho(\theta)\, d\theta, \tag{2.1}$$

where

$$k(\varphi) = \begin{cases} \cos^2 \varphi, \text{ if } \pi/2 \le \varphi \le 3\pi/2 \\ 0 \qquad \text{otherwise.} \end{cases} \tag{2.2}$$

2.1 General solution of equation (2.1)

We use the direct and inverse Fourier transformations

$$f_m = \int_0^{2\pi} f(\varphi)\, e^{-im\varphi}\, d\varphi, \qquad f(\varphi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} f_m e^{im\varphi}.$$

Applying the Fourier transformation to both parts of (2.1) and making use of (2.2), we get

$$f_m = k_m \cdot \rho_m, \tag{2.3}$$

where

$$k_m = \int_0^{2\pi} k(\varphi)\, e^{-im\varphi}\, d\varphi = \begin{cases} \frac{\sin(m\pi/2)}{m^3/4 - m}, \text{ if } m \neq 0, \pm 2 \\ \pi/2, \qquad \text{if } m = 0 \\ \pi/4, \qquad \text{if } m = \pm 2\,. \end{cases} \tag{2.4}$$

Thus, we see that $k_m = 0$, if $m\,(\neq 0, \pm 2)$ is an even number.

Equations (2.3) and (2.4) imply that $f_m = 0$ for $m\,(\neq 0, \pm 2)$ even, and thus, f can be represented in the form

$$f(\varphi) = c + a_1 \cos 2\varphi + a_2 \sin 2\varphi + \Phi(\varphi), \text{ with } \Phi(\varphi + \pi) = -\Phi(\varphi). \tag{2.5}$$

The inverse transformation, in terms of Fourier-image, has the form

$$\rho_m = \frac{1}{k_m} \cdot f_m, \tag{2.6}$$

and using (2.4), we get

$$\begin{cases} \rho_0 = \frac{2}{\pi} f_0, \quad \rho_{\pm 2} = \frac{4}{\pi} f_{\pm 2} \\ \rho_m = \left(m^3/4 - m\right) \sin \frac{m\pi}{2} \cdot f_m, \text{ if } m \text{ is odd} \\ \rho_m \text{ arbitrary, if } m\,(\neq 0, \pm 2) \text{ is even.} \end{cases} \tag{2.7}$$

Denote by $[C^\infty(S^1)]'$ the space of distributions over S^1. Taking into account (2.5) and (2.7), one comes to the following theorem.

Theorem 2.1. Let $f \in [C^\infty(S^1)]'$. Equation (2.1) is solvable in $[C^\infty(S^1)]'$ if and only if f has the form (2.5). For a function f of this form, all the solutions are given by

$$\rho(\varphi) = \frac{2}{\pi}c + \frac{4}{\pi}(a_1\cos 2\varphi + a_2 \sin 2\varphi) + \frac{1}{4}\Phi'''(\varphi+\pi/2) + \Phi'(\varphi+\pi/2) + w(\varphi), \tag{2.8}$$

where w is an arbitrary element of $[C^\infty(S^1)]'$ satisfying $w(\varphi) = w(\varphi+\pi)$ and

$$\int_0^{2\pi} w(\varphi)\,d\varphi = 0, \quad \int_0^{2\pi} w(\varphi)\,\cos 2\varphi\, d\varphi = 0, \quad \int_0^{2\pi} w(\varphi)\,\sin 2\varphi\, d\varphi = 0. \tag{2.9}$$

Besides, for any integer $n \geq 0$, f of the form (2.5) belongs to $C^{n+3}(S^1)$ if and only if there exists a solution $\rho \in C^n(S^1)$.

Proof. If f cannot be represented in the form (2.5), then obviously there is no solution.
Let f has the form (2.5); then

$$c + a_1 \cos 2\varphi + a_2 \sin 2\varphi = \frac{1}{2\pi} f_0 + \frac{1}{2\pi}\left(f_2 e^{2i\varphi} + f_{-2}e^{-2i\varphi}\right)$$

and

$$\Phi(\varphi) = \frac{1}{2\pi}\sum_{m \text{ odd}} f_m e^{im\varphi}.$$

According to (2.7), one has $\rho(\varphi) = \rho_0(\varphi) + \rho_1(\varphi) + w(\varphi)$, where

$$\rho_0(\varphi) = \frac{1}{2\pi}\left(\frac{2}{\pi}f_0 + \frac{4}{\pi}\left(f_2e^{2i\varphi} + f_{-2}e^{-2i\varphi}\right)\right) = \frac{2}{\pi}c + \frac{4}{\pi}(a_1\cos 2\varphi + a_2 \sin 2\varphi),$$

$$\rho_1(\varphi) = \frac{1}{2\pi}\sum_{m \text{ odd}}\left(\frac{m^3}{4} - m\right)\sin\frac{m\pi}{2} f_m e^{im\varphi} = \frac{1}{4}\Phi'''(\varphi+\pi/2) + \Phi'(\varphi+\pi/2),$$

and

$$w(\varphi) = \frac{1}{2\pi}\sum_{m(\neq 0,\pm 2)\text{ even}} \rho_m e^{im\varphi}$$

with ρ_m arbitrary; this implies that w is an arbitrary even function satisfying (2.9).

Finally, let f have the form (2.5) and $f \in C^{n+3}(S^1)$. Then $\Phi \in C^{n+3}(S^1)$ and, taking $w \equiv 0$ in (2.8), one immediately gets a solution $\rho \in C^n(S^1)$. Inversely, let a solution ρ belong to C^n; then its odd part $\frac{1}{2}(\rho(\varphi) - \rho(\varphi+\pi)) = \frac{1}{4}\Phi'''(\varphi+\pi/2) + \Phi'(\varphi+\pi/2)$ also belongs to C^n. This implies that Φ, and therefore f, belong to C^{n+3}. □

Remark 1. The last statement of theorem 2.1 means that smoothness of f is 3 orders higher than that of the original function ρ.

2.2 On positive solutions

Theorem 2.1 provides a necessary and sufficient condition for existence of a solution ρ for equation (2.1). Notice, however, that only *non-negative* solutions are of physical interest. In this section we obtain a refinement of that theorem establishing conditions for existence of a non-negative solution.

All functions considered below in this section are elements of $[C^\infty(S^1)]'$.

If f has the form (2.5), denote

$$g(\varphi) = \frac{1}{4}\Phi'''(\varphi+\pi/2) + \Phi'(\varphi+\pi/2) \tag{2.10}$$

and

$$q(\varphi) = |g(\varphi)| = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} q_m e^{im\varphi}.$$

Theorem 2.2. Equation (2.1) has a non-negative solution ρ, if and only if f has the form (2.5) and additionally

$$\left|\frac{4}{\pi}f_2 - q_2\right| \le \frac{2}{\pi}f_0 - q_0. \tag{2.11}$$

If the inequality is strict, there is a continuum of non-negative solutions.

If $\frac{4}{\pi}f_2 - q_2 = e^{-2i\varphi_0}(\frac{2}{\pi}f_0 - q_0) \neq 0$, where φ_0 is a real number, the unique non-negative solution is given by

$$\rho(\varphi) = 2g(\varphi)\,\theta_+(g(\varphi)) + \frac{1}{2}\Big(\frac{2}{\pi}f_0 - q_0\Big)(\delta(\varphi-\varphi_0) + \delta(\varphi-\varphi_0+\pi)). \tag{2.12}$$

If $\frac{4}{\pi}f_2 - q_2 = \frac{2}{\pi}f_0 - q_0 = 0$, the unique non-negative solution is given by

$$\rho(\varphi) = 2g(\varphi)\,\theta_+(g(\varphi)).$$

Here $\theta_+(t) = \frac{1}{2}(1+\mathrm{sgn(t)})$.

Proof. Let ρ be a non-negative solution of (2.1). Then (2.5) is satisfied, and ρ can be written as

$$\rho(\varphi) = Z(\varphi) + g(\varphi),$$

where Z is an even function, $Z(\varphi+\pi) = Z(\varphi)$, satisfying the relations

$$\int_0^{2\pi} Z(\varphi)\,d\varphi = \rho_0 = \frac{2}{\pi}f_0 \quad \text{and} \quad \int_0^{2\pi} Z(\varphi)\,e^{-2i\varphi}\,d\varphi = \rho_2 = \frac{4}{\pi}f_2 \tag{2.13}$$

and g is given by (2.10). Note that g is odd, $g(\varphi+\pi) = -g(\varphi)$, therefore

$$\min\{\rho(\varphi), \rho(\varphi+\pi)\} = Z(\varphi) - |g(\varphi)|.$$

Recall that, according to our notation, $|g(\varphi)| = q(\varphi)$. Denote $H(\varphi) = Z(\varphi) - q(\varphi)$; non-negativity of ρ implies that $H(\varphi) \ge 0$ for all φ. Applying (2.13), we get

$$\int_0^{2\pi} H(\varphi)\,d\varphi = \frac{2}{\pi}f_0 - q_0 \quad \text{and} \quad \int_0^{2\pi} H(\varphi)\,e^{-2i\varphi}\,d\varphi = \frac{4}{\pi}f_2 - q_2, \tag{2.14}$$

and using that $|\int_0^{2\pi} H(\varphi)\,e^{-2i\varphi}\,d\varphi| \le \int_0^{2\pi} H(\varphi)\,d\varphi$, we come to inequality (2.11).

Inversely, let inequality (2.11) and formula (2.5) be satisfied. Using the above notation, any solution ρ can be written down in the form

$$\rho(\varphi) = H(\varphi) + 2g(\varphi)\,\theta_+(g(\varphi)), \tag{2.15}$$

where H is an even function satisfying (2.14). The above argument implies that ρ is non-negative if and only if H is non-negative.

If $\frac{4}{\pi}f_2 - q_2 = \frac{2}{\pi}f_0 - q_0 = 0$, the unique non-negative H is $H \equiv 0$, therefore the unique non-negative solution ρ is given by $\rho(\varphi) = 2g(\varphi)\,\theta_+(g(\varphi))$.

If $\frac{4}{\pi}f_2 - q_2 = e^{-2i\varphi_0}(\frac{2}{\pi}f_0 - q_0) \geq 0$, the unique non-negative H is given by $H(\varphi) = \frac{1}{2}\Big(\frac{2}{\pi}f_0 - q_0\Big)(\delta(\varphi - \varphi_0) + \delta(\varphi - \varphi_0 + \pi))$, therefore the unique solution ρ is given by (2.12).

If the inequality (2.11) is strict, then define the real values $0 \leq r < 1$ and φ_0 by

$$\frac{4}{\pi}f_2 - q_2 = re^{-2i\varphi_0}\Big(\frac{2}{\pi}f_0 - q_0\Big)$$

and denote $\lambda_\pm = (\frac{2}{\pi}f_0 - q_0)(1 \pm r)/4$. Then a particular non-negative even function H satisfying (2.14) is given by

$$H(\varphi) = \lambda_+(\delta(\varphi - \varphi_0) + \delta(\varphi - \varphi_0 - \pi)) + \lambda_-(\delta(\varphi - \varphi_0 - \pi/2) + \delta(\varphi - \varphi_0 - 3\pi/2)),$$

and a one-parameter family of functions is given, for example, by

$$H_\varepsilon(\varphi) = \lambda_+(\varepsilon)\Big(h\big(\frac{\varphi - \varphi_0}{\varepsilon}\big) + h\big(\frac{\varphi - \varphi_0 - \pi}{\varepsilon}\big)\Big) + \lambda_-(\varepsilon)\Big(h\big(\frac{\varphi - \varphi_0 - \pi/2}{\varepsilon}\big) + h\big(\frac{\varphi - \varphi_0 - 3\pi/2}{\varepsilon}\big)\Big)$$

with ε small enough, where h is a non-negative function with bounded support, $\int_{\mathbb{R}} h(t)dt = 1$, $\lambda_+(\varepsilon) \geq \lambda_+$ is a non-decreasing function of ε, and $\lambda_+(\varepsilon) + \lambda_-(\varepsilon) = (\frac{2}{\pi}f_0 - q_0)/2$. By substituting H_ε in formula (2.15) we then get a one-parameter family of non-negative solutions ρ_ε. □

Corollary 1. Suppose that f satisfies (2.5) and inequality (2.11) is strict. Then $f \in C^3(S^1)$ if and only if there exists a non-negative solution $\rho \in C(S^1)$.

Proof. Let f be continuous; then, according to Theorem 2.1, f is in C^3.

Now, let f be in C^3; then the function g and therefore the function $g\,\theta_+(g)$ are continuous. Taking a continuous non-negative H and using (2.15), one gets a continuous ρ. □

Note that this argument does not work for higher orders of smoothness, since smoothness of g does not imply smoothness of $g\,\theta_+(g)$.

Remark 2. Note that $\Phi(\varphi) = \frac{1}{2}(f(\varphi) - f(\varphi + \pi))$, therefore one has $q(\varphi) = |\frac{1}{4}\Phi'''(\varphi + \pi/2) + \Phi'(\varphi + \pi/2)| = |\frac{1}{8}(f'''(\varphi + \pi/2) - f'''(\varphi - \pi/2)) + \frac{1}{2}(f'(\varphi + \pi/2) - f'(\varphi - \pi/2))|$. Then the conditions for existence of a non-negative solution stated in Theorem 2.2 can be rewritten in a more explicit, but also more cumbersome form

$$\frac{1}{2}(f(\varphi) + f(\varphi + \pi)) = c + a_1 \cos 2\varphi + a_2 \sin 2\varphi \quad \text{for some real } c,\ a_1,\ a_2;$$

$$\Big|\frac{4}{\pi}\int_0^{2\pi} f(\varphi)e^{-2i\varphi}d\varphi - \int_0^{2\pi}\Big|\frac{f'''(\varphi + \pi/2) - f'''(\varphi - \pi/2)}{8} + \frac{f'(\varphi + \pi/2) - f'(\varphi - \pi/2)}{2}\Big|e^{-2i\varphi}d\varphi\Big|$$
$$\leq \frac{2}{\pi}\int_0^{2\pi} f(\varphi)\,d\varphi - \int_0^{2\pi}\Big|\frac{f'''(\varphi + \pi/2) - f'''(\varphi - \pi/2)}{8} + \frac{f'(\varphi + \pi/2) - f'(\varphi - \pi/2)}{2}\Big|\,d\varphi.$$

3 The spatial case

We rewrite equation (1.2) in the form

$$(\mathcal{K}\rho)(x) := \int\limits_{S^{d-1}} k(x\cdot\sigma)\rho(\sigma)d\sigma = f(x), \ \ x \in S^{d-1} \tag{3.1}$$

where the kernel $k(t), -1 \le t \le 1$, has the form

$$k(t) = t^2\theta_-(t), \ \ \theta_-(t) = \frac{1 - sgn\, t}{2}.$$

To solve this equation, we make use of the well known notions of spherical convolution operators and their Fourier multipliers, see for instance [8], or [7], p.10.

3.1 On basics of spherical harmonic expansions

Every spherical convolution equation of form (3.1) with the kernel $k(x\cdot\sigma)$ depending on the scalar product of the arguments, when applied to the Fourier-Laplace expansion

$$\rho(\sigma) = \sum_{m=0}^{\infty} Y_m(\rho,\sigma), \ \ \sigma \in S^{d-1},$$

of a function $\rho(\sigma)$, is reduced to the multiplication by the Fourier-Laplace multiplier k_m corresponding to the kernel $k(x\cdot\sigma)$:

$$(\mathcal{K}\rho)(x) = \sum_{m=0}^{\infty} k_m Y_m(\rho,x).$$

Recall that the Fourier-Laplace components $Y_m(\rho,x), \ x \in S^{d-1}$, of a function ρ are given by the formula

$$Y_m(\rho,x) = \nu_m ⨍_{S^{d-1}} \rho(\sigma)P_m(x\cdot\sigma)d\sigma \tag{3.2}$$

where we use the notation

$$⨍_{S^{d-1}} f(\sigma)d\sigma := \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(\sigma)d\sigma,$$

$\nu_m = (d+2m-2)\frac{(d+m-3)!}{m!(d-2)!}$ and $P_m(t)$ are Chebyshev polynomials in the plane case $d = 2$, and the Gegenbauer polynomials

$$P_m(t) = c_m C_m^{\frac{d-2}{2}}(t), \quad c_m = \frac{1}{\binom{m+d-3}{m}} = \frac{\Gamma(m+1)\Gamma(d-2)}{\Gamma(m+d-2)} \tag{3.3}$$

in the spatial case $d \ge 3$; the polynomials $P_m(t)$ are even or odd functions when m is even or odd, respectively.

The Funk-Hekke formula (see for instance [7], p. 13) states that

$$\int\limits_{S^{d-1}} k(x\cdot\sigma)Y_m(\sigma)d\sigma = k_m Y_m(\sigma)$$

and

$$k_m = |S^{d-2}|\int\limits_{-1}^{1} k(t)P_m(t)(1-t^2)^{\frac{d-3}{2}}\, dt. \tag{3.4}$$

Note that

$$|P_m(t)| \le 1 \quad \text{for} \quad -1 \le t \le 1. \tag{3.5}$$

3.2 Calculation of the multiplier $\{k_m\}$ of the operator $\mathcal{K}$

For further goals, we consider a slightly more general operator

$$(\mathcal{K}_j\rho)(x) := \int\limits_{S^{d-1}} k_j(x\cdot\sigma)\rho(\sigma)d\sigma, \quad k_j(t) = t^j\theta_-(t), \quad j = 0,1,2,..., \tag{3.6}$$

but we will use only the result for the cases $j = 2$ and $j = 0$, the former corresponding to our operator (3.1) and the latter will be used in calculating the kernel of the inversion operator $\mathcal{L}$, see (3.22).

Lemma 1. The multiplier k_m^j corresponding to the operator $\mathcal{K}_j$ is calculated by the formula

$$k_m^j = \begin{cases} \frac{\pi^{\frac{d}{2}-1} j!\Gamma\left(\frac{m-j}{2}\right)}{2^j\Gamma\left(\frac{m+d+j}{2}\right)} \sin\frac{(j-m)\pi}{2}, & m \geq j+1 \\ \frac{(-1)^{m+j}\pi^{\frac{d}{2}} j!}{2^j\Gamma\left(1+\frac{j-m}{2}\right)\Gamma\left(\frac{m+d+j}{2}\right)}, & 0 \leq m \leq j \end{cases} \tag{3.7}$$

Proof. For the kernel $k_j(t) = t^j\theta_-(t)$, by formula (3.4) we have

$$k_m^j = |S^{d-2}|(-1)^{m+j}\int\limits_0^1 t^j P_m(t)(1-t^2)^{\frac{d-3}{2}}\,dt.$$

For definiteness, we take $d \geq 3$, the case $d = 2$ being similarly treated with the Gegenbauer polynomials replaced by the Chebyshev polynomials.

To calculate k_m^j, we make use of formula 7.311.2 from [2] which we transform as follows:

$$\int\limits_0^1 t^{m+2\rho}(1-t^2)^{\frac{d-3}{2}} C_m^{\frac{d-2}{2}}(t)\,dt = \frac{\pi}{\Gamma\left(\frac{d-2}{2}\right)} \frac{\Gamma(m+d-2)\Gamma(m+2\rho+1)}{2^{m+d+2\rho-2}m!\Gamma(\rho+1)\Gamma\left(m+\rho+\frac{d}{2}\right)}. \tag{3.8}$$

Note that this formula is valid for all ρ such that $m+2\rho > -1$ (we escaped the restriction $2\rho > -1$ which accompanies formula 7.311.2 in [2]) by removing indeterminacy of the right-hand side of that formula by means of the duplication formula for the gamma function and analytical continuation with respect to ρ. Now we choose ρ by the rule $2\rho + m = j$ and get

$$\int\limits_0^1 t^j(1-t^2)^{\frac{d-3}{2}} C_m^{\frac{d-2}{2}}(t)\,dt = \frac{\pi}{\Gamma\left(\frac{d-2}{2}\right)} \frac{j!\Gamma(m+d-2)}{2^{d+j-2}m!\Gamma\left(\frac{j-m}{2}+1\right)\Gamma\left(\frac{m+d+j}{2}\right)}.$$

Then, in view of (3.3) after easy calculations with the help of duplication and complement formulas for the Gamma function, we arrive at (3.7).

Corollary 2. The multiplier $\{k_m\} = \{k_m^2\}$ of the operator $\mathcal{K}$ is given by the formulas

$$k_m = \frac{2\pi^{\frac{d}{2}-1}}{(m-2)(m+d)} \cdot \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+d}{2}\right)} \cdot \sin\frac{m\pi}{2}, \quad m \neq 0, m \neq 2, \tag{3.9}$$

$$k_0 = \frac{\pi^{\frac{d}{2}}}{2\Gamma\left(\frac{d+2}{2}\right)}, \quad k_2 = \frac{\pi^{\frac{d}{2}}}{2\Gamma\left(\frac{d+4}{2}\right)}. \tag{3.10}$$

Remark 3. In the plane case $d = 2$, from (3.9) we obtain

$$k_m = \frac{\sin\frac{m\pi}{2}}{\frac{m^3}{4} - m}$$

Corollary 3. For further goals we found it also convenient to read formula (3.7) with $j = 0$ as

$$\sin\frac{m\pi}{2} = -\frac{\Gamma\left(\frac{m+d}{2}\right)}{\pi^{\frac{d}{2}-1}\Gamma\left(\frac{m}{2}\right)} \cdot k_m^0. \tag{3.11}$$

3.3 General solution of equation (3.1)

We arrive at the following statement, where by $[C^\infty(S^{d-1})]'$ we denote the space of distributions (generalized functions) over S^{d-1}; we refer to [4] for the study of the space $[C^\infty(S^{d-1})]'$, see also [7], p. 149; note that distributions $f \in [C^\infty(S^{d-1})]'$ have finite order in the sense that their Fourier-Laplace coefficients f_m satisfy the estimate $|f_m| \le Cm^N$, $N = N(f)$. Note also that equation (3.1) when treated in $[C^\infty(S^{d-1})]'$ is correspondingly interpreted in the distributional sense.

By

$$\mathbb{P}_2^0(x,x) = \sum_{j,k=1}^{d} a_{jk}x_jx_k$$

we denote a harmonic quadratic form, that is, a form with $\mathrm{tr}(P_2) := \sum_{j=1}^{d} a_{jj} = 0$.

Theorem 3.1. Let $f \in [C^\infty(S^{d-1})]'$. Equation (3.1) is solvable in $[C^\infty(S^{d-1})]'$ if and only if f has the form

$$f(x) = c_0 + \mathbb{P}_2^0(x,x) \ + \ f_0(x), \quad x = (x_1,...,x_d) \in S^{d-1}, \tag{3.12}$$

where $f_0(x)$ is an arbitrary odd element in $[C^\infty(S^{d-1})]'$, c_0 is a constant and $\mathbb{P}_2^0(x,x)$ is an arbitrary harmonic quadratic form. For f of form (3.12), all the solutions are given by

$$\rho(x) = \frac{c_0}{k_0} + \frac{\mathbb{P}_2^0(x,x)}{k_2} \ + \ \mathcal{L}f_0(x) + \varphi(x), \tag{3.13}$$

where $\varphi(x)$ is an arbitrary even function with vanishing spherical moments of orders 0 and 2:

$$\int_{S^{d-1}} \varphi(\sigma)d\sigma = 0, \quad \int_{S^{d-1}} \varphi(\sigma)\sigma_j\sigma_k d\sigma = 0, \quad k,j = 1,...,d, \tag{3.14}$$

and $\mathcal{L}$ is the spherical operator defined by the multiplier

$$\lambda_m = \frac{(m-2)(m+d)}{2\pi^{\frac{d}{2}-1}} \cdot \frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \cdot \sin\frac{m\pi}{2}. \tag{3.15}$$

Proof. It suffices to refer to formula (3.9) for the multiplier of the operator $\mathcal{K}$. □

In Theorem 3.2 in the sequel we give an explicit formula for the operator $\mathcal{L}$.

3.4 Effective construction of the operator $\mathcal{L}$

Multiplier (3.15) defining the operator $\mathcal{L}$ has the behaviour

$$|\lambda_m| \leq C \sim m^{\frac{d}{2}+2} \quad \text{as} \quad m \to \infty$$

so that the operator $\mathcal{L}$ is a kind of spherical pseudo-differential type operator of order $\frac{d}{2}+2$. As shown in Theorem below, it may be represented as composition of a usual differential operator (a polynomial of the Beltrami-Laplace operator) and an integral operator over the sphere S^{d-1}, which is an integral operator (spherical convolution) with an explicitly calculated kernel. The concrete form of such a representation is different for odd and even dimensions d. Because of our application to the aerodynamics problem, in the final representations of Theorem 3.2 we deal only with the cases $d=3$ and $d=2$, although the corresponding formulas may be written down for an arbitrary dimension d.

The proof of Theorem 3.2 given in Subsection 3.4, in the $3d$-case is based on some auxiliary constructions presented in the next subsection.

Auxiliary lemmas

In this subsection we follow ideas and approaches developed in [6]. Let

$$I^\alpha f(x) = \frac{1}{\gamma_d(\alpha)} \int\limits_{\mathbb{R}^d} \frac{f(y)\,dy}{|x-y|^{d-\alpha}}, \quad x \in \mathbb{R}^d$$

be the Riesz potential operator with the standard normalizing constant $\gamma_d(\alpha) = \frac{2^\alpha \pi^{\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$. We introduce the following modification of this operator for functions homogeneous of degree 0:

$$\mathfrak{K}_{\alpha,\gamma} f(x) = |x|^\gamma \int\limits_{\mathbb{R}^d} \frac{f(y')}{|y|^{\alpha+\gamma}|x-y|^{n-\alpha}}\,dy, \tag{3.16}$$

where $y' = \frac{y}{|y|}$, $0 < \alpha < d, \gamma > 0, \alpha + \gamma < d$. The operator $\mathfrak{K}_{\alpha,\gamma}$ transforms the class of functions homogeneous of degree 0 to itself; as an operator on functions defined on S^{d-1}, it is a spherical convolution operator.

Lemma 2. The Fourier-Laplace multiplier k_m of the operator $\mathfrak{K}_{\alpha,\gamma}$ is calculated by the formula

$$k_m = 2^{-\alpha-\gamma} \frac{\Gamma\left(\frac{m+\gamma}{2}\right)\Gamma\left(\frac{m+n-\alpha-\gamma}{2}\right)}{\Gamma\left(\frac{m+n-\gamma}{2}\right)\Gamma\left(\frac{m+\alpha+\gamma}{2}\right)}, \quad m = 0,1,2,3,... \tag{3.17}$$

P r o o f. We calculate the multiplier k_m directly from the "spatial" definition of the operator as an operator on functions defined on $\mathbb{R}^d$ and calculate $\mathfrak{K}_{\alpha,\gamma} Y_m(x)$ for a spherical harmonic $Y_m(x')$. We have

$$\mathfrak{K}_{\alpha,\gamma} Y_m(x) = F^{-1} F \mathfrak{K}_{\alpha,\gamma} Y_m(x)$$

where $F\varphi(\xi) = \int\limits_{\mathbb{R}^d} e^{i\xi y} \varphi(y)\,dy$ is the Fourier transform in $\mathbb{R}^d$. Since

$$\mathfrak{K}_{\alpha,\gamma} f(x) = |x|^\gamma I^\alpha \left(\frac{f(y)}{|y|^{\alpha+\gamma}} \right)(x) \quad \text{and} \quad F(I^\alpha f)(\xi) = \frac{1}{|\xi|^\alpha} Ff(\xi),$$

we obtain

$$F\mathfrak{K}_\alpha Y_m(\xi) = \frac{1}{|\xi|^\alpha} F\left(\frac{Y_m(y')}{|y|^\alpha}\right)(\xi).$$

By the Calderon formula (see [8], or for instance [7], formula 4.92), we have

$$F\left(\frac{Y_m(y')}{|y|^{\alpha+\gamma}}\right)(\xi) = i^m a_n(m,\alpha+\gamma)|\xi|^{\alpha+\gamma-n}Y_m(\xi')$$

where $a_n(m,\alpha) = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{m+n-\alpha}{2}\right)}{2^{\alpha-n}\Gamma\left(\frac{m+\alpha}{2}\right)}$. Hence

$$\mathfrak{K}_{\alpha,\gamma}Y_m(x) = i^m a_n(m,\alpha+\gamma)F^{-1}\left(\frac{Y_m(\xi')}{|\xi|^n}\right)(x) = \frac{(-i)^m a_n(m,\alpha+\gamma)}{(2\pi)^n} F\left(\frac{Y_m(\xi')}{|\xi|^n}\right)(x).$$

Making use of the same Calderon formula, we obtain

$$\mathfrak{K}_{\alpha,\gamma}Y_m(x) = (-i)^m \frac{a_n(m,\alpha+\gamma)b_n(m)}{(2\pi)^n} Y_m(x)$$

with $b_n(m) = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)}$, which proves (3.17). □

Corollary 4. In the $3d$-case, the operator

$$\mathfrak{K}f(x') = \frac{2}{\pi^2}\int_{\mathbb{R}^3} \frac{f(y')\,dy}{|y|^2|x'-y|^2} \tag{3.18}$$

has the multiplier

$$k_m = \left[\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\right]^2.$$

We are interested in the direct spherical representation of the operator (3.18). It is provided by the following lemma.

Lemma 3. The operator $\mathfrak{K}$ has the representation

$$\mathfrak{K}f(x') = \frac{2}{\pi^2}\int_{\mathbb{S}^2} \frac{\pi - \arccos x'\cdot\sigma}{\sqrt{1-(x'\cdot\sigma)^2}} f(\sigma)d\sigma. \tag{3.19}$$

Proof. We pass to polar coordinates in (3.18) and obtain

$$\mathfrak{K}f(x') = \frac{2}{\pi^2}\int_{\mathbb{S}^2} f(\sigma)d\sigma \int_0^\infty \frac{dr}{r^2 - 2rx'\cdot\sigma + 1}.$$

It remains to observe that

$$\int_0^\infty \frac{dr}{r^2-2rt+1} = \frac{1}{\sqrt{1-t^2}}\left(\frac{\pi}{2} + arctg\frac{t}{\sqrt{1-t^2}}\right) = \frac{\pi - \arccos t}{\sqrt{1-t^2}}.$$

Theorem on the explicit representation of the operator $\mathcal{L}$ in the $3d$-case

By

$$\delta f(x') = |x|^2 \Delta f\left(\frac{x}{|x|}\right)$$

we denote the Beltrami-Laplace operator (*the restriction onto the unit sphere of the Laplace operator applied to functions homogeneous of degree 0*), it has the multiplier $-m(m+n-2)$. We also introduce the notation

$$\mathcal{K}_0 f(x') = \int\limits_{x'\cdot\sigma<0} f(\sigma)d\sigma. \tag{3.20}$$

Theorem 3.2. In the case $d=3$, the operator $\mathcal{L}$ has the representation

$$\mathcal{L}f(x') = \frac{1}{16\pi^3}\delta^2(\delta+6)\int\limits_{S^2}\frac{\pi-\arccos(x'\cdot\sigma)}{\sqrt{1-(x'\cdot\sigma)^2}}\left(\int\limits_{\sigma\cdot s<0} f(s)ds\right)d\sigma \tag{3.21}$$

where δ is the Beltrami-Laplace operator.

Proof. We substitute $\sin\frac{m\pi}{2}$ from formulas (3.11) into (3.15) and obtain the following representation for the multiplier λ_m:

$$\lambda_m = -\frac{(m-2)(m+d)}{2\pi^{d-2}}\cdot\left[\frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right]^2\cdot k_m^0. \tag{3.22}$$

By Lemma 1 we know that the spherical convolution operator corresponding to the multiplier k_m^0, is realized in form (3.20). The multiplier

$$(m-2)(m+d) = m(m+d-2)-2d$$

leads to the Beltrami-Laplace operator, corresponding to the operator $-\delta-2d$. It remains to manage with the multiplier $\left[\frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right]^2$. By properties of the Gamma-function, we have

$$\left[\frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right]^2 = \frac{1}{16}[m(m+d-2)]^2\left[\frac{\Gamma\left(\frac{m+d-2}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\right]^2, \tag{3.23}$$

where the multiplier in the brackets on the right-hand side has the multiplier vanishing at infinity when $d-2<2$, that is, in the cases $d=2$ and $d=3$, which is sufficient for our purposes (if we could wish to cover the case of any dimension, then the procedure in (3.23) should be repeated N times with $N>\frac{d-2}{2}$). Then from (3.22) in the case $d=3$ we have

$$\lambda_m = \frac{1}{32\pi}[m(m+1)-6][m(m+1)]^2\left[\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\right]^2\cdot k_m^0. \tag{3.24}$$

By Corollary 4 and Lemma 3 we then obtain

$$\mathcal{L}f(x') = -\frac{1}{32\pi}\delta^2(\delta+6)\ (\mathfrak{K}\mathcal{K}_0 f)(x'),$$

where $\mathfrak{K}$ is the spherical convolution operator defined in (3.19). Formula (3.21) has been proved. □

3.5 Refinement of Theorem 3.1

Theorem 3.1 provides a general solution of equation (3.1) either in the case of very nice functions ρ, f in the class $C^\infty(S^{d-1})$ or in the class of distributions $[C^\infty(S^{d-1})]'$. The following theorem represents a refinement in terms of solutions $\rho \in L^q(S^{d-1})$ and right-hand sides f in the Sobolev space $W^{q,N}(S^{d-1})$. The Sobolev space $W^{q,N}(S^{d-1})$ is defined as the closure of $C^\infty(S^{d-1})$ with respect to the norm

$$\|f\|_{W^{q,N}(S^{d-1})} = \|(-\delta)^{\frac{N}{2}} f\|_{L^q(S^{d-1})}, \quad 1 \le q < \infty,$$

with the Beltrami-Laplace operator δ treated in the distributional sense. It is known that in the case $1 < q < \infty$ this definition is equivalent to

$$W^{q,N}(S^{d-1}) = \left\{ f \in L^q(S^{d-1}) : |x|^{|j|} D^j \left[f\left(\frac{x}{|x|}\right)\right] \in L^q(S^{d-1}), 0 < |j| \le N \right\}.$$

Theorem 3.3. Let $1 < q < \infty,\ d = 2,3,4,...$ and

$$f \in W^{q,d+2}(S^{d-1}).$$

Equation (3.1) is solvable in $L^q(S^{d-1})$ if and only if f has the form (3.12) where $f_0(x)$ is an odd function. For f of form (3.12), all the solutions are given by formula (3.13) where $\varphi(x)$ is an arbitrary even function in $L^q(S^{d-1})$ with conditions (3.14). In the $3d$-case the inversion operator $\mathcal{L}$ is given by formula (3.21).

Proof. It suffices to prove that the operator $\mathcal{L}$ is a bounded operator from $W^{q,d+2}(S^{d-1})$ to $L^q(S^{d-1})$. We base ourselves on representation (3.22) of the multiplier λ_m of this operator. In (3.22) the last factor k_m^0 corresponds to the operator (3.20) which is obviously bounded in $L^q(S^{d-1})$. It remains to verify that the multiplier

$$(m-2)(m+d)\left[\frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right]^2$$

corresponds to the operator bounded from $W^{q,d+2}(S^{d-1})$ to $L^q(S^{d-1})$. By definition of the space $W^{q,d+2}(S^{d-1})$, this means that the multiplier

$$\frac{(m-2)(m+d)}{[m(m+d-2)]^{1+\frac{d}{2}}}\left[\frac{\Gamma\left(\frac{m+d}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right]^2$$

should correspond to an operator bounded in $L^q(S^{d-1})$. The latter is easily verified by means of the Strichartz criterion for spherical Fourier q-multipliers ([9], see also for instance [7], p. 150) and properties of the quotients of Gamma function.

Remark 4. In the $2d$-case, due to the specificity of the plane case the obtained solution required differentiability only up to order 3, see Theorem 2.1, not up to order 4, as stated in general multidimensional Theorem 3.3. Note that the order $d+2$ of smoothness in the general case may be weakened, which may be seen from the fact that in the proof of Theorem 3.3 we made use just of the boundedness of the operator $\mathcal{K}_0$ in the space $L^q(S^{d-1})$, while this operator has some smoothness improvement properties. However, the operator $\mathcal{K}_0$ has oscillating multiplier, so that to catch its smoothness improvement properties in the multidimensional case, we need other means. We do not touch such a possibility.

3.6 On positive solutions

We follow in main the scheme of arguments in Subsection 2.2. We base ourselves on Theorem 3.1 and assume *a priori* that the right-hand side of equation (3.1) has form (3.12), so that solutions exist and are given by (3.13).

To formulate the statement on the existence of positive solutions we need the following notation

$$G(x) = \fint_{S^{d-1}} |\mathcal{L} f_0(\sigma)|(x\cdot\sigma)^2 d\sigma, \quad x \in S^{d-1},$$

and

$$H_0 = \frac{c_0}{k_0} - \fint_{S^{d-1}} |\mathcal{L} f_0(\sigma)| d\sigma$$

where the operator $\mathcal{L}$ is defined by its multiplier (3.15), see the explicit expression for $\mathcal{L}$ in $3d$-case in (3.21). We also denote

$$\alpha = \frac{c_0 - k_0 H_0}{d k_0}, \quad \beta = \frac{d-1}{d k_2}$$

Theorem 3.4. Let $f(x)$ have form (3.12). The condition

$$\sup_{x\in S^{d-1}} \left| G(x) - \alpha - \beta \mathbb{P}_2^0(x,x) \right| \le \frac{d-1}{d} H_0 \tag{3.25}$$

is necessary for the equation (3.1) to have a non-negative integrable solution and the condition

$$\sup_{x\in S^{d-1}} \left(|\mathcal{L} f_0(x)| - \frac{\mathbb{P}_2^0(x,x)}{k_2} \right) \le H_0 \tag{3.26}$$

is sufficient. Under condition (3.26), the function

$$\rho(x) = H_0 + \frac{\mathbb{P}_2^0(x,x)}{k_2} - \mathcal{L} f_0(x) \tag{3.27}$$

is one of the non-negative solutions; in case (3.26) holds with strict inequality, equation has infinitely many positive solutions.

Proof.
1^0. *Preliminary constructions*. We denote

$$Z(x) = \frac{\mathbb{P}_2^0(x,x)}{k_2} + \varphi(x),$$

$\varphi(x)$ being an arbitrary even function with the Fourier-Laplace components $Y_2(\varphi,x) \equiv Y_0(\varphi) = 0$. According to (3.13), the assumption for the solution to be non-negative is equivalent to the condition

$$H(x) := Z(x) + \frac{c_0}{k_0} - |\mathcal{L} f_0(x)| \ge 0, \tag{3.28}$$

where we took into account that the function $\mathcal{L} f_0(x)$ is odd. Recall that $c_0 = \fint_{S^{d-1}} f(\sigma) d\sigma$ is the constant from (3.12) and k_0 is given by (3.10).

Note that

$$\fint_{S^{d-1}} H(\sigma) d\sigma = \frac{c_0}{k_0} - \fint_{S^{d-1}} |\mathcal{L} f_0(\sigma)| d\sigma,$$

so that the condition

$$\fint_{S^{d-1}} |\mathcal{L}f_0(\sigma)|d\sigma < \frac{c_0}{k_0} \tag{3.29}$$

is necessary for the existence of a non-negative solution $\rho(x)$ different from identical zero, which preassumes that c_0 must be positive. We normalize the function $H(x)$:

$$h(x) := \frac{H(x)}{H_0} = \frac{1}{H_0}\left(Z(x) + \frac{c_0}{k_0} - |\mathcal{L}f_0(x)|\right),$$

where

$$H_0 = \fint_{S^{d-1}} H(\sigma)d\sigma = \frac{c_0}{k_0} - \fint_{S^{d-1}} |\mathcal{L}f_0(\sigma)|d\sigma. \tag{3.30}$$

Note that the function $Z(x)$ has the uniquely determined components $Y_0(Z,x) \equiv 0$ and $Y_2(Z,x) \equiv \frac{\mathbb{P}_2^0(x,x)}{k_2}$. The former has already been used in (3.30) and the latter provides the corresponding restriction on the component $Y_2(h,x)$ of the function $h(x)$:

$$Y_2(h,x) = \frac{1}{H_0}\left(\frac{\mathbb{P}_2^0(x,x)}{k_2} - Y_2(|\mathcal{L}f_0|,x)\right). \tag{3.31}$$

We make use of formula (3.2) with $m = 2$ and $P_2(t) = \frac{dt^2-1}{d-1}$ and get

$$\fint_{S^{d-1}} h(\sigma)P_2(x\cdot\sigma)d\sigma = \mu(x), \;\; x \in S^{d-1}, \tag{3.32}$$

where $Y_2(|\mathcal{L}f_0|,x)$ is the Fourier-Laplace component of order $m = 2$ (see (3.2)) of the function $|\mathcal{L}f_0(\sigma)|$ and

$$\mu(x) = \frac{1}{H_0\nu_2}\left(\frac{\mathbb{P}_2^0(x,x)}{k_2} - Y_2(|\mathcal{L}f_0|,x)\right), \quad \nu_2 = \frac{(d-1)(d+2)}{2}. \tag{3.33}$$

Making use of formula (3.2) with $m = 2$ again, we transform (3.33) to

$$\mu(x) = \frac{1}{H_0\nu_2}\left(\frac{\mathbb{P}_2^0(x,x)}{k_2} - \frac{\nu_2}{d-1}\fint_{S^{d-1}} |\mathcal{L}f_0(\sigma)|\left[d(x\cdot\sigma)^2 - 1\right]d\sigma\right).$$

In view of (3.30) this may be also transformed to

$$\mu(x) = \gamma + \frac{\mathbb{P}_2^0(x,x)}{H_0 k_2 \nu_2} - \frac{d}{H_0(d-1)}\fint_{S^{d-1}} |\mathcal{L}f_0(\sigma)|(x\cdot\sigma)^2 d\sigma \tag{3.34}$$

where $\gamma = \frac{c_0 - k_0 H_0}{H_0 k_0 (d-1)}$.

2^0. *Necessity part*. From (3.32) by (3.5) we get

$$|\mu(x)| \le \fint_{S^{d-1}} h(\sigma)d\sigma = 1,$$

so that the condition

$$\sup_{x\in S^{d-1}} |\mu(x)| \le 1 \tag{3.35}$$

necessarily holds when $h(x)$ is non-negative. It is easy to see that inequality (3.35) is nothing else but (3.25).

3^0. *Sufficiency part*. Since $\frac{\mathbb{P}(x,x)}{k_2} = Y_2\left(\frac{\mathbb{P}_2^0}{k_2}, x\right)$, from (3.31) we observe that the function

$$h(x) = 1 + \frac{1}{H_0}\left(\frac{\mathbb{P}_2^0(x,x)}{k_2} - |\mathcal{L} f_0(x)|\right)$$

is one of the solutions of equation (3.32) and satisfies the condition $⨍_{S^{d-1}} h(\sigma)d\sigma = 1$. It remains to note that (3.26) is the condition of non-negativity of this function, and by relation (3.28) between $h(x)$ and $\rho(x)$ we arrive at (3.27). The non-uniqueness of positive solutions in case of strict inequality in (3.26) is seen from the structure (3.13) of solutions: for (3.27) we then have $m_0 := \inf_{x\in S^{d-1}} \rho(x) > 0$ and consequently the function $\rho(x) + \varphi(x)$ is also a non-negative solution with an arbitrary even function $\varphi(x)$ satisfying conditions (3.14) and having small values $|\varphi(x)| \le m_0$.

4 Generalizations of the problem

The above consideration is related to the idealized model where the particle-surface interaction is perfectly elastic. However, in practice the interaction is normally non-elastic, with the elasticity rate strongly depending on the angle of incidence (see, e.g., [3, 1]). This implies that the kernel k in the formula for normal pressure (3.1) should be modified as follows:

$$k(t) = \frac{2-\alpha(t)}{2}\, t^2\,\theta_-(t).$$

Here $\alpha(t) \in [0,\,1]$ is the *accommodation coefficient*; the case $\alpha \equiv 0$ corresponds to specular reflection, and $\alpha \equiv 1$, to diffuse reflection. Recall that t means cosine of the angle between the velocity of incidence and the normal to the surface.

We have seen that in the case of *elastic interaction* the medium structure cannot be uniquely reconstructed from the distribution of pressure. Surprisingly enough, such a reconstruction is generally possible in the case of *non-elastic interaction*. Consider for simplicity the case where $\alpha(t)$ is a linear function, that is,

$$\frac{2-\alpha(t)}{2} = a + bt,$$

where a and b are constants. Note that the assumption $0 \le \alpha(t) \le 1$ for all $t \in [-1,\,1]$ implies two conditions on a and b:

$$1/2 \le a \le 1 \quad \text{and} \quad 1/2 \le a - b \le 1. \tag{4.1}$$

Below we briefly consider the inverse problem in the two-dimensional and multidimensional cases. We show that typically ρ is uniquely determined by f. In the two-dimensional case the explicit inversion formulas are provided.

4.1 The two-dimensional case

In this case the Fourier multipliers

$$k_m = \int_0^{2\pi} k(\cos\varphi)\, e^{-im\varphi}\, d\varphi$$

take the form

$$k_m = ak_m^0 + bk_m^1, \tag{4.2}$$

where

$$k_m^0 = \begin{cases} \frac{\sin(m\pi/2)}{m^3/4-m}, & \text{if } m \text{ is odd} \\ 0, & \text{if } m\,(\neq 0,\, \pm 2) \text{ is even,} \end{cases} \tag{4.3}$$

$$k_0^0 = \pi/2, \quad k_{\pm 2}^0 = \pi/4,$$

$$k_m^1 = \begin{cases} \frac{-12\cos(m\pi/2)}{(m^2-1)(m^2-9)}, & \text{if } m \text{ is even} \\ 0, & \text{if } m\,(\neq \pm 1,\, \pm 3) \text{ is odd,} \end{cases} \tag{4.4}$$

$$k_{\pm 1}^1 = 3\pi/8, \quad k_{\pm 3}^1 = \pi/8.$$

The conditions (4.1) imply that $a + \kappa b > 0$ for any $-2 < \kappa < 1$, and therefore, the special values of k_m for $m = 0, \pm 1, \pm 2, \pm 3$ are nonzero:

$$k_0 = \frac{\pi a}{2} - \frac{4b}{3} > 0, \quad k_{\pm 1} = -\frac{4a}{3} + \frac{3\pi b}{8} < 0,$$

$$k_{\pm 2} = \frac{\pi a}{4} - \frac{4b}{5} > 0, \quad k_{\pm 3} = -\frac{4a}{15} + \frac{\pi b}{8} < 0.$$

From now on in this subsection we assume that $b \neq 0$; otherwise the density f is just proportional to the function given by (2.1). From the above formulas (4.2), (4.3), (4.4) one easily sees that the rest of the values k_m are also nonzero, therefore ρ can be uniquely reconstructed from f. Moreover, the exact formula takes place

$$\frac{1}{k_m} = \frac{1}{a}\Big(\frac{m^3}{4} - m\Big)\sin(m\pi/2) - \frac{1}{12b}(m^2-1)(m^2-9)\cos(m\pi/2) \quad \text{for } m \neq 0, \pm 1, \pm 2, \pm 3.$$

Then using this formula and the relation $\rho_m = \frac{1}{k_m} f_m$, one easily obtains the inversion formulas

$$\rho(\varphi) = \frac{1}{a}\rho^0(\varphi) + \frac{1}{b}\rho^1(\varphi) + \sum_{m=0,\pm 1,\pm 2,\pm 3} \frac{f_m}{k_m},$$

with

$$\rho^0(\varphi) = \frac{1}{8}\left[f'''(\varphi+\pi/2) - f'''(\varphi-\pi/2)\right] + \frac{1}{2}\left[f'(\varphi+\pi/2) - f'(\varphi-\pi/2)\right]$$

and

$$\rho^1(\varphi) = -\frac{1}{24}\left[f^{IV}(\varphi+\pi/2) + f^{IV}(\varphi-\pi/2)\right] - \frac{5}{12}\left[f''(\varphi+\pi/2) + f''(\varphi-\pi/2)\right] - $$
$$-\frac{3}{8}\left[f(\varphi+\pi/2) + f(\varphi-\pi/2)\right].$$

4.2 The multidimensional case

Now we consider equation (3.1) with the kernel $k(t) := (a+bt)t^2\theta_-(t),\;\; a, b \in R_+^1 \setminus \{0\}$. In notation (3.6) we have

$$k(x) = ak_2(t) + bk_3(t).$$

Therefore, to calculate the Fourier-Laplace multiplier k_m corresponding to the kernel $k(t)$ we can use formula (3.7) with $j = 2$ and $j = 3$. This yields

$$k_m = \pi^{\frac{d}{2}-1}\left[\frac{2a\sin\frac{m\pi}{2}}{m(m-2)}\frac{\Gamma\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{m+d+2}{2}\right)}\theta_{odd}(m) - \frac{3b\cos\frac{m\pi}{2}}{(m-1)(m-3)}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+d+3}{2}\right)}\theta_{even}(m)\right] \tag{4.5}$$

for $m \neq 0,1,2,3$, where we denoted $\theta_{odd}(m) = \begin{cases} 1, & m \text{ is odd} \\ 0, & m \text{ is even} \end{cases}$ and $\theta_{even}(m) = 1 - \theta_{odd}(m)$. For the excluded values $m = 0,1,2,3$ we have

$$k_0 = \pi^{\frac{d}{2}}\left[\frac{a}{2\Gamma\left(\frac{d+2}{2}\right)} - \frac{b}{\sqrt{\pi}\Gamma\left(\frac{d+3}{2}\right)}\right],$$

$$k_1 = \pi^{\frac{d}{2}}\left[-\frac{a}{\sqrt{\pi}\Gamma\left(\frac{d+3}{2}\right)} + \frac{3b}{4\Gamma\left(\frac{d+4}{2}\right)}\right],$$

$$k_2 = \frac{\pi^{\frac{d}{2}}}{2}\left[\frac{a}{\Gamma\left(\frac{d+4}{2}\right)} - \frac{3b}{\sqrt{\pi}\Gamma\left(\frac{d+5}{2}\right)}\right]$$

$$k_3 = \frac{\pi^{\frac{d}{2}}}{2}\left[-\frac{a}{\sqrt{\pi}\Gamma\left(\frac{d+5}{2}\right)} + \frac{3b}{2\Gamma\left(\frac{d+6}{2}\right)}\right].$$

It is obvious that $k_m \neq 0$ for $m \geq 4$, while the conditions $k_m \neq 0, m = 0,1,2,3$, lead to the following assumptions on a and b:

$$\frac{a}{b} \neq \frac{2\Gamma\left(\frac{d+2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+3}{2}\right)}, \quad \frac{a}{b} \neq \frac{3\Gamma\left(\frac{d+3}{2}\right)}{4\sqrt{\pi}\Gamma\left(\frac{d+4}{2}\right)}, \tag{4.6}$$

$$\frac{a}{b} \neq \frac{3\Gamma\left(\frac{d+4}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+5}{2}\right)}, \quad \frac{a}{b} \neq \frac{3\Gamma\left(\frac{d+5}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d+6}{2}\right)}. \tag{4.7}$$

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References

[1] P. D. Fieseler. *A method for solar sailing in a low Earth orbit*. Acta Astronautica **43**, 531-541 (1998).

[2] I.S. Gradshtein and I.M. Ryzhik. *Tables of Integrals, Sums, Series and Products.* 5th Edition, Academic Press, Inc., 1994. 1204 pages.

[3] I K. Harrison and G. G. Swinerd. *A free molecule aerodynamic investigation using multiple satellite analysis*. Planet. Space Sci. **44**, 171-180 (1996).

[4] C. Lemoine. Fourier transforms of homogeneous distribution. *Ann Scuola norm. super. Pisa Sci. fiz. e mat.*, 26(1):117–149, 1972.

[5] I. Newton, *Philosophiae naturalis principia mathematica*. 1687.

[6] S.G. Samko. Spherical potentials, spherical Riesz differentiation and their applications (russian). *Izv. Vuzov. Matematika*, (2):135–139, 1977. Transl. in *Soviet Math. (Izvestiya VUZ. Matematika)*, 1977, 21:2, 106110.

[7] S.G. Samko. *Hypersingular Integrals and their Applications.* London-New-York: Taylor & Francis, Series *"Analytical Methods and Special Functions*", vol. 5, 2002. 358 + xvii pages.

[8] E.M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Space*. Princeton Univ. Press, 1971.

[9] R.S. Strichartz. Multipliers for spherical harmonic expansions. *Trans. Amer. Math. Soc.*, 167(1):115–124, 1972.