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Pseudodifferential Operators Approach to Singular Integral Operators in Weighted Variable Exponent Lebesgue Spaces on Carleson Curves

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Dedicated to the memory of Professor I. Simonenko

Abstract. The main results of the paper are: (1) The boundedness of singular integral operators in the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma, w)$ on a class of composed Carleson curves Γ where the weights w have a finite set of oscillating singularities. The proof of this result is based on the boundedness of Mellin pseudodifferential operators on the spaces $L^{p(\cdot)}(\mathbb{R}_+, d\mu)$ where $d\mu$ is an invariant measure on multiplicative group $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$. (2) Criterion of local invertibility of singular integral operators with piecewise slowly oscillating coefficients acting on $L^{p(\cdot)}(\Gamma, w)$ spaces. We obtain this criterion from the corresponding criteria of local invertibility at the point 0 of Mellin pseudodifferential operators on $\mathbb{R}_+$ and local invertibility of singular integral operators on $\mathbb{R}$. (3) Criterion of Fredholmness of singular integral operators in the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma, w)$ where Γ belongs to a class of composed Carleson curves slowly oscillating at the nodes, and the weight w has a finite set of slowly oscillating singularities.

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1. Introduction

Last decade there arose a big interest to investigation of the classical operators of Analysis, i.e. singular and maximal operators, Hardy operators, pseudodifferential operators, in the $L^{p(\cdot)}$-spaces with variable exponents $p(\cdot)$. Many papers have been devoted to the extension of various results on the boundedness of operators, well known for the constant p, to the case of variable $p(\cdot)$.

This extension is essentially nontrivial and demands new ideas and methods, see for instance [7–11, 28, 44] and references therein.

Similar to the case of the constant p, the Fredholm theory of the mentioned operators in spaces related to $L^{p(\cdot)}$ has also a big interest. With respect to one-dimensional singular integral operators in variable exponent Lebesgue spaces we refer, for instance, to [17–26, 41].

In our paper [41] we proved the boundedness of pseudodifferential operators of the class $OPS^0_{1,0}$ acting in the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and obtained the necessary and sufficient conditions of the Fredhom property of operators of the class $OPS^0_{1,0}$ with symbols slowly oscillating at infinity, in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$. The proof of the sufficiency of conditions of the Fredholmness is more or less standard being based on the calculus of pseudodifferential operators, the boundedness theorems and the interpolation in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$, while the proof of the necessity of those conditions meet big difficulties. (In particular, they are connected with the fact that the shift and dilation operators are unbounded in $L^{p(\cdot)}$).

The main aim of the paper is the Fredholm theory of singular integral operators (SIOs) on composed curves Γ with whirling points and coefficients having slowly oscillating discontinuities acting in the weighted spaces $L^{p(\cdot)}(\Gamma, w)$. Applying results from [41] we prove that singular integral operators are bounded in $L^{p(\cdot)}(\Gamma, w)$ and they are the local type operators in the Simonenko sense [46–48]. Consequently, for the investigation of the Fredholm property we can apply the Simonenko local principle. This principle reduces the investigation of the Fredholm property of local type operators to the investigation of the local invertibility of their local representatives which are simpler operators than the original one.

For instance, the investigation of the Fredholm property of the SIO

$$A = aI + bS_\Gamma,$$

with continuous coefficients a and b and a Lyapunov curve Γ, in the space $L^p(\Gamma), 1 < p < \infty$, is known to be reduced to investigation of local representatives at every point $t_0 \in \Gamma$ which are operators of the type $A_{t_0} = a(t_0)I + b(t_0)S_{\mathbb{R}}$. Their local invertibility in $L^p(\mathbb{R})$ coincides with the invertibility which is equivalent to the condition $a(t_0) \pm b(t_0) \neq 0$. The investigation of the Fredholm property of the operator $A = aI + bS_\Gamma$ with piece-wise continuous coefficients on a simple Lyapunov curve Γ in the space $L^p(\Gamma, w)$ with power weight w, is reduced to the investigation of the local invertibility of the homogeneous operators of the form $aI + bS_{\mathbb{R}}$ acting in $L^p(\mathbb{R})$, where a, b are piecewise constant functions with the only discontinuity at the origin and infinity. These operators are realized as Mellin convolutions and conditions of their invertibility are given in the terms of the Mellin transform of the kernel.

In [2–5, 33–36, 38], the Simonenko local method was applied to SIO on some composed Carleson curves with discontinuous coefficients acting on weighted L^p-spaces, and in the paper [39] for SIO acting on weighted Hölder spaces. In this case the local representatives are *Mellin pseudodifferential operators* with variable symbols. The symbols of local representatives (the local symbols) explain the appearance of the logarithmic double spirales and spiral horns in the local spectrums of SIO. Note that in the theory of Gohberg et al. [14] and Spitkovsky [49] for SIO on Lyapunov curves in L^p-spaces with Muckenhoupt weights, the typical local spectra are circular arcs and circular horns.

We extend here the results of the mentioned papers to the case of variable exponent $p(\cdot)$. The local representatives of the SIO at the singular points $t \in \Gamma$ appear as Mellin pseudodifferential operators with a symbol depending on the curve, weight and coefficients and also on the values of $p(\cdot)$ at singular points t. Making use of the results on local invertibility of Mellin pseudodifferential operators, we obtain necessary and sufficient conditions of the local invertibility of SIOs at singular points of the curves, weights and coefficients. Finally, the application of the Simonenko local principle allows to obtain the necessary and sufficient conditions of Fredhomness in $L^p(\Gamma, w)$.

The methods of localization developed in the paper can be applied to the study of the Fredholm property of multidimensional SIOs and pseudodifferential operators on compact and noncompact manifolds, boundary value problems in Sobolev and Besov spaces connected with $L^{p(\cdot)}$. We hope to do this in forthcoming papers.

Another approach to the investigation of the algebra of operators generated by the operator S_Γ of singular integration along a general composed Carleson curve Γ and operators of multiplication by piece-wise continuous functions, acting in $L^p(\Gamma, w)$, where $1 < p < \infty$, and w is a Muckenhoupt weight, based on the Wiener–Hopf factorization and theory of submultiplicative functions was given by Böttcher and Karlovich (see book [1] and references therein). In [18–20], some results of the book [1] were transferred to algebras of SIO acting in the Lebesgue spaces with variable exponents.

The paper is organized as follows. In Sect. 2 we consider pseudodifferential operators on $\mathbb{R}$ acting in the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$. The main result of this section is a criterion of local invertibility, at the point $+\infty$, of pseudodifferential operators with slowly oscillating symbols, and a criterion of local invertibility of pseudodifferential operators and singular integral operators at the point $x_0 \in \mathbb{R}$.

In Sect. 3 the results of Sect. 2 are reformulated for the Mellin pseudodifferential operators acting on $L^{p(\cdot)}(\mathbb{R}_+, d\mu)$ with the invariant measure $d\mu = \frac{dr}{r}$ on the multiplicative group $\mathbb{R}_+$.

In Sect. 4 we apply the results of Sects. 2 and 3 to the investigation of boundedness, local invertibility and Fredholmness of singular integral operators on composed Carleson curves acting on the Lebesgue spaces $L^{p(\cdot)}(\Gamma, w)$ with weights having a finite set of oscillating singularities. We obtain here the following results:

(1) Theorem on the boundedness of SIO on composed Carleson curves Γ acting on the Lebesgue spaces $L^p(\Gamma, w)$ with weights having a finite set of oscillating singularities. The proof of this theorem is based on the local boundedness of Mellin pseudodifferential operators on the spaces $L^{p(\cdot)}(\mathbb{R}_+, d\mu)$ and an admissible partition of unity on the curve

Γ. The pseudodifferential operators approach demands that the curve near every node is infinitely smooth. But in fact we use the existence of only a finite number of derivatives.

(2) Criterion of the local invertibility and Fredholmness of SIOs on slowly oscillating composed curves with piecewise slowly oscillating coefficients, in the spaces $L^{p(\cdot)}(\Gamma, w)$ with the weight w slowly oscillating at the nodes. The main tools of this section is the local principle of Simonenko and necessary and sufficient conditions of local invertibility of Mellin pseudodifferential operators acting in $L^{p(\cdot)}\left(\mathbb{R}_+, \frac{dr}{r}\right)$ at the point 0, and pseudodifferential and singular integral operators acting in $L^{p(\cdot)}(\mathbb{R})$ at the point $x_0 \in \mathbb{R}$.

Section 5 is devoted to a comparison of the used class of oscillating weights with the Bary-Stechkin type weights. In particular, we show in Lemma 56 that our assumption on the differentiability of weights near the nodes is inessential in the sense that any function in the Bary-Stechkin class is equivalent to N times differentiable function in this class, for any given finite N, the Matuszewska-Orlicz indices coinciding under the equivalence, as is known. However, the conditions on the weights in terms of the Simonenko indices are somewhat stricter than in terms of the Matuszewska-Orlicz indices, see Remark 57.

We will use the following notations:

- for a Banach space X, $\mathcal{B}(X)$ stands for the space of all bounded operators in X,
- $C^\infty(\mathbb{R})$ is the linear space of infinitely differentiable functions on $\mathbb{R}$,
- $C_0^\infty(\mathbb{R})$ is a subspace of $C^\infty(\mathbb{R})$ of functions with compact support,
- $C_b^\infty(\mathbb{R})$ is a subspace of $C^\infty(\mathbb{R})$ of functions bounded on $\mathbb{R}$ with all their derivatives,
- $S(\mathbb{R})$ is the L. Schwartz space of functions in $C^\infty(R)$ decreasing at infinity with all their derivatives faster than every power $|x|^{-n}, n \in \mathbb{N}$.
- If a is a function or matrix, by aI we denote the operator of multiplication by a.

2. Pseudodifferential Operators on $\mathbb{R}$

2.1. Some Properties

In this section we give an auxiliary material on pseudodifferential operators (more information may be found for instance in [38, Chapter 4], or [37]).

Definition 1. (i) We say that a function $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ is a symbol of the class $S_{1,0}^m$ if

$$|a|_{l_1,l_2} = \sum_{\alpha \leq l_1, \beta \leq l_2} \sup_{(x,\xi) \in \mathbb{R}^2} \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \langle \xi \rangle^{-m+\alpha} < \infty, \tag{2.1}$$

for every $l_1, l_2 \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2}$. To a symbol a we relate the pseudodifferential operator (ψdo)

$$Op(a)u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a(x,\xi)u(y)e^{i(x-y)\cdot\xi}dy, \tag{2.2}$$

where $u \in C_0^\infty(\mathbb{R})$; by $OPS_{1,0}^0$ we denote the class of $\psi do's$ with symbols in $S_{1,0}^0$.

(ii) We say that a function $a \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ is a double symbol of the class $S_{1,0,0}^m$ if

$$|a|_{l_1,l_2,l_3} = \sum_{\alpha \le l_1, \beta \le l_2, \gamma \le l_3} \sup_{(x,y,\xi)\in\mathbb{R}^3} \left|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi)\right| \langle \xi \rangle^{\alpha - m} < \infty \tag{2.3}$$

for every $l_1, l_2, l_3 \in \mathbb{N}_0$. To a symbol a we relate the pseudodifferential operator with *double* symbol

$$Op_d(a)u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} a(x,y,\xi)u(y)e^{i(x-y)\cdot\xi}dy, \tag{2.4}$$

where $u \in C_0^\infty(\mathbb{R})$, and we denote the class of $\psi do's$ with symbols in $S_{1,0,0}^m$ by $OPS_{1,0,0}^m$.

Proposition 2. *(Calderon–Vaillancourt, see for instance [38, Theorem 4.1.12]). Let $Op(a) \in OPS_{1,0}^0$. Then the operator $Op(a)$ is bounded in $L^2(\mathbb{R})$ and*

$$\|Op(a)\|_{\mathcal{B}(L^2(\mathbb{R}))} \le C\,|a|_{2,2}\,, \tag{2.5}$$

where C does not depend on a.

Proposition 3. *(see [38, Chapter 4])*

(i) *Let $a_j \in S_{1,0}^{m_j}, j = 1,2$ and $C = Op(a_1)Op(a_2)$. Then $C \in OPS_{1,0}^{m_1+m_2}, C = Op(c)$ where*

$$c(x,\xi) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} a(x,\xi+\eta)b(x+y,\xi)e^{-iy\cdot\eta}dyd\eta. \tag{2.6}$$

Moreover,

$$c(x,\xi) = a(x,\xi)b(x,\xi) + t(x,\xi), \tag{2.7}$$

where $t \in S_{1,0}^{m_1+m_2-1}$.

(ii) *Let $a \in S_{1,0,0}^m$. Then $Op_d(a) \in OPS_{1,0}^m$, $Op_d(a) = OP(a^\#)$ where*

$$a^\#(x,\xi) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} a(x,x+y,\xi+\eta)e^{-iy\cdot\eta}dyd\eta. \tag{2.8}$$

Moreover, $a^\#(x,\xi) = a(x,x,\xi) + t(x,\xi)$, where $t \in S_{1,0}^{m-1}$.

Note that the operators in $OPS^m_{1,0}$ are bounded in $S(\mathbb{R})$ (see for instance [38, Proposition 4.1.5]).

We say that an operator A^τ is formally adjoint to $A = Op(a) \in OPS^m_{1,0}$ if

$$(A^\tau u, v) = (u, Av)$$

for all $u, v \in S(\mathbb{R}^n)$, where $(\cdot,\cdot)$ is the standard scalar product corresponding to $L^2(\mathbb{R})$.

Proposition 4. *Let $a \in S^m_{1,0}$ Then the operator A^τ formally adjoint to $A = Op(a)$ belongs to $OPS^m_{1,0}$ and $A^\tau = Op(a^\tau)$ with*

$$a^\tau(x,\xi) = \frac{1}{2\pi}\int\int_{\mathbb{R}^2} \bar{a}(x+y, \xi+\eta)e^{-iy\cdot\eta}dyd\eta. \tag{2.9}$$

$a^\tau(x,\xi) = \bar{a}(x,\xi) + t(x,\xi)$, *where* $t \in S^{m-1}_{1,0}$.

The integrals in (2.6), (2.8), (2.9) are understood as oscillatory (see [38, Chap. 4.1.2], [37, Chap. 2]).

Definition 5. (i) We say that a symbol $a \in S^0_{1,0}$ is slowly oscillating at the point $+\infty$, if

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \le C_{\alpha\beta}(x)\left\langle\xi\right\rangle^{-\alpha}, \tag{2.10}$$

and $\lim_{x\to+\infty} C_{\alpha\beta}(x) = 0$ for all $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$. We denote this class by $SO_{+\infty}$ and the corresponding class of $\psi do's$ by $OPSO_{+\infty}$.

(ii) We say that a double symbol $a \in S^0_{1,0,0}$ is slowly oscillating at the point $+\infty$, if

$$\left|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi)\right| \le C_{\alpha\beta\gamma}(x,y)\left\langle\xi\right\rangle^{-\alpha}$$

where $\lim_{x\to+\infty} C_{\alpha\beta\gamma}(x,y) = 0$ uniformly with respect y for all $\alpha,\gamma \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$, and $\lim_{y\to+\infty} C_{\alpha\beta\gamma}(x,y) = 0$ uniformly with respect x for all $\alpha,\beta \in \mathbb{N}_0$ and $\gamma \in \mathbb{N}$. We denote this class by $SO_{+\infty,d}$ and the corresponding class of $\psi do's$ by $OPSO_{+\infty,d}$.

(iii) We say that $a \in \mathring{S}_{+\infty}$, if the coefficient $C_{\alpha\beta}(x)$ in estimate (2.10) satisfies the condition $\lim_{x\to+\infty} C_{\alpha\beta}(x) = 0$ for all $\alpha,\beta \in \mathbb{N}_0$. The corresponding class of $\psi do's$ is denoted by $OP\mathring{S}_{+\infty}$.

Proposition 6. *([38, Chap. 4])*

(i) *Let $Op(a_j) \in OPSO_{+\infty}, j = 1,2$ and $B = Op(a_1)Op(a_2)$. Then $B \in OPSO_{+\infty}$ and $B = Op(b)$ with*

$$b(x,\xi) = a_1(x,\xi)a_2(x,\xi) + q(x,\xi),$$

where $q \in \mathring{S}_{+\infty}$.

(ii) *Let $Op_d(a) \in OPSO_{+\infty,d}$. Then $Op_d(a) = Op(a^\#) \in OPSO_{+\infty}$, where*

$$a^\#(x,\xi) = a(x,x,\xi) + q(x,\xi),$$

and $q \in \mathring{S}_{+\infty}$.

(iii) *Let* $Op(a) \in OPSO_{+\infty}$. *Then the formal adjoint operator* $(Op(a))^\tau = Op(a^\tau)$ *is in* $OPSO_{+\infty}$ *with*

$$a^\tau(x,\xi) = \bar{a}(x,x,\xi) + q(x,\xi),$$

and $q \in \mathring{S}^m_{+\infty}$.

2.2. Pseudodifferential Operators on Lebesgue Space with Variable Exponent

We give the definition of variable exponent Lebesgue spaces for the general case where the underlying space is an arbitrary quasimetric measure space, because such spaces will be used in various settings in this paper.

Let (X, d, μ) be a quasimetric measure space, i.e. a topological space endowed with the quasimetric $d : X \times X \to \mathbb{R}^1_+$ and nonnegative Borel measure μ (we refer to [6,12,15] for quasimetric measure spaces). Let $p : X \to (1,\infty)$ be a measurable function on X.

Definition 7. The variable exponent Lebesgue space $L^{p(\cdot)}(X)$ is introduced via the modular

$$I_X^{p(\cdot)}(f) = \int\limits_X |f(x)|^{p(x)}\, d\mu(x) < \infty \tag{2.11}$$

by the norm

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\left\{\lambda > 0 : I_X^{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

We also use a similar space $L_n^{p(\cdot)}(X)$ of vector-functions on X with values in $\mathbb{C}^n$, defined via norm

$$\|f\|_{L_n^{p(\cdot)}(X)} = \inf\left\{\lambda > 0 : I_n^{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\},$$

where $I_n^{p(\cdot)}(f) := \int_X \|f(x)\|_{\mathbb{C}^n}^{p(x)}\, d\mu(x) < \infty$.

Everywhere in the sequel we assume that $p(\cdot)$ satisfies the conditions:

(i) there exists numbers $p_-, p_+ \in (1,\infty)$ such that

$$1 < p_- \le p(x) \le p_+ < \infty. \tag{2.12}$$

(ii) there holds the log-condition

$$|p(x) - p(y)| \le \frac{A}{\log\frac{1}{d(x,y)}}, \qquad x, y \in \mathbb{R}, \qquad d(x,y) \le \frac{1}{2}, \tag{2.13}$$

Under condition (2.12) the space $L^{p(\cdot)}(X)$ is reflexive and $\left(L^{p(\cdot)}(X)\right)^* = L^{q(\cdot)}(X)$ where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1, x \in X$.

The case $X = \mathbb{R}^n$ will be the main one in this paper and in this case we also suppose that

(iii) there exists the limit $\lim\limits_{|x|\to\infty} p(x) = p(\infty)$ and

$$|p(x) - p(\infty)| \le \frac{A}{\log(2 + |x|)}, \qquad x \in \mathbb{R}^n. \tag{2.14}$$

Note that under condition (2.12) for a function $a \in L^\infty(X)$ we have

$$\|aI\|_{\mathcal{B}(L^{p(\cdot)}(X))} \le \|a\|_{L^\infty(X)} \tag{2.15}$$

which follows from the definition of the norm in $L^{p(\cdot)}(X)$, and that the modular convergence is equivalent to the norm convergence. The latter follows from the properties:

$$c_1 \le \|f\|_{L^{p(\cdot)}(X)} \le c_2 \quad \Longrightarrow c_3 \le I^{p(\cdot)}(f) \le c_4 \tag{2.16}$$

and

$$C_1 \le I^{p(\cdot)}(f) \le C_2 \quad \Longrightarrow C_3 \le \|f\|_{L^{p(\cdot)}(\mathbb{R})} \le C_4 \tag{2.17}$$

with $c_3 = \min\left(c_1^{p_-}, c_1^{p_+}\right), c_4 = \max\left(c_2^{p_-}, c_2^{p_+}\right), C_3 = \min\left(C_2^{1/p_-}, C_2^{1/p_+}\right)$, $C_4 = \max\left(C_2^{1/p_-}, C_2^{1/p_+}\right)$.

In the case $X = \mathbb{R}^n$ the imbeddings

$$C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n)$$

hold; they are dense under under assumptions (2.12), (2.13), (2.14) (see, for instance, Theorem 2.11 in [29]).

Proposition 8. *([10]) Let $p_j : \mathbb{R}^n \to [1, \infty), j = 1, 2,$ be bounded measurable functions, A be a linear operator defined on $L^{p_1(\cdot)}(\mathbb{R}^n) \cap L^{p_2(\cdot)}(\mathbb{R}^n)$ and*

$$\|Au\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \le C_j \|u\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}, \quad j = 1, 2. \tag{2.18}$$

Then A is also bounded on the intermediate space $L^{p_\theta(\cdot)}(\mathbb{R}^n)$, where

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)},\ \theta \in [0, 1],$$

and

$$\|A\|_{\mathcal{B}(L^{p_\theta(\cdot)})} \le \|A\|^\theta_{\mathcal{B}(L^{p_1(\cdot)})} \|A\|^{1-\theta}_{\mathcal{B}(L^{p_2(\cdot)})}.$$

The following proposition is an extension of the well-known theorem of Krasnosel′skii [30] on the interpolation of the compactness property in L^p-spaces with a constant p.

Proposition 9. *([41, Proposition 2.2]) Let $p_j : \mathbb{R}^n \to [1, \infty), j = 1, 2,$ be bounded measurable functions satisfying assumptions (2.12)–(2.14) and let a linear operator A defined on $L^{p_1(\cdot)}(\mathbb{R}^n) \cap L^{p_2(\cdot)}(\mathbb{R}^n)$ satisfy the boundedness assumptions in (2.18). If*

$$A : L^{p_1(\cdot)}(\mathbb{R}^n) \to L^{p_1(\cdot)}(\mathbb{R}^n)$$

is a compact operator, then

$$A : L^{p_\theta(\cdot)}(\mathbb{R}^n) \to L^{p_\theta(\cdot)}(\mathbb{R}^n)$$

is a compact operator in every intermediate space $L^{p_\theta(\cdot)}(\mathbb{R}^n), \theta \in (0, 1]$.

Theorem 10. *([41, Theorem 5.1]) An operator $Op(a) \in OPS^0_{1,0}$ is bounded in $L^{p(\cdot)}(\mathbb{R})$ and there exists $M > 0$ and $C > 0$ not depending on a such that*

$$\|Op(a)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \le C\,|a|_{M,M}$$

Proposition 3 and Theorem 10 imply the following

Corollary 11. *An operator* $Op(a) \in OPS^0_{1,0,0}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R})$ *and there exists* $M > 0$ *and* $C > 0$ *not depending on* A *such that*

$$\|Op(a)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \leq C\,|a|_{M,M,M}\,,$$

where $C > 0$ *and* $M > 0$ *do not depend on* a.

Note that, because $S(\mathbb{R})$ is dense in $L^{p(\cdot)}(\mathbb{R})$, the formal adjoint A^τ to the operator $A = Op(a) \in OPS^0_{1,0}$ coincides with the operator A^* adjoint to the operator A acting in $L^{p(\cdot)}(\mathbb{R})$. Hence $A^* = Op(a^\tau) \in OPS^0_{1,0}$, where a^τ is defined by (2.9).

Proposition 12. *Let* χ_R *be the characteristic function of the segment* $[R, +\infty)$, $Q = Op(q) \in OP\mathring{S}_{+\infty}$. *Then*

$$\lim_{R\to+\infty} \|\chi_R Q\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{R\to+\infty} \|Q\chi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0. \tag{2.19}$$

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be a real-valued function such that

$$\varphi(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x \leq 1/2 \end{cases},$$

and $\varphi_R(x) = \varphi(\frac{x}{R}), R > 0$. We have $\varphi_R Q = Op(\varphi_R q)$. Since $q \in \mathring{S}_{+\infty}$, we have

$$\lim_{R\to\infty} |\varphi_R q|_{l_1,l_2} = 0$$

for every $l_1, l_2 \in \mathbb{N}_0$. Applying Theorem 10, we obtain that $\lim_{R\to\infty} \|\varphi_R Q\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0$. Now we will prove that $\lim_{R\to+\infty} \|Q\varphi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0$. We have $\|Q\varphi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \|\varphi_R Q^*\|_{\mathcal{B}(L^{q(\cdot)}(\mathbb{R}))}$, where $Q^* \in OP\mathring{S}_{+\infty}$ by statement (iii) of Proposition 6. Hence

$$\lim_{R\to\infty} \|Q\varphi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{R\to\infty} \|\varphi_R Q^*\|_{\mathcal{B}(L^{q(\cdot)}(\mathbb{R}))} = 0. \tag{2.20}$$

Since $\varphi_R\chi_R = \chi_R$, equality (2.20) implies (2.19). □

2.3. Local Invertibility at $+\infty$

Definition 13. We say that an operator $A \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ is locally invertible at the point $+\infty$, if there exist operators $\mathcal{L}_R$ and $\mathcal{R}_R$ such that

$$\mathcal{L}_R\, A\chi_R I = \chi_R I, \chi_R A\mathcal{R}_R = \chi_R I. \tag{2.21}$$

We also need the following propositions, where by

$$V_h u(x) = u(x-h)$$

we denote the translation operator.

Proposition 14. *([41, Proposition 6.3]) Let a sequence* $(\mathbb{R} \ni)\, h_m \to +\infty$, *and* $w_m\, (\in C(\mathbb{R}))$ *be a sequence converging in the* sup-*norm on* $\mathbb{R}$ *to a function* $w \in C(\mathbb{R})$. *Suppose also that there exists a constant* $C > 0$ *such that* $|w_m(x)| \leq \frac{C}{|x|}$ *for every* $m \in \mathbb{N}$ *and* $|w(x)| \leq \frac{C}{|x|}$. *Then*

$$\lim_{m\to\infty} \|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R})} = \|w\|_{L^{p(+\infty)}(\mathbb{R})}\,. \tag{2.22}$$

Proposition 15. *([41, Proposition 6.4]) Let* $Op(a) \in OPSO_{+\infty}$*, and a sequence* $h_m \to +\infty$*. Then there exists a subsequence* h_{m_k} *of* h_m *and a symbol* $a_{(h)} \in OPS^0_{1,0}$ *not depending on* x*, such that for every function* $u \in C_0^\infty(\mathbb{R})$

$$\lim_{k\to\infty} V_{-h_{m_k}} Op(a) V_{h_{m_k}} u = Op(a_{(h)})u$$

in the topology of $S(\mathbb{R})$*.*

In what follows, if a is a symbol and $h \in \mathbb{R}$, then a^h denotes the symbol shifted in x, that is, $a^h(x,\xi) = a(x+h,\xi)$. Note that $V_{-h}Op(a)V_h = Op(a^h)$.

Proposition 16. *Let* $Op(a) \in OPS\mathring{O}_{+\infty}$*, and a sequence* $h_m \to +\infty$*. Then for every function* $u \in C_0^\infty(\mathbb{R})$

$$\lim_{m\to\infty} \|V_{-h_m} Op(a) V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R})} = 0. \tag{2.23}$$

Proof. We have $V_{-h_m}Op(a)V_{h_m} = Op(a^{h_m})$. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi u = u$. Hence

$$V_{-h_m} Op(a) V_{h_m} u = Op_d(a^{h_m}\varphi)u.$$

Applying formula (2.8) we obtain that $Op_d(a^{h_m}\varphi) = Op(b_m)$ where

$$b_m(x,\xi) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} a(x+h_m,\xi+\eta)\varphi(x+y)e^{-iy\cdot\eta}dyd\eta. \tag{2.24}$$

Then applying the definition of the oscillatory integral in (2.24) we obtain that

$$\lim_{m\to\infty} \sup_{(x,\xi)\in\mathbb{R}^2} \left|\partial_x^\beta \partial_\xi^\alpha b_m(x,\xi)\right| = 0$$

for all $\alpha, \beta \in \mathbb{N}_0$. Theorem 10 implies that

$$\lim_{m\to\infty} \|Op(b_m)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{m\to\infty} \|Op_d(a_{h_m}\varphi)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0. \tag{2.25}$$

Hence the statement of the proposition follows from formula (2.25). □

Theorem 17. *Let* $Op(a) \in OPSO_{+\infty}$*. Then the operator* $Op(a)$ *:* $L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$ *is locally invertible at the point* $+\infty$ *if and only if*

$$\liminf_{x\to+\infty} \inf_{\xi\in\mathbb{R}} |a(x,\xi)| > 0. \tag{2.26}$$

Proof. (a) First we prove that condition (2.26) is sufficient for the local invertibility of $Op(a)$ at the point $+\infty$. Let φ_R be the function from the proof of Proposition 16. Condition (2.26) implies that there exists an $R_0 > 0$ such that $b_{R_0} = \varphi_{R_0}a^{-1} \in SO_{+\infty}$. Hence by Proposition 6

$$Op(b_{R_0})Op(a) = \varphi_{R_0}I + Q_{R_0}, \tag{2.27}$$

where $Q_{R_0} \in OP\mathring{S}_{+\infty}$. Equality (2.27) implies that

$$Op(b_{R_0})Op(a)\chi_R I = (I + Q_{R_0}\chi_R I)\chi_R I, \tag{2.28}$$

where R is such that $\varphi_{R_0}\chi_R = \chi_R$. By Proposition 12 we can choose an R such that $\|Q\chi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < 1$. Hence

$$(I + Q\chi_R I)^{-1} Op(b_R)Op(a)\chi_R I = \chi_R I.$$

Thus, the operator $Op(a)$ is locally left invertible at the point $+\infty$. In the same way we prove that $Op(a)$ is locally right invertible at the point $+\infty$.

(b) Now we prove the necessity of condition (2.26) for the local invertibility of $Op(a)$ at the point $+\infty$. Let $Op(a) : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$ be a locally invertible operator. Then there exists $C > 0$ and $R > 0$ such that

$$\|Op(a)\chi_R u\|_{L^{p(\cdot)}(\mathbb{R})} \geq C \|\chi_R u\|_{L^{p(\cdot)}(\mathbb{R})} \tag{2.29}$$

for every $u \in C_0^\infty(\mathbb{R})$.

Let a sequence $h_m \in \mathbb{R}$ tend to $+\infty$, and $u \in C_0^\infty(\mathbb{R})$. Then for a fixed $R > 0$ there exists $m_0 > 0$ such that $\chi_R V_{h_m} u = V_{h_m} u$ for $m \geq m_0$. Hence for such m

$$\begin{aligned} \|V_{h_m}(V_{-h_m} Op(a) V_{h_m} u)\|_{L^{p(\cdot)}(\mathbb{R})} &= \|Op(a)\chi_R V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R})} \\ &\geq C \|V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R})}. \end{aligned} \tag{2.30}$$

Let h_{m_k} be a subsequence of h_m defined in Proposition 15 and let

$$w_k = V_{-h_{m_k}} Op(a) V_{h_{m_k}} u = Op\left(a^{h_{m_k}}\right) u.$$

Applying Proposition 15, we obtain that $w_k \to w = Op(a_{(h)})u$ in the space $S(\mathbb{R})$. Then we can use Proposition 14 to pass to the limit in the inequality

$$\left\|V_{h_{m_k}} w_k\right\|_{L^{p(\cdot)}(\mathbb{R})} \geq C \left\|V_{h_{m_k}} u\right\|_{L^{p(\cdot)}(\mathbb{R})},$$

and obtain that

$$\left\|Op(a_{(h)})u\right\|_{L^{p(+\infty)}(\mathbb{R})} \geq C \|u\|_{L^{p(+\infty)}(\mathbb{R})}, \tag{2.31}$$

where the symbol $a_{(h)}$ depends only on ξ. Estimate (2.31) implies the condition

$$\inf_{\xi \in \mathbb{R}} \left|a_{(h)}(\xi)\right| > 0. \tag{2.32}$$

Thus, we proved that for every sequence $h_m \to +\infty$ there exists a subsequence h_{m_k} and a limit symbol $a_{(h)} \in S_{1,0}^0$ such that the sequence $a(h_{m_k}, \xi)$ converges uniformly on $\mathbb{R}$ to the limit function $a_{(h)}(\xi)$ for which condition (2.32) holds. Suppose now that condition (2.26) is not satisfied. Then there exists a sequence $(h_m, \xi_m), h_m \to +\infty$ such that

$$\lim_{m \to \infty} a(h_m, \xi_m) = 0. \tag{2.33}$$

Choose a subsequence h_{m_k} of the sequence h_m such that $a(h_{m_k}, \xi)$ converges uniformly with respect to $\xi \in \mathbb{R}$ to the limit function $a_h(\xi)$ for which condition (2.32) holds. Then

$$\lim_{k \to \infty} a(h_{m_k}, \xi_{m_k}) = 0 \tag{2.34}$$

and

$$\lim_{k \to \infty} \left|a(h_{m_k}, \xi_{m_k}) - a_{(h)}(\xi_{m_k})\right| = 0. \tag{2.35}$$

Hence (2.34) and (2.35) contradict to (2.32). □

By $OPS^0_{1,0}(n)$ $(OPSO_{+\infty}(n))$ we denote the class of $\psi do's$ $Op(a)$, where a is a matrix with entries $a_{ij} \in S^0_{1,0}$ $(SO_{+\infty})$.

Theorem 17 is reformulated for the matrix case in the following form.

Theorem 18. *Let $Op(a) \in OPSO_{+\infty}(n)$. Then $Op(a) : L^{p(\cdot)}_n(\mathbb{R}) \to L^{p(\cdot)}_n(\mathbb{R})$ is locally invertible at the point $+\infty$ if and only if*

$$\liminf_{x\to+\infty} \inf_{\xi\in\mathbb{R}} |\det(a(x,\xi))| > 0. \tag{2.36}$$

2.4. Local Invertibility at the Point $x_0 \in \mathbb{R}$

Definition 19. We say that $A \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ is locally invertible at the point $x_0 \in \mathbb{R}$, if there exist an interval $\mathcal{I}_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon\ )$ and operators $\mathcal{L}_{x_0,\varepsilon}, \mathcal{R}_{x_0,\varepsilon} \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ such that

$$\mathcal{L}_{x_0,\varepsilon} A \chi^{x_0}_\varepsilon I = \chi^{x_0}_\varepsilon I, \chi^{x_0}_\varepsilon A \mathcal{R}_{x_0,\varepsilon} = \chi^{x_0}_\varepsilon I,$$

where $\chi^{x_0}_\varepsilon = \chi_{\mathcal{I}_\varepsilon(x_0)}$ is the characteristic function of $\mathcal{I}_\varepsilon(x_0)$. The operators $\mathcal{L}_{x_0,\varepsilon}(\mathcal{R}_{x_0,\varepsilon})$ are called left (right) locally inverse operators.

We consider a subclass $\tilde{S}^0_{1,0}$ of symbols in $S^0_{1,0}$ for which there exist functions $a^\pm \in C^\infty_b(\mathbb{R})$ such that

$$\lim_{\xi\to\pm\infty} \sup_{x\in\mathbb{R}} \left|a(x,\xi) - a^\pm(x)\right| = 0. \tag{2.37}$$

Let $Op(a) \in OP\tilde{S}^0_{1,0}$. Then we set

$$\sigma_{x_0}(A) = \left\{a^+(x_0), a^-(x_0)\right\}$$

and say that $\sigma_{x_0}(Op(a))$ is the local symbol of the operator $Op(a)$ at the point $x_0 \in \mathbb{R}$. Note that if $Op(a_j) \in OP\tilde{S}^0_{1,0}, j = 1, 2$, then

$$\begin{aligned}\sigma_{x_0}(Op(a_1)Op(a_2)) &= \sigma_{x_0}(Op(a_1))\sigma_{x_0}(Op(a_2)) \\ &:= \left\{a_1^+(x_0)a_2^+(x_0), a_1^-(x_0)a_1^-(x_0)\right\}.\end{aligned}$$

The ψdo $Op(a) \in OP\tilde{S}^0_{1,0}$ is called elliptic at the point x_0, if the local symbol $\sigma_{x_0}(Op(a))$ is invertible, that is, $a^\pm(x_0) \neq 0$.

In this section we also need the following propositions.

Proposition 20. *Let $t \in S^0_{1,0}$ and*

$$\lim_{(x,\xi)\to 0} t(x,\xi) = 0. \tag{2.38}$$

Then $Op(t)$ is a compact operator in $L^{p(\cdot)}(\mathbb{R})$, where $p(\cdot)$ satisfies conditions (2.12)–(2.14).

Proof. Condition (2.38) implies that $Op(t)$ is compact in $L^2(\mathbb{R})$ (see [37, Theorem 5.8.3]). We can find a function $r : \mathbb{R} \to (1, \infty)$ satisfying (2.12)–(2.14) such that $L^{p(\cdot)}(\mathbb{R})$ is an intermediate space between $L^2(\mathbb{R})$ and $L^{r(\cdot)}(\mathbb{R})$. Hence $Op(t)$ is a compact operator in $L^{p(\cdot)}(\mathbb{R})$ by Proposition 9. □

Let $\varphi \in C^\infty_0(\mathbb{R}), \varphi(x) = 1$ if $|x| \leq \frac{1}{2}$, $supp\,\varphi = [-1, 1]$, and $0 \leq \varphi(x) \leq 1$. We set $\varphi^{x_0}_\varepsilon(x) = \varphi(\frac{x-x_0}{\varepsilon})$.

Proposition 21. *Let $t \in S^0_{1,0}$ and*

$$\lim_{\xi\to\infty} \sup_{x\in\mathbb{R}^n} |t(x,\xi)| = 0. \tag{2.39}$$

Then

$$\lim_{\varepsilon\to 0} \|Op(t)\chi_\varepsilon^{x_0} I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{\varepsilon\to 0} \|\chi_\varepsilon^{x_0} Op(t)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0 \tag{2.40}$$

and

$$\lim_{\varepsilon\to 0} \|Op(t)\varphi_\varepsilon^{x_0} I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{\varepsilon\to 0} \|\varphi_\varepsilon^{x_0} Op(t)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0. \tag{2.41}$$

Proof. Fix $\varepsilon_0 > 0$ and let $0 < \varepsilon < \varepsilon_0$. Then $Op(t)\chi_\varepsilon^{x_0} I = Op(t)\chi_{\varepsilon_0}^{x_0}\chi_\varepsilon^{x_0} I$. The operator $Op(t)\chi_{\varepsilon_0}^{x_0} I$ is compact by Proposition 20, and $\chi_\varepsilon^{x_0} I \to 0$ if $\varepsilon \to 0$ strongly in $L^{p(\cdot)}(\mathbb{R})$. Hence $\lim_{\varepsilon\to 0} \|Op(t)\chi_\varepsilon^{x_0} I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0$. Passing to the adjoint operators and taking into account that (2.39) implies the convergence

$$\lim_{\xi\to\infty} \sup_{x\in\mathbb{R}^n} |t^\tau(x,\xi)| = 0,$$

we obtain that

$$\lim_{\varepsilon\to 0} \|\chi_\varepsilon^{x_0} Op(t)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \left\|(Op(t))^* \chi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{q(\cdot)}(\mathbb{R}))} = 0.$$

Formula (2.41) follows from (2.40). □

Proposition 22. *Let $(\tau_{x_0,\delta} u)(x) = \delta^{-\frac{1}{p(x)}} u\left(\frac{x-x_0}{\delta}\right), \delta > 0$. Then*

$$\lim_{\delta\to 0} \|\tau_{x_0,\delta} u\|_{L^{p(\cdot)}(\mathbb{R})} = \|u\|_{L^{p(x_0)}(\mathbb{R})}$$

for every function $u \in C_0^\infty(\mathbb{R})$.

Proof. Fix a function $u \in C_0^\infty(\mathbb{R})$ and set

$$F(\lambda,\delta) = I_\lambda^{p(\cdot)}(\tau_{x_0,\delta} u) = \int_{\mathbb{R}} \left|\frac{u\left(\frac{x-x_0}{\delta}\right)}{\lambda}\right|^{p(x)} \delta^{-1} dx, \quad \lambda > 0$$

After the change of the variables $\frac{x-x_0}{\delta} = y$ we get

$$F(\lambda,\delta) = \int_{\mathbb{R}} \left|\frac{u(y)}{\lambda}\right|^{p(x_0+\delta y)} dy. \tag{2.42}$$

Passing to the limit in (2.42) as $\delta \to 0$, we obtain

$$\lim_{\delta\to 0} F(\lambda,\delta) = \int_{\mathbb{R}} \left|\frac{u(y)}{\lambda}\right|^{p(x_0)} dx := F(\lambda,0) \tag{2.43}$$

where the convergence is uniform with respect to $\lambda > 0$ on every segment $[a,b] \subset \mathbb{R}$.

Note that $F : (0,+\infty)\times[0,1] \to \mathbb{R}_+$ is a continuous function. Moreover, there exists a partial derivative $F'_\lambda(\lambda,\delta) < 0$ for every $(\lambda,\delta) \in (0,+\infty)\times[0,1]$. Hence for every fix $\delta \in [0,1]$, $F(\cdot,\delta)$ is a monotonically decreasing function of λ on $(0,\infty)$. It implies that

$$\|\tau_{x_0,\delta} u\|_{L^{p(\cdot)}(\mathbb{R})} = \inf\{\lambda > 0 : F(\lambda,\delta) \le 1\} = \lambda(\delta)$$

where $\lambda(\delta)$ is a solution of the equation $F(\lambda, \delta) = 1$. One can see that for $\delta = 0$ the equation $F(\lambda, 0) = 1$ has an unique solution $\lambda(0) = \|u\|_{L^{p(x_0)}(\mathbb{R})}$. Moreover,

$$F'_\lambda \left(\|u\|_{L^{p(x_0)}(\mathbb{R})}, 0 \right) \neq 0.$$

Hence by the Implicit Function Theorem we obtain that there exists a unique solution $\lambda(\delta)$ of the equation $F(\lambda, \delta) = 1$ for small δ and $\lambda(\delta)$ is a continuous function in a neighborhood of the point 0.

Hence

$$\|u\|_{L^{p(x_0)}(\mathbb{R}^n)} = \lambda(0) := \lim_{\delta \to 0} \lambda(\delta) = \lim_{\delta \to 0} \|\tau_{x_0,\delta} u\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for every function $u \in C_0^\infty(\mathbb{R})$. □

Let $\phi \in C_0^\infty(\mathbb{R})$ be a real-valued function such that $\phi(\xi) = 1$ if $|\xi| \leq 1$, $\phi(\xi) = 0$ if $|\xi| \geq 2$, and $0 \leq \phi(\xi) \leq 1$. Let also $\phi_R(\xi) = \phi(\xi/R)$ and $\psi_R = 1 - \phi_R$.

Theorem 23. *Let* $a \in \tilde{S}^0_{1,0}$. *Then* $Op(a) : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$ *is locally invertible at a point* $x_0 \in \mathbb{R}$ *if and only if* $Op(a)$ *is an elliptic operator at the point* x_0.

Proof. (i) First we prove that *the local ellipticity of* $Op(a)$ at the point x_0 implies the local invertibility at this point. Let $a^0(x, \xi) = a^+(x)\theta(\xi) + a^-(\xi)(1 - \theta(\xi))$, where θ is the characteristic function of $\mathbb{R}_+$. Since $a \in \tilde{S}^0_{1,0}$, we then obtain

$$\lim_{R \to +\infty} \sup_{(x,\xi) \in \mathbb{R}^2} \left| (a(x, \xi) - a^0(x, \xi))\psi_R(\xi) \right| = 0. \tag{2.44}$$

and

$$\lim_{\varepsilon \to 0} \sup_x \left| \left(a^\pm(x) - a^\pm(x_0) \right) \varphi^{x_0}_\varepsilon(x) \right| = 0. \tag{2.45}$$

Hence

$$\lim_{\varepsilon \to 0, R \to +\infty} \sup_{(x,\xi) \in \mathbb{R}^2} \left| (a(x, \xi) - a^0(x_0, \xi))\varphi^{x_0}_\varepsilon(x)\psi_R(\xi) \right| = 0. \tag{2.46}$$

In view of the ellipticity of $Op(a)$ at the point x_0 and relation (2.46) we obtain that there exist ε_0 and R_0 such that the symbol $b(x, \xi) = a^{-1}(x, \xi)\varphi^{x_0}_{\varepsilon_0}(x)\psi_{R_0}(\xi)$ is in $S^0_{1,0}$. Then

$$Op(b)Op(a) = Op(\varphi^{x_0}_{\varepsilon_0}\psi_{R_0}) + Op(t_{\varepsilon_0,R_0}), \tag{2.47}$$

where $t_{\varepsilon_0,R_0} \in S^{-1}_{1,0}$ by formula (2.7).

Formula (2.47) implies that

$$Op(b)Op(a) = \varphi^{x_0}_{\varepsilon_0} I + \varphi^{x_0}_{\varepsilon_0} Op(\phi_{R_0}) + Op(t_{\varepsilon_0,R_0}). \tag{2.48}$$

Choose $\varepsilon > 0$ such that $\chi^{x_0}_\varepsilon \varphi^{x_0}_{\varepsilon_0} = \chi^{x_0}_\varepsilon$. Then from (2.48) we get

$$Op(b)Op(a)\chi^{x_0}_\varepsilon I = \chi^{x_0}_\varepsilon I + Q_\varepsilon, \tag{2.49}$$

where

$$Q_\varepsilon = \varphi^{x_0}_{\varepsilon_0} Op(\phi_{R_0})\chi^{x_0}_\varepsilon I + Op(t_{\varepsilon_0,R_0})\chi^{x_0}_\varepsilon I$$

is a compact operator in $L^{p(\cdot)}(\mathbb{R})$ by Proposition 20. Since we have the strong convergence $\chi_\varepsilon^{x_0} I \to 0$ in $L^{p(\cdot)}(\mathbb{R})$, we can choose $\varepsilon' > 0$ small enough such that $\|Q_\varepsilon \chi_{\varepsilon'}^{x_0} I\| < 1$. Hence

$$(I + Q_\varepsilon \chi_{\varepsilon'}^{x_0} I)^{-1} Op(b) Op(a) \chi_{\varepsilon'}^{x_0} I = \chi_{\varepsilon'}^{x_0} I.$$

Hence $\mathcal{L} = (I + Q_\varepsilon \chi_{\varepsilon'}^{x_0} I)^{-1} Op(b)$ is the left locally inverse operator at the point $x_0 \in \mathbb{R}$. In the same way we prove that there exists a right locally inverse operator at the point x_0.

(ii) Now we prove that the local invertibility of $A = Op(a)$ at the point x_0 implies the local ellipticity of $Op(a)$ at this point. We denote

$$\begin{aligned} A^0 &= a^+ P_+ + a^- P_-, \\ A^{x_0} &= a^+(x_0) P_+ + a^-(x_0) P_-, \end{aligned}$$

where

$$P_\pm = \frac{1}{2}(I \pm S_\mathbb{R}) \quad \text{and} \quad (S_\mathbb{R} u)(x) = \frac{1}{\pi i} \int\limits_\mathbb{R} \frac{u(y)dy}{y - x}.$$

Note that the SIOs A^0 and A^{x_0} are bounded in $L^{p(\cdot)}(\mathbb{R})$ (see for instance [41]). By the multiplicative inequality (see for instance [50, p. 22], or [37, Proposition 5.8.1]) formula (2.44) implies that

$$\lim_{R \to +\infty} \sup_{(x,\xi) \in \mathbb{R}^2} \left| \partial_x^\beta \partial_\xi^\alpha ((a(x,\xi) - a^0(x,\xi)) \psi_R(\xi)) \right| = 0,$$

By Theorem 10, for each $\eta > 0$ we can find an $R_0 > 0$ such that

$$\lim_{R \to \infty} \left\| (A - A^0) Op(\psi_{R_0}) \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta. \tag{2.50}$$

Continuity of the coefficients $a^\pm$ at the point x_0 implies that for every $\eta > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\left\| (A^0 - A^{x_0}) \varphi_\varepsilon^{x_0} I \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta. \tag{2.51}$$

Furthermore,

$$\begin{aligned} \left\| (A - A^0) \varphi_\varepsilon^{x_0} I \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} &\le \left\| (A - A^0) Op(\psi_{R_0}) \varphi_\varepsilon^{x_0} I \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \\ &\quad + \left\| (A - A^0) Op(\phi_{R_0}) \varphi_\varepsilon^{x_0} I \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} &\left\| (A - A^0) Op(\phi_{R_0}) \varphi_\varepsilon^{x_0} I \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \\ &\quad \le \left\| (A - A^0) \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \| Op(\phi_{R_0}) \varphi_\varepsilon^{x_0} I \|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}. \end{aligned}$$

By Proposition 21, for small $\varepsilon > 0$ we have

$$\| Op(\phi_{R_0}) \varphi_\varepsilon^{x_0} I \|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \frac{\eta}{\| (A - A^0) \|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}}.$$

Hence

$$\left\| (A - A^0) Op(\phi_{R_0}) \varphi_\varepsilon^{x_0} I \right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta. \tag{2.53}$$

Thus, estimates (2.50), (2.52) and (2.53) yield that

$$\| (A - A^{x_0}) \varphi_\varepsilon^{x_0} I \|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < 3\eta \tag{2.54}$$

for small $\varepsilon > 0$. Let A be locally invertible at x_0. Then there exist $\varepsilon' > 0$ and $C > 0$ such that the following estimate holds

$$\|A\chi_{\varepsilon'}^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})} \geq C\,\|\chi_{\varepsilon'}^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})} \tag{2.55}$$

for every $u \in C_0^\infty(\mathbb{R})$. Note that for $\varepsilon > 0$ small enough $\chi_{\varepsilon'}^{x_0}\varphi_\varepsilon^{x_0} = \varphi_\varepsilon^{x_0}$. Then (2.55) implies

$$\|A\varphi_\varepsilon^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})} \geq C\,\|\varphi_\varepsilon^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})}, \quad u \in C_0^\infty(\mathbb{R}). \tag{2.56}$$

Let $\eta = \frac{C}{6}$. Then (2.54) and (2.56) yield that

$$\|A^{x_0}\varphi_\varepsilon^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})} \geq \frac{C}{2}\,\|\varphi_\varepsilon^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})}, \quad u \in C_0^\infty(\mathbb{R}). \tag{2.57}$$

We replace u in (2.57) by $\tau_{x_0,\delta}u$ where $\delta > 0$. Then for $\delta > 0$ small enough $\varphi_\varepsilon^{x_0}(\tau_{x_0,\delta}u) = \tau_{x_0,\delta}u$. Since A_{x_0} commutes with the operator $\tau_{x_0,\delta}$, from (2.57) we obtain

$$\|\tau_{x_0,\delta}A^{x_0}u\|_{L^{p(\cdot)}(\mathbb{R})} \geq \frac{C}{2}\,\|\tau_{x_0,\delta}u\|_{L^{p(\cdot)}(\mathbb{R})}. \tag{2.58}$$

Passing to the limit as $\delta \to 0$ in (2.58) and applying Proposition 22, we obtain the estimate

$$\|A^{x_0}u\|_{L^{p(x_0)}(\mathbb{R})} \geq \frac{C}{2}\,\|u\|_{L^{p(x_0)}(\mathbb{R})} \tag{2.59}$$

for every $u \in C_0^\infty(\mathbb{R})$. In the same way, from the estimate

$$\|A^*\chi_\varepsilon^{x_0}u\|_{L^{q(\cdot)}(\mathbb{R})} \geq C\,\|\chi_\varepsilon^{x_0}u\|_{L_n^{q(\cdot)}(\mathbb{R})}, \quad u \in C_0^\infty(\mathbb{R}) \tag{2.60}$$

we obtain that

$$\|(A^{x_0})^*v\|_{L^{q(x_0)}(\mathbb{R})} \geq \frac{C}{2}\,\|v\|_{L^{q(x_0)}(\mathbb{R})}, v \in C_0^\infty(\mathbb{R}). \tag{2.61}$$

Since $C_0^\infty(\mathbb{R})$ is dense in $L^{p(x_0)}(\mathbb{R})$, estimates (2.60) and (2.61) imply the invertibility of A^{x_0} in $L^{p(x_0)}(\mathbb{R})$. It remains to note that the invertibility of the SIO A^{x_0} in the space $L^p(\mathbb{R})$ with constant $p \in (1,\infty)$ implies, as is well known, the condition $a_\pm(x_0) \neq 0$ (see for instance [47]). □

Theorem 24. *Let $A^0 = a^+P_+ + a^-P_-$ be a SIO with coefficients $a^\pm \in L^\infty(\mathbb{R})$ continuous at a point $x_0 \in \mathbb{R}$. Then $A : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$ is locally invertible at the point x_0, if and only if $a^\pm(x_0) \neq 0$.*

Proof. (i) Let the condition $a^\pm(x_0) \neq 0$ hold. By the continuity of $a^\pm$ at the point x_0, for every $\eta > 0$ we can find an $\varepsilon > 0$ such that

$$\left\|\left(A^0 - A^{x_0}\right)\varphi_\varepsilon^{x_0}I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta. \tag{2.62}$$

Hence

$$A^0\varphi_\varepsilon^{x_0}I = A^{x_0}\varphi_\varepsilon^{x_0}I + T_\varepsilon, \tag{2.63}$$

where $\|T_\varepsilon\| < \eta$. The condition $a^\pm(x_0) \neq 0$ implies that there exists the inverse operator $(A^{x_0})^{-1} = a^+(x_0)^{-1}P_+ + a^-(x_0)^{-1}P_-$. Let $\eta < \left\|(A^{x_0})^{-1}\right\|$. Then there exists an ε' such that $\varphi_\varepsilon^{x_0}\chi_{\varepsilon'}^{x_0} = \chi_{\varepsilon'}^{x_0}$. From formula (2.63) we get

$$(I + T_\varepsilon\chi_{\varepsilon'}^{x_0}I)^{-1}\,(A^{x_0})^{-1}\,A^0\chi_{\varepsilon'}^{x_0}I = \chi_{\varepsilon'}^{x_0}I.$$

Hence there exists a left locally inverse operator for A^0 at the point x_0. In the same way we prove that there exists a right locally inverse operator.

(ii) Let A^0 be a locally invertible operator at the point x_0. Then (2.62) implies that A^{x_0} is also locally invertible at the point x_0. Hence for every $u \in C_0^\infty(\mathbb{R})$ estimate (2.59) holds. As in the part (ii) of the proof of Theorem 23 we obtain that $a_\pm(x_0) \neq 0$. □

3. Mellin Pseudodifferential Operators

3.1. Main Property

In this section we reformulate the results of Sect. 2 for the Mellin pseudodifferential operators. (See, for instance [38, Chapter 4.5]).

Definition 25. (i) We say that a matrix-function $a = (a_{ij})_{i,j=1}^n$ belongs to $\mathcal{E}(n)$, if $a_{ij} \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and

$$|a|_{l_1,l_2} = \max_{1\le i,j\le n} \sup_{(r,\xi)\in\mathbb{R}_+\times\mathbb{R}} \sum_{\alpha\le l_1,\beta\le l_2} \left|(r\partial_r)^\beta \partial_\xi^\alpha a_{ij}(r,\xi)\right| \langle\xi\rangle^\beta < \infty, \quad \langle\xi\rangle = (1+\xi^2)^{1/2} \tag{3.1}$$

for all $l_1, l_2 \in \mathbb{N}_0$.

(ii) We say that a matrix-function $a = (a_{ij})_{i,j=1}^n$ belongs to $\mathcal{E}_d(n)$, if $a_{ij} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R})$ and

$$|a|_{l_1,l_2,l_3} = \max_{1\le i,j\le n} \sup_{(r,\rho,\xi)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}} \sum_{\alpha\le l_1,\beta\le l_2,\gamma\le l_3} \left|(r\partial_r)^\alpha (\rho\partial_\rho)^\gamma \partial_\xi^\beta a_{ij}(r,\rho,\xi)\right| \langle\xi\rangle^\beta < \infty, \tag{3.2}$$

for all $l_1, l_2, l_3 \in \mathbb{N}_0$.

(iii) Let $a \in \mathcal{E}(n)$. The operator

$$(Op(a)u)(r) = (2\pi)^{-1} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}_+} a(r,\xi)\,(r\rho^{-1})^{i\xi} u(\rho)\rho^{-1} d\rho, \tag{3.3}$$

where $u \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^n)$, is called the Mellin pseudodifferential operator ($M\psi do$) with symbol $a \in \mathcal{E}(n)$. We denote by $OP\mathcal{E}(n)$ the class of all such operators and by $OP\mathcal{E}_d(n)$ the class of the double $M\psi do's$ $Op_d(a)$ with symbols $a \in \mathcal{E}_d(n)$ which are defined by formula (3.3) with the symbol a of two variables replaced by the double symbol a of three variables.

(iv) We say that a matrix-function a ($\in \mathcal{E}(n)$) is slowly oscillating at the point $r = 0$ and belongs to $\mathcal{E}_{sl}(n)$, if

$$\lim_{r\to+0} \sup_{\xi\in\mathbb{R}} |(r\partial_r)^\beta \partial_\xi^\alpha a_{ij}(r,\xi)| \langle\xi\rangle^\alpha = 0, \tag{3.4}$$

for all $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$. By $\mathcal{E}_0(n)$ we denote the set of matrix-functions satisfying condition (3.4) for all $\alpha, \beta \in \mathbb{N}_0$.

We say that the matrix-function $a = (a_{ij})_{i,j=1}^n \in \mathcal{E}_d(n)$ is slowly oscillating at the point 0 and belongs to $\mathcal{E}_{sl,d}(n)$ if

$$\lim_{r\to+0} \sup_{(\rho,\xi)\in\mathbb{R}_+\times\mathbb{R}} |(r\partial_r)^\beta(\rho\partial_\rho)^\gamma \partial_\xi^\alpha a_{ij}(r,\rho,\xi)|\langle\xi\rangle^\alpha = 0$$

for all $\beta \in \mathbb{N}$ and every $\gamma, \alpha \in \mathbb{N}_0$, and

$$\lim_{\rho\to+0} \sup_{(r,\xi)\in\mathbb{R}_+\times\mathbb{R}} |(r\partial_r)^\beta(\rho\partial_\rho)^\gamma \partial_\xi^\alpha a_{ij}(r,\rho,\xi)|\langle\xi\rangle^\alpha = 0$$

for all $\gamma \in \mathbb{N}$ and every $\beta, \alpha \in \mathbb{N}_0$. The corresponding classes of Mellin $\psi do's$ are denoted by $OP\mathcal{E}_{sl}(n), OP\mathcal{E}_{sl,d}(n), OP\mathcal{E}_0(n)$.

Remark 26. Note that the Mellin $\psi do's$ are $\psi do's$ on the multiplicative group $\mathbb{R}_+$ with the invariant measure $d\mu = \frac{dr}{r}$. The $M\psi do's$ are obtained from $\psi do's$ on $\mathbb{R}$ by means of the change of the variables : $\mathbb{R}_+ \ni r = e^{-x}, x \in \mathbb{R}$ which maps the point $+\infty$ to the point 0. The main properties of $M\psi do's$ easily follow from the corresponding properties of $\psi do's$ on $\mathbb{R}$ (see [38, Chap. 4.5]).

By $L_n^2(\mathbb{R}_+, d\mu)$ we denote the space of measurable $\mathbb{C}^n$-valued functions u on $\mathbb{R}_+$ with the norm

$$\|u\|_{L_n^2(\mathbb{R}_+,d\mu)} = \left(\int_{\mathbb{R}_+} \|u(r)\|_{\mathbb{C}^n}^2 \, d\mu \right)^{1/2}.$$

Proposition 27. ([38, *Chap.* 4]) *Let $A = Op(a) \in OP\mathcal{E}(n)$. Then the operator A is bounded in $L_n^2(\mathbb{R}_+, d\mu)$ and there exists $C > 0$ not depending on A such that*

$$\|A\|_{\mathcal{B}(L_n^2(\mathbb{R}_+,d\mu))} \le C\, |a|_{2,2}\,. \tag{3.5}$$

Proposition 28. ([38, *Chap.* 4])

(i) *Let $Op(a), Op(b) \in OP\mathcal{E}(n)$. Then $C = Op(a)Op(b) \in OP\mathcal{E}(n)$, and $C = Op(c)$ with*

$$c(r,\xi) = \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} a(r,\xi+\eta)b(r\rho,\xi)\rho^{-i\eta} d\rho d\eta. \tag{3.6}$$

(ii) *Let $Op_d(a) \in OP\mathcal{E}_d(n)$. Then $Op_d(a) \in OP\mathcal{E}(n), Op_d(a) = Op(a^\#)$ and*

$$a^\#(r,\xi) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} a(r,r\rho,\xi+\eta)\rho^{-i\eta} d\rho d\eta. \tag{3.7}$$

(iii) *Let $A = Op(a) \in OP\mathcal{E}(n)$ and acting in $L_n^2(\mathbb{R}_+, d\mu)$. Then the adjoint operator $A^* \in OP\mathcal{E}(n)$, and $A^* = Op(b)$*

$$b(r,\xi) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} a^*(r\rho,\xi+\eta)\rho^{-i\eta} d\rho d\eta, \tag{3.8}$$

where $a^*(r,\xi)$ is the Hermite adjoint matrix to $a(r,\xi)$.

The integrals in (3.6), (3.7), (3.8) are understood as oscillatory (see [38, Chap. 4]).

Proposition 29. (i) *Let* $Op(a), Op(b) \in OP\mathcal{E}_{sl}(n)$. *Then* $Op(a)Op(b) = Op(c) \in OP\mathcal{E}_{sl}(n)$, *where*

$$c(r,\xi) = a(r,\xi)b(r,\xi) + q(r,\xi),$$

and $q(r,\xi) \in \mathcal{E}_0(n)$. (ii) *Let* $Op_d(a) \in OP\mathcal{E}_{d,sl}(n)$. *Then* $Op_d(a) = Op(a^{\#}) \in OP\mathcal{E}_{sl}(n)$, *where*

$$a^{\#}(r,\xi) = a(r,r,\xi) + q(r,\xi)$$

and $q(r,\xi) \in \mathcal{E}_0(n)$.

(ii) *Let* $Op(a) \in OP\mathcal{E}_{sl}(n)$ *and act in* $L^2(\mathbb{R}_+, d\mu, \mathbb{C}^n)$. *Then the adjoint operator* $Op(a)^* = Op(b) \in OP\mathcal{E}_{sl}(n)$ *and*

$$b(r,\xi) = a^*(r,\xi) + q(r,\xi),$$

where $a^*(r,\xi)$ *is the Hermite adjoint matrix to* $a^*(r,\xi)$, *and* $q \in \mathcal{E}_0(n)$.

Let $w{=}\exp v$, where $v \in C^\infty(\mathbb{R}_+)$ is a real valued function such that

$$\sup_{r\in\mathbb{R}_+} \left| \left(r\frac{d}{dr} \right)^k v(r) \right| < \infty \tag{3.9}$$

for every $k \in \mathbb{N}$. Moreover, we assume that there exists an interval $(c,d) \ni 0$ such that the function $\varkappa_v = rv'$ satisfies the condition

$$c < \inf_{r\in\mathbb{R}_+} \varkappa_v(r) \le \sup_{r\in\mathbb{R}_+} \varkappa_v(r) < d. \tag{3.10}$$

We say that $w = e^v$ is the weight of the class $\mathcal{R}(c,d)$, if conditions (3.9) and (3.10) hold, and of the class $\mathcal{R}_{sl}(c,d)$, if $w \in \mathcal{R}(c,d)$ and

$$\lim_{r\to 0} r\varkappa_v'(r) = 0. \tag{3.11}$$

The weights in $\mathcal{R}_{sl}(c,d)$ are called *slowly oscillating at the point* 0.

Definition 30. We say that a symbol a defined on $\mathbb{R}_+ \times \mathbb{R}$ belongs to $\mathcal{E}(n,(c,d))$, if a is analytically extended with respect to the second variable ξ into the strip $\Pi = \{\xi \in \mathbb{C} : \Im(\xi) \in (c,d)\}$ and

$$\sup_{(r,\xi+i\eta)\in\mathbb{R}_+\times\mathbf{\Pi}} \left| (r\partial_r)^\beta \partial^\alpha a_{ij}(r,\xi+i\eta) \right| < \infty$$

for all $\alpha, \beta \in \mathbb{N}_0$. By $OP\mathcal{E}(n,(c,d))$ we denote the corresponding class of $M\psi do's$ with analytical symbols.

The class $OP\mathcal{E}_d(n,(c,d))$ of $M\psi do's$ with double symbols defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ and analitically extended, with respect to the third variable, into the strip Π is introduced in the obvious way.

Proposition 31. ([38, *Chap.* 4]) . (i) *Let* $a \in \mathcal{E}(n,(c,d))$ *and* $w = e^v \in \mathcal{R}(c,d)$. *Then*

$$wOp(a)w^{-1} = Op_d(a_w), \tag{3.12}$$

where $a_w(r,\rho,\xi) = a(r,\rho,\xi + i\vartheta_v(r,\rho))$ *and*

$$\vartheta_v(r,\rho) = \int_0^1 \varkappa_v(r^{1-\tau}\rho^\tau)d\tau.$$

(Note that condition (3.10) *yields that* $\vartheta_v(r,\rho) \in (c,d)$ *for all* $r,\rho \in \mathbb{R}_+$*).*
(ii) *Let* $A = Op(a) \in OP\mathcal{E}_{sl}(n,(c,d))$ *,* $w \in \mathcal{R}_{sl}(c,d)$*. Then* $wOp(a)w^{-1} \in OP\mathcal{E}_{sl}(n)$ *and*

$$wOp(a)w^{-1} = Op(\tilde{a}_w) + Op(q) \tag{3.13}$$

where $\tilde{a}_w(r,\xi) = a(r,\xi + i\varkappa_v(r))$ *and* $q \in \mathcal{E}_0(n)$*.*

3.2. Mellin ψdo in the Spaces $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$

In this subsection we deal with the space $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$ with variable exponent, defined in a general form by Definition 7; now we take

$$X = \mathbb{R}_+ \quad \text{and} \quad d\mu(r) = \frac{dr}{r}.$$

Let $p : \mathbb{R}_+ \to (1,\infty)$ be a measurable function satisfying condition (2.12) on $X = \mathbb{R}^1_+$. We suppose that the function p satisfies the log-condition of the form

$$|p(r) - p(\rho)| \le \frac{A}{\log \frac{1}{\left|\log \frac{r}{\rho}\right|}} \tag{3.14}$$

for all $r,\rho \in \mathbb{R}_+$ such that $\frac{1}{\sqrt{e}} \le \frac{r}{\rho} \le \sqrt{e}$. Note that condition (3.14) is nothing else but condition (2.13) with the metric $d(r,\rho) = \left|\log \frac{r}{\rho}\right|$. Correspondingly to (2.14) we also suppose that there exist the coinciding limits

$$p(0) := \lim_{r\to+0} p(r) = p(\infty) := \lim_{r\to+\infty} p(r)$$

and

$$|p(r) - p(0)| \le \frac{C}{\log\,(2 + |\log r|)}, \quad C > 0, \;\; r \in \mathbb{R}_+. \tag{3.15}$$

Note that the mapping $\mathbb{R} \ni x \to \exp x \in \mathbb{R}_+$ generates an isomorphism of the spaces $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$ and $L_n^{\tilde{p}(\cdot)}(\mathbb{R})$, where $p(r) = \tilde{p}(\log r)$, so that conditions (3.14) and (3.15) have their obvious origin in conditions (2.13) and (2.14) with $d(x,y) = |x-y|$ on $\mathbb{R}^1$.

Theorem 32. *Let* p *satisfy assumption* (2.13) *and conditions* (3.14) *and* (3.15)*. Then every operator* $Op(a) \in OP\mathcal{E}(n)$ $(Op_d(a) \in OP\mathcal{E}_d(n))$ *is bounded in* $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$ *and there exist* $M > 0$ *such that*

$$\|Op(a)\|_{\mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+,d\mu))} \le C\,|a|_{M,M}\,, \tag{3.16}$$

$$(\|Op_d(a)\|_{\mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+,d\mu))} \le C\,|a|_{M,M,M}). \tag{3.17}$$

Proof. Let u be a measurable function on $\mathbb{R}$ with values in $\mathbb{C}^n$. We set $(\Psi u)(r) = u(-\log r), r \in \mathbb{R}_+$. It is evident that the mapping

$$\Psi : L_n^{\tilde{p}(\cdot)}(\mathbb{R}) \to L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$$

where $p(r) = \tilde{p}(-\log r), r \in \mathbb{R}_+$, is an isomorphism between the Banach spaces. This isomorphism generates the isomorphism of the spaces of operators

$$\tilde{\Psi} : \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)) \to \mathcal{B}(L_n^{\tilde{p}(\cdot)}(\mathbb{R}))$$

by the formula $\tilde{\Psi}(A) = \Psi^{-1} A \Psi, A \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu))$. Moreover, $\tilde{\Psi}$ $(OP\mathcal{E}(n)) = OPS_{1,0}^0(n)$. Hence Theorem 32 follows from Theorem 10 and Corollary 11. □

Let w be a weight, that is, a.e. positive measurable function on $\mathbb{R}_+$. We introduce the weighted space $L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)$ by the norm

$$\|u\|_{L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)} = \|wu\|_{L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)} < \infty.$$

Theorem 33. *Let $Op(a) \in OP\mathcal{E}(n, (c,d)), w = e^v \in \mathcal{R}(c,d)$. Then $Op(a)$ is bounded in $L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)$ and there exist constants $M > 0, C > 0$, not depending of a such that*

$$\|Op(a)\|_{\mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu))} \leq C \, |a|_{M,M} \, |v|_M \,, \tag{3.18}$$

where

$$|v|_M = \sum_{k=1}^{M} \sup_{r \in \mathbb{R}_+} \left| v^{(k)}(r) \right|.$$

Proof. The boundedness of A in $L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)$ is equivalent to the boundedness of wAw^{-1} in $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$. Applying formula (3.12) and Theorem 32 we obtain estimate (3.18). □

3.2.1. Local Invertibility of Mellin Pseudodifferential Operators. Let $A \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu))$. We say that A is locally invertible at the point 0, if there exists an $R > 0$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu))$ such that

$$\mathcal{L}_R A \chi_{[0,R]} I = \chi_{[0,R]} I, \quad \chi_{[0,R]} A \mathcal{R}_R = \chi_{[0,R]} I.$$

Theorem 34. *Let $Op(a) \in OP\mathcal{E}_{sl}(n)$ and act in $L_n^{p(\cdot)}(\mathbb{R}_+, \mu)$. Then $Op(a)$ is locally invertible at the point 0 , if and only if*

$$\liminf_{r \to +0} \inf_{\xi \in \mathbb{R}} |\det a(r, \xi)| > 0.$$

Proof. Note that the operator $A \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, \mu))$ is locally invertible at the point 0, if and only if the operator $\Psi A \Psi^{-1} \in \mathcal{B}(L_n^{\tilde{p}(\cdot)}(\mathbb{R}))$ is locally invertible at the point $+\infty$. Moreover, $\tilde{\Psi}(OP\mathcal{E}_{sl}(n)) = OPSO_{+\infty}(n)$. Hence Theorem 34 follows from Theorem 18. □

4. Singular Integral Operators on Some Carleson Curves

4.1. Curves, Weights, Coefficients

We say that a complex-valued function $a \in \mathcal{C}^m(0,\varepsilon), \varepsilon > 0$, if $a \in C^m(0,\varepsilon)$ and

$$\sup_{r\in(0,\varepsilon)} \left| \left(r\frac{d}{dr} \right)^j a(r) \right| < \infty$$

for every $j = 0, 1, \ldots, m$, and $a \in \mathcal{C}^\infty(0,\varepsilon)$ if $m = \infty$. We say that $a \in \tilde{\mathcal{C}}^m(0,\varepsilon)$ if $\varkappa_a = r\frac{da}{dr} \in \mathcal{C}^m(0,\varepsilon)$, and $a \in \tilde{\mathcal{C}}^\infty(0,\varepsilon)$ if $m = \infty$. A function $a \in \mathcal{C}^m(0,\varepsilon), m \geq 1$ is said to be slowly oscillating at the point 0 and belong to the class $\mathcal{C}^m_{sl}(0,\varepsilon)$ if

$$\lim_{r\to 0} \varkappa_a(r) = 0.$$

We write $\mathcal{C}^\infty_{sl}(0,\varepsilon)$ if $m = \infty$. We denote by $\tilde{\mathcal{C}}^m_{sl}(0,\varepsilon), m \geq 1$ the class of functions $a \in \tilde{\mathcal{C}}^m(0,\varepsilon)$ such that $\varkappa_a \in \mathcal{C}^m_{sl}(0,\varepsilon)$. We write $\tilde{\mathcal{C}}^\infty(0,\varepsilon)$ if $m = \infty$. If $a \in \tilde{\mathcal{C}}^m(0,\varepsilon), m \geq 1$ we set

$$\vartheta_a(r,\rho) = \int_0^1 \varkappa_a(r^{1-\tau}\rho^\tau)d\tau.$$

A set $\gamma \subset \mathbb{C}$ is called a *simple locally Lyapunov arc*, if there exists a homeomorphism $\varphi : [0,1] \to \gamma$ such that $\varphi \in C^1((0,1))$, $\varphi'(r) \neq 0$ for all $r \in (0,1)$, and for every segment $[a,b] \subset (0,1)$ there exist $C > 0$ and $\alpha \in (0,1]$ such that

$$|\varphi'(r) - \varphi'(\rho)| \leq C\,|r-\rho|^\alpha \quad \text{for all} \quad r,\rho \in [a,b].$$

The points $\varphi(0)$ and $\varphi(1)$ are called the *endpoints* of γ. We refer to a set $\Gamma(\subset \mathbb{C})$ as a *composed curve* if $\Gamma = \cup_{k=1}^K \Gamma_k$, where $\Gamma_1, \ldots, \Gamma_K$ are oriented and rectifiable simple *locally Lyapunov* arc, each pair of which has at most endpoints in common. A *node* of Γ is a point which is endpoint of at least one of the arcs $\Gamma_1, \ldots, \Gamma_K$. The set of all the nodes is denoted by $\mathcal{F}$.

Let $t_0 \in \mathcal{F}$. We suppose that there exists an $\varepsilon > 0$ such that the portion

$$\Gamma(t_0,\varepsilon) = \{t \in \Gamma : |t_0 - t| < \varepsilon\}$$

is of the form

$$\Gamma(t_0,\varepsilon) = \{t_0\} \cup \Gamma^1_{t_0} \cup \cdots \cup \Gamma^{n(t_0)}_{t_0}$$

where

$$\Gamma^j_{t_0} = \left\{ z \in \mathbb{C} : z = t_0 + re^{i\varphi_{t_0,j}(r)} : r \in (0,\varepsilon), (j = 1,\ldots,n(t_0)) \right\},$$

and

$$\varphi_{t_0,j}(r) = \psi_{t_0}(r) + \psi_{t_0,j}(r),$$

where $\psi_{t_0}, \psi_{t_0,1}, \ldots, \psi_{t_0,n(t_0)}$ are real-valued functions such that: $\psi_{t_0} \in \tilde{\mathcal{C}}^\infty(0,\varepsilon)$, $\psi_{t_0,j} \in \mathcal{C}^\infty(0,\varepsilon)$, and

$$\begin{aligned} 0 \le m_1 < \psi_{t_0,1}(r) < M_1 < m_2 < \psi_{t_0,2}(r) < M_2 \\ < \cdots < m_{n_{t_0}} < \psi_{t_0,n_{t_0}}(r) < M_{n_{t_0}} < 2\pi \end{aligned}$$

for all $r \in (0,\varepsilon)$ with certain constants m_j, M_j. Note that the function ψ_{t_0} defines the *rotation*, and the functions $\psi_{t_0,j}$ define the *oscillations* of the curves $\Gamma^j_{t_0}$ near the node t_0.

We suppose that these conditions hold for every node, and we denote such class of curves by $\mathcal{L}$.

If $\psi_{t_0} \in \tilde{\mathcal{C}}^\infty_{sl}(0,\varepsilon), \psi_{t_0,j} \in \mathcal{C}^\infty_{sl}(0,\varepsilon)$ in the above conditions for every node $t_0 \in \mathcal{F}$, then we say that the curve Γ is *slowly oscillating* at every node t_0. We denote such class of curves by $\mathcal{L}_{sl}$.

For example, if

$$\varphi_{t_0,,j}(r) = \delta_{t_0} \log r + \mu_{t_0,j}, \qquad j = 1, \ldots, n(t_0), \qquad r \in (0,\varepsilon)$$

with $0 \le \mu_{t_0 1} < \mu_{t_0 2} < \cdots < \mu_{t_0 n_{t_0}} < 2\pi$, then the above conditions on the node 0 are fulfilled.

Let Γ be a locally rectifiable composed curve with Lebesgue length measure. The curve Γ is said to be a *Carleson curve* (*an Ahlfors–David regular curve*) (see for instance [1, p. 2]) if

$$C_\Gamma = \sup_{t\in\Gamma} \sup_{\varepsilon>0} \frac{|\Gamma(t,\varepsilon)|}{\varepsilon} < \infty,$$

where $|\Gamma(t,\varepsilon)|$ is the length of the portion $\Gamma(t,\varepsilon)$.

Taking into account that

$$\left| d(re^{i\varphi(r)}) \right| = \sqrt{(1 + (r\varphi'(r))^2} dr,$$

it easy to see that $\Gamma \in \mathcal{L}$ is a *Carleson curve*. Throughout what follows we assume that $\Gamma \in \mathcal{L}$ and we suppose for simplicity that Γ is a compact curve.

Let $p : \Gamma \to (1,\infty)$ be a measurable function satisfying assumption (2.13) on $X = \Gamma \backslash \mathcal{F}$. For $t_0 \in \mathcal{F}$ we suppose that there exist an $\varepsilon > 0$ such that the functions

$$p_{t_0 j}(r) := p(t_0 + re^{i\varphi_{t_0,j}(r)}) = p_{t_0}(r), \qquad r \in (0,\varepsilon) \tag{4.1}$$

do not depend on j and belong to $\mathcal{C}^\infty(0,\varepsilon)$ and satisfy assumption (2.12) and conditions (3.14) and (3.15). It follows from condition (3.15) that p_{t_0} is a continuous function at the origin and

$$\lim_{r\to 0} p_{t_0}(r) = p_{t_0}(0) = p(t_0).$$

Let $w : \Gamma \to [0,\infty]$ be a measurable function referred in the sequel as *a weight*. The weighted variable exponent Lebesgue space $L^{p(\cdot)}(\Gamma, w)$ is defined as the space of functions f such that $[w(x)]^{\frac{1}{p(x)}} \in L^{p(\cdot)}(\Gamma)$, the latter being introduced by Definition 7. We write $L^{p(\cdot)}(\Gamma)$ if $w \equiv 1$.

We consider *weights* on $\mathbb{R}_+$ of the form

$$w = \exp v, \tag{4.2}$$

where a real-valued function $v \in \tilde{\mathcal{C}}^\infty(0,\varepsilon)$. We denote such a class of weights by $\mathcal{R}_0$. Let $\varkappa_v = rv'$, and

$$\varkappa_w^+ = \limsup_{r\to 0} \varkappa_v(r) = \limsup_{r\to 0} \frac{rw'(r)}{w(r)}, \tag{4.3}$$

and

$$\varkappa_w^- = \liminf_{r\to 0} \varkappa_v(r) = \liminf_{r\to 0} \frac{rw'(r)}{w(r)}. \tag{4.4}$$

By $\mathcal{R}_0^{sl}$ we denote the class of weights $w = \exp v$ with $v \in \tilde{\mathcal{C}}_{sl}^\infty(0,\varepsilon)$. For instance, if

$$v(r) = f(\log(-\log r))\log r, \qquad r \in (0,\varepsilon)$$

and $f \in C_b^\infty(\mathbb{R})$, then $w \in \mathcal{R}_0^{sl}$. For $f = \sin x$

$$\begin{aligned}\varkappa_v(r) &= \cos(\log(-\log r)) + \sin(\log(-\log r))\\ &= \sqrt{2}\cos\left(\log(-\log r) - \frac{\pi}{2}\right)\end{aligned}$$

and $\varkappa_w^+ = \sqrt{2}, \varkappa_w^- = -\sqrt{2}$.

Proposition 35. *Let $w = e^v \in \mathcal{R}_0$. Then for every $\delta > 0$ there exists an $\varepsilon' \in (0,\varepsilon)$ such that*

$$\begin{aligned} w(\rho)r^{\varkappa_w^+ + \delta} \le w(r) \le w(\rho)r^{\varkappa_w^- - \delta}\\ \text{for } \rho, r \in (0,\varepsilon'). \end{aligned} \tag{4.5}$$

Proof. Let

$$\vartheta_v(r,\rho) := \int_0^1 \varkappa_v(\rho^{1-\tau}r^\tau)d\tau = \frac{1}{\ln\frac{r}{\rho}}\int_\rho^r \frac{\varkappa_v(t)}{t}dt = \frac{v(r)-v(\rho)}{\ln\frac{r}{\rho}}.$$

Then

$$w(r)w^{-1}(\rho) = e^{v(r)-v(\rho)} = e^{\vartheta_v(r,\rho)(\log r - \log\rho)} = \left(r\rho^{-1}\right)^{\vartheta_v(r,\rho)}. \tag{4.6}$$

For every $\delta > 0$ we can find $\varepsilon' \in (0,\varepsilon)$ such that

$$\varkappa_w^- - \delta < \vartheta_v(r,\rho) < \varkappa_w^+ + \delta \tag{4.7}$$

for all $r, \rho \in (0,\varepsilon')$. Estimate (4.5) follows then from (4.6) and (4.7). □

Let w be a weight on the curve Γ. We suppose that for every point $t_j \in \mathcal{F}$ there exists a neighborhood U_j such that w and w^{-1} belong $L^\infty(\Gamma\backslash \cup_{t_j\in\mathcal{F}} (\Gamma \cap U_j))$. We say that $w \in \mathcal{R}_\Gamma$, if for every point $t_0 \in \mathcal{F}$ and for every $j \in \{1,\dots,n(t_0)\}$ the function

$$w_{t_0}(r) = w(t_0 + re^{i\varphi_{t_0,j}(r)}) = e^{v_{t_0}(r)}, \qquad r \in (0,\varepsilon) \tag{4.8}$$

does not depend on j and $w_{t_0} = e^{v_{t_0}} \in \mathcal{R}_0$. By $\mathcal{A}_{p(\cdot)}(\Gamma)$ we denote the class of weights in $\mathcal{R}_\Gamma$ such that

$$-\frac{1}{p(t_0)} < \liminf_{r\to 0} \varkappa_{v_{t_0}}(r) \le \limsup_{r\to 0} \varkappa_{v_{t_0}}(r) < 1 - \frac{1}{p(t_0)}, \tag{4.9}$$

for every node $t_0 \in \mathcal{F}$, and by $\mathcal{A}^{sl}_{p(\cdot)}(\Gamma)$ the class of weights in $\mathcal{A}_{p(\cdot)}(\Gamma)$ such that $w_{t_0} \in \mathcal{R}^{sl}_0$ for every node $t_0 \in \mathcal{F}$.

Proposition 36. *If* $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$, *then* $w \in L^{p(\cdot)}(\Gamma)$ *and* $w^{-1} \in L^{q(\cdot)}(\Gamma)$.

Proof. First we prove that if $t_0 \in \mathcal{F}$, then there exists an $\varepsilon > 0$ such that $w \in L^{p(\cdot)}(\Gamma(t_0, \varepsilon))$. We will prove that

$$I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(w) = \int\limits_{\Gamma(t_0,\varepsilon)} w(t)^{p(t)}\,|dt| < \infty. \tag{4.10}$$

Applying expressions (4.1) and (4.8) for the weight w and exponent p on the portion $\Gamma(t_0, \varepsilon)$, we obtain that

$$\begin{aligned} I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(w) &= \sum_{j=1}^{n_{t_0}} \int\limits_{\Gamma^j_{t_0}} w(t)^{p(t)}\,|dt| \\ &= \sum_{j=1}^{n_{t_0}} \int\limits_0^{\varepsilon} w_{t_0}^{p_{t_0}(r)}(r)\sqrt{1 + (r\varphi_{t_0,j}(r))^2}\,dr. \end{aligned}$$

Applying Proposition 35 we obtain that for every $\delta > 0$ there exist an $\varepsilon \in (0, 1)$ such that

$$w_{t_0}(r) \le C r^{\varkappa^-_{w_{t_0}} - \delta}, \qquad r \in (0, \delta),$$

where

$$\varkappa^-_{w_{t_0}} = \liminf_{r\to 0} \varkappa_{v_{t_0}}(r) > -\frac{1}{p(t_0)}. \tag{4.11}$$

Note that p_{t_0} is a continuous function and $p_{t_0}(0) = p(t_0)$. Then applying estimate (4.5) we can find first a $\delta > 0$ and then an $\varepsilon > 0$ such that

$$\gamma_{t_0} = \inf_{r\in(0,\varepsilon)} p_{t_0}(r)(\varkappa^-_{w_{t_0}} - \delta) > -1. \tag{4.12}$$

Estimate (4.12) yields that

$$w_{t_0}^{p_{t_0}(r)}(r) \le C^{p_{t_0}(r)} r^{\left(\varkappa^-_{w_{t_0}} - \delta\right)p_{t_0}(r)} = C_1 r^{\gamma_{t_0}}. \tag{4.13}$$

Hence $I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(w) < \infty$, because $\gamma_{t_0} > -1$. In the same way applying the right hand side inequality from (4.9), we obtain that $w^{-1} \in L^{q(\cdot)}(\Gamma(t_0, \varepsilon))$ for sufficiently small $\varepsilon > 0$. Since w and w^{-1} are $L^\infty-$functions outside the union of small neighborhoods of the nodes $t_k \in \mathcal{F}$, and $L^{p(\cdot)}(K) \supset L^\infty(K)$ for every compact set K, we obtain that $w \in L^{p(\cdot)}(\Gamma)$ and $w^{-1} \in L^{q(\cdot)}(\Gamma)$. □

A function $a : \Gamma \to \mathbb{C}$ is said to be *piecewise slowly oscillating* on Γ, if $a \in C(\Gamma\backslash\mathcal{F})$ and for each node $t_0 \in \mathcal{F}$ we have

$$a(t_0 + re^{i\varphi_{t_0,j}(r)}) = a_{t_0,j}(r), \quad r \in (0, \varepsilon), \quad j \in \{1, \ldots, n(t_0)\},$$

and $a_{t_0,j} \in \mathcal{C}^\infty_{sl}(0, \varepsilon)$.

We denote the class of piecewise slowly oscillating functions by $PSO(\Gamma)$.

4.2. Representation of a Singular Integral Operator at the Node as a Mellin Pseudodifferential Operators

We suppose that Γ is a compact Carleson curve of the class $\mathcal{L}$ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$ is a weight satisfying conditions given in Sect. 4.1. We consider the Cauchy SIO defined on Γ as

$$(S_\Gamma f)(t) = \lim_{\varepsilon\to 0} \int\limits_{\Gamma\setminus\Gamma(t_0,\varepsilon)} \frac{f(\tau)d\tau}{\tau - t}, \quad t \in \Gamma. \tag{4.14}$$

For the point $t_0 \in \mathcal{F}$ we introduce the mapping

$$\Phi_{t_0} : L^{p(\cdot)}(\Gamma(t_0,\varepsilon), w) \to L^{p_{t_0}(\cdot)}_{n(t_0)}((0,\varepsilon), d\mu), \tag{4.15}$$

where

$$(\Phi_{t_0} f)(r) = \begin{pmatrix} r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0 + re^{i\varphi_{t_0,1}(r)}) \\ \cdot \\ \cdot \\ \cdot \\ r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0 + re^{i\varphi_{t_0,n(t_0)}(r)}) \end{pmatrix} = \tilde{f}(r), \quad r \in (0,\varepsilon).$$

The inverse mapping $\Phi_{t_0}^{-1}$ transforms the vector-function $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_{n(t_0)}) \in L^{p_{t_0}(\cdot)}_{n(t_0)}((0,\varepsilon), d\mu)$ to the function f on the curve $\Gamma(t_0,\varepsilon) = \cup_{j=1}^{n(t_0)} \Gamma_{t_0 j}$ by the rule

$$f\,|_{\Gamma_{t_0 j}} (t_0 + re^{i\varphi_{t_0,j}(r)}) = r^{-\frac{1}{p_{t_0}(r)}} w_{t_0}^{-1}(r) \tilde{f}_j(r).$$

Proposition 37. *The mapping Φ_{t_0} is an isomorphism between the corresponding Banach spaces.*

Proof. We have

$$I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(f,w) = \int\limits_{\Gamma(t_0,\varepsilon)} |w(\tau)f(\tau)|^{p(\tau)}\, |d\tau| = \sum_{j=1}^{n(t_0)} \int\limits_{\Gamma_{t_0 j}} |w(\tau)f(\tau)|^{p(\tau)}\, |d\tau|. \tag{4.16}$$

After the change of variables $\tau = t_0 + re^{i\varphi_{t_0,j}(r)}$ we obtain

$$\int\limits_{\Gamma(t_0,\varepsilon)} |w(\tau)f(\tau)|^{p(\tau)}\, |d\tau| \tag{4.17}$$

$$= \sum_{j=1}^{n(t_0)} \int\limits_0^\varepsilon |w(t_0 + re^{i\varphi_{t_0,j}(r)}) f(t_0 + re^{i\varphi_{t_0,j}(r)})|^{p_{t_0}(r)} \sqrt{1 + \left(r\varphi'_{t_0 j}(r)\right)^2} dr$$

$$= \sum_{j=1}^{n(t_0)} \int\limits_0^\varepsilon |r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0 + re^{i\varphi_{t_0,j}(r)})|^{p_{t_0}(r)} \sqrt{1 + \left(r\varphi'_{t_0 j}(r)\right)^2} d\mu(r).$$

Since

$$0 < \inf_{(0,\varepsilon)} \sqrt{1 + \left(r\varphi'_{t_0 j}(r)\right)^2} \le \sup_{(0,\varepsilon)} \sqrt{1 + \left(r\varphi'_{t_0 j}(r)\right)^2} < \infty, \tag{4.18}$$

estimate (4.18) implies that the modular $\int_{\Gamma(t_0,\varepsilon)} |w(\tau)f(\tau)|^{p(\tau)} \quad |d\tau|$ is bounded if and only the modulars

$$\int_0^{\varepsilon} |r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0 + re^{i\varphi_{t_0,j}(r)})|^{p_{t_0}(r)} \sqrt{1 + \left(r\varphi'_{t_0 j}(r)\right)^2} d\mu(r)$$

are bounded for every $j = 1, 2, \dots, n(t_0)$. Hence the mapping

$$\Phi_{t_0} : L^{p(\cdot)}(\Gamma(t_0, \varepsilon), w) \to L^{p_{t_0}(\cdot)}_{n(t_0)}((0, \varepsilon), d\mu)$$

is bounded.

In the same way we show that

$$\Phi^{-1}_{t_0} : L^{p_{t_0}(\cdot)}_{n(t_0)}((0, \varepsilon), d\mu) \to L^{p(\cdot)}(\Gamma(t_0, \varepsilon), w)$$

is bounded. Hence Φ_{t_0} is an isomorphism between the corresponding Banach spaces. □

To formulate the main results, we need the following notation. Put $\varepsilon_k = 1$, if t_0 is the starting point of an oriented arc $\Gamma_{t_0 k}$ and $\varepsilon_k = -1$, if t_0 is its ending point. Define

$$\nu : [0, 2\pi) \times (\mathbb{C} \backslash i\mathbb{Z}) \to \mathbb{C}$$

by

$$\nu(\delta, z) = \begin{cases} \coth(\pi z), & \delta = 0 \\ \frac{e^{(\pi-\delta)z}}{\sinh(\pi z)}, & \delta \in (0, 2\pi). \end{cases} \tag{4.19}$$

Let $\phi_{t_0} \in C_0^\infty(\Gamma(t_0, \varepsilon))$ and equal to 1 in a smaller neighborhood of t_0.

Proposition 38. *Let* Γ *be a composed compact curve of the class* $\mathcal{L}$, *the exponent* $p(\cdot)$ *satisfy the above conditions on* Γ *and* $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$. *Then for every point* $t_0 \in \mathcal{F}$ *the operator*

$$S^{t_0} := \Phi_{t_0} \phi_{t_0} S_\Gamma \phi_{t_0} \Phi^{-1}_{t_0} = Op(s^{t_0}),$$

is a Mellin ψ*do in the class* $OP\mathcal{E}_d(n)$ *with the double symbol* $s^{t_0} = (s^{t_0}_{jk})^{n(t_0)}_{j,k=1}$ *where*

$$s^{t_0}_{jk}(r, \rho, \xi)$$
$$= \begin{cases} \varepsilon_k \tilde{\phi}_{j,t_0}(r) \tilde{\phi}_{k,t_0}(\rho) \frac{1+i\rho\varphi'_{t_0,k}(\rho)}{1+i\vartheta_{\psi_{t_0}}(r,\rho)} \nu(2\pi + \psi_{t_0,j}(r) - \psi_{t_0,k}(\rho), \frac{\xi + i\left(\frac{1}{p_{t_0}(r)} + \vartheta_{v_{t_0}}(r,\rho)\right)}{1+i\vartheta_{\psi_{t_0}}(r,\rho)}), & j < k, \\ \tilde{\phi}_{j,t_0}(r) \tilde{\phi}_{j,t_0}(\rho) \varepsilon_k \frac{1+i\rho\varphi'_{t_0,k}(\rho)}{1+i\vartheta_{\varphi_{t_0,k}}(r,\rho)} \nu(0, \frac{\xi + i(\frac{1}{p_{t_0}(r)} + \vartheta_{v_{t_0}}(r,\rho))}{1+i\vartheta_{\psi_{t_0}}(r,\rho)}), & j = k; \\ \varepsilon_k \tilde{\phi}_{jt_0}(r) \tilde{\phi}_{kt_0}(\rho) \frac{1+i\rho\varphi'_{t_0,k}(\rho)}{1+i\vartheta_{\psi_{t_0},k}(r,\rho)} \nu(\psi_{t_0,j}(r) - \psi_{t_0,k}(\rho), \frac{\xi + i(\frac{1}{p_{t_0}(r)} + \vartheta_{v_{t_0}}(r,\rho))}{1+i\vartheta_{\psi_{t_0}}(r,\rho)}), & j > k, \end{cases} \tag{4.20}$$

and $\tilde{\phi}_{j,t_0}(r) = \phi_{t_0}(t_0 + re^{i\varphi_{t_0,j}(r)})$.

Remark 39. Proposition 38 has been proved first in [35] for the constant $p : 1 < p < \infty$ (see the proof of Proposition 3.4 in [35]). The detailed proof is contained in the book [38, Chapter 4.6]. The proof for the variable exponents uses Propositions 28, 29 , 31 and repeats, word for word, the proof for the constant p.

4.3. Boundedness of the Singular Integral Operator in $L^{p(\cdot)}(\Gamma, w)$

Note that in the case of simple Carleson curves and the weights of the form

$$w(t) = \prod_{j=1}^{N} \omega_j(|t - t_j|)$$

where ω_j may grow and have oscillations at the point 0, the boundedness of $S_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ has been established in [23], but we may not use this result for composed Carleson curves.

In the following theorem we give some conditions of the boundedness of $S_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ which are based on the boundedness of the Mellin pseudodifferential operators.

We say that a nonnegative function $\phi_{t_0} \in C_0^\infty(\Gamma(t_0, \varepsilon))$ is a smooth cut-off function of a neighborhood $\Gamma(t_0, \varepsilon)$ of the point t_0, if there exists an $\varepsilon' < \varepsilon$ such that $\phi_{t_0}(t) = 1$ for all $t \in \Gamma(t_0, \varepsilon')$.

Theorem 40. *Let Γ be a composed compact curve of the class $\mathcal{L}$, and let $p(\cdot)$ satisfy the above conditions on Γ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$. Then $S_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is a bounded operator.*

Proof. Let

$$\sum_{k=0}^{N} \phi_k(t) = 1, \qquad t \in \Gamma \tag{4.21}$$

be a partition of unity on Γ, where N is a number of nodes on $\Gamma, \phi_0 \in C_0(\Gamma)$ (the class of continuous functions with a compact support), $\phi_j, j = 1, \ldots, N$, be smooth cut-off functions such that *supp* ϕ_j contains only one node t_j, and let mapping (4.15) be defined on *supp* ϕ_j, $j = 1, \ldots, N$. It is clear that $\Gamma \cap$ *supp* ϕ_0 is a Lyapunov curve, and w and w^{-1} belong $L^\infty(\mathit{supp}\ \phi_0)$. Let ψ_j be another smooth cut-off function of a neighborhood of the point t_j with *supp* ψ_j in a small neighborhood of *supp* φ_j and $\psi_j(t) = 1$ for $t \in$ *supp* φ_j. Then

$$S_\Gamma = \sum_{j=0}^{N} \psi_j S_\Gamma \varphi_j I + \sum_{j=0}^{N} (1 - \psi_j) S_\Gamma \varphi_j I. \tag{4.22}$$

The boundedness $\varphi_0 S_\Gamma \psi_0 I : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ follows from [23] because $\varphi_0 S_\Gamma \psi_0 I$ is defined on a simple Lyapunov portion of Γ, and w and w^{-1} belong L^∞ on this portion.

It follows from Proposition 38 that for every $j = 1, \ldots, N$ the operator $S^{t_j} := \Phi_{t_j} \psi_j S_\Gamma \varphi_j \Phi_{t_j}^{-1} I$ is the Mellin pseudodifferential operator in $OP\mathcal{E}_d(n(t_j))$ with a double symbol defined by formulas (4.19) and (4.20). By Theorem 32 S^{t_j} is bounded on $L_{n(t_j)}^{p_{t_j}(\cdot)}(\mathbb{R}_+, d\mu)$. Hence $\psi_j S_\Gamma \varphi_j I$ is a bounded operator in $L^{p(\cdot)}(\Gamma, w)$. Let us consider the operator $K_{ij} = (1 - \psi_j) S_\Gamma \varphi_j I$. Since $\mathit{supp}(1 - \psi_j) \cap \mathit{supp} \varphi_j = \emptyset$, the operator K_{ij} has a smooth kernel, and K_{ij} is bounded from $L^1(\Gamma)$ in $L^\infty(\Gamma)$. By Proposition 36, $w \in L^{p(\cdot)}(\Gamma)$ and

$w^{-1} \in L^{q(\cdot)}(\Gamma)$. Applying the Hölder inequality for the space $L^{p(\cdot)}$ with variable exponents $p(\cdot)$, we obtain that the operator $u \to w^{-1}u$ is a bounded operator from $L^{p(\cdot)}(\Gamma)$ into $L^1(\Gamma)$. Since the operator K_{ij} is bounded from $L^1(\Gamma)$ into $L^\infty(\Gamma)$ and the operator $v \to wv$ is bounded from $L^\infty(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$, we obtain that $wK_{ij}w^{-1}I$ is a bounded operator in the space $L^{p(\cdot)}(\Gamma)$. This concludes the proof. □

4.4. The Fredholm Property of Singular Integral Operators in $L^{p(\cdot)}(\Gamma, w)$

4.5. Local Invertibility

Definition 41. We say that an operator $A \in \mathcal{B}(L^{p(\cdot)}(\Gamma, w))$ is locally invertible at the point $t_0 \in \Gamma$, if there exist a neighborhood $U_{t_0}(\subset \Gamma)$ of the point t_0, and operators $R_{U_{t_0}}, L_{U_{t_0}} \in \mathcal{B}(L^{p(\cdot)}(\Gamma, w))$ such that

$$R_{U_{t_0}} A\chi_{U_{t_0}} I = \chi_{U_{t_0}} I \quad \text{and} \quad AL_{U_{t_0}}\chi_{U_{t_0}} I = \chi_{U_{t_0}} I,$$

where $\chi_{U_{t_0}}$ is a characteristic function of U_{t_0}.

We set

$$\tilde{\sigma}^{t_0}(S_\Gamma) = (\tilde{s}^{t_0}_{jk})^m_{j,k},$$

where

$$\tilde{s}^{t_0}_{jk}(r,\xi) = s^{t_0}_{jk}(r,r,\xi)$$
$$= \begin{cases} \varepsilon_k \nu(2\pi + \psi_{t_0,j}(r) - \psi_{t_0,k}(r), \frac{\xi + i\left(\frac{1}{p(t_0)} + rv'_{t_0}(r)\right)}{1 + ir\psi'_{t_0}(r)}), & j < k, \\ \varepsilon_k \nu \left(0, \frac{\xi + i\left(\frac{1}{p(t_0)} + rv'_{t_0}(r)\right)}{1 + ir\psi'_{t_0}(r)}\right), & j = k, \\ \nu \left(\psi_{t_0,j}(r) - \psi_{t_0,k}(r), \frac{\xi + i\left(\frac{1}{p(t_0)} + rv'_{t_0}(r)\right)}{1 + ir\psi'_{t_0}(r)}\right), & j > k, \end{cases} .$$

If $a \in PSO(\Gamma)$ and $t_0 \in \mathcal{F}$, then we set

$$\tilde{\sigma}^{t_0}(aI)(r) = \begin{pmatrix} a_{t_0,1}(r) & & & \\ & a_{t_0,2}(r) & & \\ & & \cdot & \\ & & & \cdot \\ & & & & a_{t_0,n(t_0)}(r) \end{pmatrix}.$$

Let

$$A_\Gamma = aI + bS_\Gamma, \qquad a, b \in PSO(\Gamma). \tag{4.23}$$

Then we define

$$\tilde{\sigma}^{t_0}(A_\Gamma)(r,\xi) = \tilde{\sigma}^{t_0}(aI)(r) + \tilde{\sigma}^{t_0}(bI)(r)\tilde{\sigma}^{t_0}(S_\Gamma)(r,\xi), \qquad r \in (0,\varepsilon), \qquad \xi \in \mathbb{R}, \tag{4.24}$$

and

$$\tilde{\sigma}^{t_0}(A_\Gamma) = \{a(t_0) + b(t_0), a(t_0) - b(t_0)\} \tag{4.25}$$

if $t_0 \in \Gamma \backslash \mathcal{F}$.

In the following theorem we deal with the class $\mathcal{L}_{sl}$ of slowly oscillating curves and the class $\mathcal{A}^{sl}_{p(\cdot)}(\Gamma)$ of weights slowly oscillating at every node of Γ.

Theorem 42. *Let* $\Gamma \in \mathcal{L}_{sl}$ *,* $w = \exp v \in \mathcal{A}^{sl}_{p(\cdot)}(\Gamma)$ *and* A_Γ *be an operator of form* (4.23) *which acts in* $L^{p(\cdot)}(\Gamma, w)$*. Then*:

(i) A_Γ *is locally invertible at the point* $t_0 \in \mathcal{F}$*, if and only if*

$$\liminf_{r\to 0} \inf_{\xi\in\mathbb{R}} \left|\det \tilde{\sigma}^{t_0}(A_\Gamma)(r,\xi)\right| > 0. \tag{4.26}$$

(ii) A_Γ *is locally invertible at the point* $t_0 \in \Gamma\backslash\mathcal{F}$*, if and only if* $\tilde{\sigma}^{t_0}(A_\Gamma)$ *is invertible, that is*

$$a(t_0) \pm b(t_0) \neq 0. \tag{4.27}$$

Proof. (i) Note that $A_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is locally invertible at the point $t_0 \in \Gamma$, if and only if the operator

$$A^{t_0}_\Gamma = \Phi_{t_0}\phi_{t_0}A\phi_{t_0}\Phi^{-1}_{t_0} : L^{p_{t_0}(\cdot)}_{n_{t_0}}\left((0,\varepsilon), \frac{dr}{r}\right) \to L^{p_{t_0}(\cdot)}_{n_{t_0}}\left(0,\varepsilon), \frac{dr}{r}\right)$$

is locally invertible at the point 0, where the operator $A^{t_0}_\Gamma$ is a Mellin ψdo with double symbol in the class $OP\mathcal{E}_d(n(t_0))$ given by formulas (4.19) and (4.20). The conditions $\Gamma \in \mathcal{L}_{sl}$, $w \in \mathcal{A}^{sl}_{p(\cdot)}(\Gamma)$ and $a, b \in PSO(\Gamma)$ and Proposition 38 imply that $A^{t_0}_\Gamma \in OP\mathcal{E}_{d,sl}(n(t_0))$ (see for instance [38, Chapter 4.6.5]). It follows from statement (ii) of Proposition 27 that the Mellin symbol $\sigma(A^{t_0}_\Gamma)$ of $A^{t_0}_\Gamma$ is of the form

$$\sigma(A^{t_0}_\Gamma)(r,\xi) = \tilde{\sigma}^{t_0}(A_\Gamma)(r,\xi) + q_{t_0}(r,\xi),$$

where $q_{t_0} = (q^{ij}_{t_0})^{n(t_0)}_{i,j=1}$ and

$$\lim_{r\to 0} \sup_{\xi\in\mathbb{R}} \left|\partial^\alpha_\xi (r\partial_r)^\beta q^{ij}_{t_0}(r,\xi)\right| = 0$$

for all $\alpha, \beta \in \mathbb{N}_0$. By Theorem 42 condition (4.26) is necessary and sufficient for the local invertibility of the Mellin ψdo $A^{t_0}_\Gamma$ at the point 0. Hence condition (4.26) is necessary and sufficient for the local invertibility of A_Γ : $L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ at the point $t_0 \in \mathcal{F}$.

Note that the condition of the local invertibility in the spaces $L^{p(\cdot)}(\Gamma, w)$ depends on the value $p(\cdot)$ only at the point t_0.

(ii) Let $t_0 \in \Gamma\backslash\mathcal{F}$. Then there exist a simple locally Lyapunov curve $\Gamma_j \subset \Gamma$ such that $t_0 \in int\,\Gamma_j$, where $\varphi_j : (0,1) \to int\,\Gamma_j$ is the parametrization of the curve $int\,\Gamma_j$. Let $\varphi_j(r_0) = t_0$, and $\varphi'_j(r_0) = 1$. Let $\varepsilon > 0$ be sufficiently small and $\Gamma^{t_0,\varepsilon}_j = \varphi_j(\mathcal{I}_{t_0,\varepsilon})$, $\mathcal{I}_{t_0,\varepsilon} = (r_0 - \varepsilon, r_0 + \varepsilon)$. The restriction $\varphi^{t_0,\varepsilon}_j$ of the mapping φ_j on $\mathcal{I}_{t_0,\varepsilon}$ is the homeomorphism $\mathcal{I}_{t_0,\varepsilon}$ on $\Gamma^{t_0,\varepsilon}_j$. Let

$$\Phi^{t_0,\varepsilon}_j : L^{p(\cdot)}(\Gamma^{t_0,\varepsilon}_j) \to L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}),$$

with $\tilde{p}(x) = p(\varphi_j(x))$ be the isomorphism defined as

$$(\Phi^{t_0,\varepsilon}_j u)(x) = u(\varphi^{t_0,\varepsilon}_j(x)),$$

and $\left(\Phi^{t_0,\varepsilon}_j\right)^{-1} : L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}) \to L^{p(\cdot)}(\Gamma^{t_0,\varepsilon}_j)$ be the inverse mapping.

It is well known (see for instance [2]) that

$$\Phi^{t_0,\varepsilon}_j \chi_\varepsilon S_\Gamma \chi_\varepsilon \left(\Phi^{t_0,\varepsilon}_j\right)^{-1} = \tilde{\chi}_\varepsilon S_\mathbb{R} \tilde{\chi}_\varepsilon I + T_\varepsilon, \tag{4.28}$$

where χ_ε and $\tilde{\chi}_\varepsilon$ are the characteristic functions of $\Gamma_j^{t_0,\varepsilon}$ and $\mathcal{I}_{t_0,\varepsilon}$, respectively, T_ε is a compact operator in $L^p(\mathcal{I}_{t_0,\varepsilon})$ for every constant $p \in (1,\infty)$. Moreover, it follows from (4.28) and boundedness of $\Phi_j^{t_0,\varepsilon}\chi_\varepsilon S_\Gamma \chi_\varepsilon \left(\Phi_j^{t_0,\varepsilon}\right)^{-1}$ and $\tilde{\chi}_\varepsilon S_\mathbb{R} \tilde{\chi}_\varepsilon I$ in $L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$ that T_ε is also a bounded operator in $L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$ if $p(\cdot)$ satisfies conditions (2.12), (2.13). By Proposition 9 we obtain that T_ε is a compact operator in $L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$. Let $\phi \in C_0((-1,1))$ and $\phi(0) = 1$. We set

$$\phi_\delta(x) = \phi\left(\frac{x - x_0}{\delta}\right), \quad \tilde{\phi}_\delta(t) = \phi_\delta(\varphi_j^{-1}(t)).$$

Then $\phi_\delta \chi_\varepsilon = \phi_\delta$ for sufficiently small $\delta > 0$. Hence we obtain from (4.28) that

$$\begin{aligned} \Phi_j^{t_0,\varepsilon}\phi_\delta S_\Gamma \phi_\delta \left(\Phi_j^{t_0,\varepsilon}\right)^{-1} &= \tilde{\phi}_\delta S_\mathbb{R} \tilde{\phi}_\delta I + \tilde{\phi}_\delta T_\varepsilon \tilde{\phi}_\delta I, \\ \tilde{\phi}_\delta(t) &= \phi_\delta(\varphi_j^{-1}(t)). \end{aligned} \tag{4.29}$$

The sequence $\tilde{\phi}_\delta I$ strongly converges to 0 in $L^{p(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$ as $\delta \to 0$. Hence

$$\lim_{\delta\to 0} \left\|\tilde{\phi}_\delta T_\varepsilon \tilde{\phi}_\delta I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathcal{I}_{t_0,\varepsilon}))} = 0. \tag{4.30}$$

It yields that $A_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is locally invertible at the point t_0, if and only if the operator

$$\tilde{\phi}_\delta A_\mathbb{R}^{t_0} \tilde{\phi}_\delta I : L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}) \to L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}),$$

with $A_\mathbb{R}^{t_0} = (a \circ \varphi_j^{t_0,\varepsilon})I + (b \circ \varphi_j^{t_0,\varepsilon})S_\mathbb{R}$, is locally invertible at the point $x_0 = \Phi_j^{t_0,\varepsilon}(t_0) \in \mathbb{R}$. Applying Theorem 24 we obtain that $\tilde{\phi}_\delta A_\mathbb{R}^{t_0}\tilde{\phi}_\delta I$ is locally invertible at the point $x_0 \in \mathbb{R}$, if and only if

$$\begin{aligned} &(a \circ \varphi_j^{t_0,\varepsilon})(r_0) \pm (b \circ \varphi_j^{t_0,\varepsilon})(r_0) \\ &\quad = a(t_0) \pm b(t_0) \neq 0. \end{aligned}$$

□

4.6. Simonenko's Local Principle in $L^{p(\cdot)}(X)$

We prove here Simonenko's local principle in variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ in the general setting where the underlying space X is a quasimetric measure space, as introduced by Definition 7. In this subsection we assume that X is a Hausdorff compact space.

Definition 43. An operator $A \in \mathcal{B}(L^{p(\cdot)}(X))$ is called an operator of local type, if for every two closed set F_1 and F_2 such that $F_1 \cap F_2 = \varnothing$, the operator $\chi_{F_1} A \chi_{F_2} I$ is compact.

Definition 44. An operator $A \in \mathcal{B}(L^{p(\cdot)}(X))$ is called locally Fredholm at the point $x_0 \in X$, if there exist a neighborhood U of the point x_0 and operators $L^{x_0}, R^{x_0} \in \mathcal{B}(L^{p(\cdot)}(X))$ such that

$$L^{x_0} A \chi_U I = \chi_U I + T_1 \quad \text{and} \quad \chi_U A R^{x_0} = \chi_U I + T_2, \tag{4.31}$$

where T_1, T_2 are compact operators in $L^{p(\cdot)}(X)$. If $T_1 = 0$ and $T_2 = 0$, A is called a locally invertible operator at the point x_0.

Remark 45. We say that the space X does not have discrete components, if for every point $x_0 \in X$ there exists a sequence $U_1 \supset U_2 \supset \cdots \supset U_j \supset \cdots$. of neighborhoods of the point x_0 such that

$$\lim_{j\to\infty} \mu(U_j) = 0. \tag{4.32}$$

If X does not have discrete components, the local Fredholmness coincides with the local invertibility. Indeed, let $L^{x_0} A\chi_U I = \chi_U I + T_1$, and $U \supset U_1 \supset U_2 \supset \cdots \supset U_j \supset \cdots$, then we obtain

$$L^{x_0} A\chi_{U_j} I = (I + T_1\chi_{U_j} I)\chi_{U_j} I. \tag{4.33}$$

Condition (4.32) implies that the sequence $\chi_{U_j} I$ strongly tends to 0 in $L^{p(\cdot)}(X)$. Hence

$$\lim_{j\to\infty} \left\| T_1\chi_{U_j} I\right\|_{\mathcal{B}(L^{p(\cdot)}(X))} = 0.$$

It implies that the operators $I + T_1\chi_{U_j} I$ are invertible for sufficiently large j. Then $(I + T_1\chi_{U_{j_j}} I)^{-1} L^{x_0}$ is a left local inverse operator at x_0. In the same way one can prove the existence of a right local inverse operator.

Theorem 46. (*Simonenko's local principle* [46–48]) *Let $A \in \mathcal{B}(L^{p(\cdot)}(X, \mu))$ be an operator of local type. Then A is a Fredholm operator if and only if A is a locally Fredholm operator at every point $x \in X$. If the space X does not have discrete components, we can replace the local Fredholmness by the local invertibility.*

The proof of Theorem 46 for variable $p(\cdot)$ repeats word by word the Simonenko's proof for a constant p (See for instance [48, pp 21–24]).

4.7. Fredholmness of SIO

Theorem 47. *Let Γ be a composed compact curve of the class $\mathcal{L}$, let $p(\cdot)$ satisfy the above conditions on Γ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$. Then $S_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is a local type operator in the sense of Simonenko, that is, for every closed set $F_1, F_2 \subset \Gamma$ such that $F_1 \cap F_2 = \varnothing$ the operator $\chi_{F_1} S_\Gamma \chi_{F_2} I$ is a compact operator in $L^{p(\cdot)}(\Gamma, w)$.*

Proof. The operator $\chi_{F_1} S_\Gamma \chi_{F_2} I$ has a kernel $k \in C^\infty(\Gamma \times \Gamma)$. Hence $\chi_{F_1} S_\Gamma \chi_{F_2} I : L^1(\Gamma) \to L^\infty(\Gamma)$ is a compact operator. Because $u \to w^{-1}u$ is a bounded operator from $L^{p(\cdot)}(\Gamma, w)$ in $L^1(\Gamma)$ and $v \to wv$ is a bounded operator from $L^\infty(\Gamma)$ to $L^{p(\cdot)}(\Gamma, w)$, the operator $\chi_{F_1} S_\Gamma \chi_{F_2} I$ is compact in $L^{p(\cdot)}(\Gamma, w)$. □

Theorem 48. *Let A_Γ be an operator of form (4.23) and Γ and w satisfy the assumptions of Theorem 42. Then*

$$A_\Gamma : \; L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$$

is a Fredholm operator, if and only if there hold condition (4.26) for every point $t_0 \in \mathcal{F}$ and condition (4.27) for every point $t_0 \in \Gamma \backslash \mathcal{F}$.

Proof. Make use of Theorems 42, 46 and 47. □

Remark 49. If we freeze the variable exponent $p(\cdot)$ at the point t_0, condition (4.26) coincides with the Fredholmness condition obtained in paper [3] for the case of the constant Lebesgue exponent $p \in (1, \infty)$, while condition (4.27) is classical and does not depend on $p(\cdot)$.

Let

$$A_\Gamma^{MN} = \sum_{j=1}^{N} \prod_{k=1}^{M} A_\Gamma^{jk}, \tag{4.34}$$

where $A_\Gamma^{jk} = a_{jk} I + b_{jk} S_\Gamma$, and $a_{jk}, b_{jk} \in PSO(\Gamma)$.

We define the *local symbol* of A_Γ^{MN} at the point $t \in \Gamma$ by the formula

$$\tilde{\sigma}^t(A_\Gamma^{MN}) = \sum_{j=1}^{N} \prod_{k=1}^{M} \tilde{\sigma}^t(A_\Gamma^{jk}),$$

where $\tilde{\sigma}^t(A_\Gamma^{jk})$ are defined by formulas (4.24) and (4.25). Note that

$$\tilde{\sigma}^t(A_\Gamma^{MN}) = \left\{ \tilde{\sigma}_+^t(A_\Gamma^{MN}), \tilde{\sigma}_-^t(A_\Gamma^{MN}) \right\},$$

in the case $t \in \Gamma \backslash \mathcal{F}$, where

$$\tilde{\sigma}_\pm^t(A_\Gamma^{MN}) = \sum_{j=1}^{N} \prod_{k=1}^{M} (a_{jk}(t) \pm b_{jk}(t)).$$

We say that the symbol $\tilde{\sigma}^t(A_\Gamma^{MN})$ is invertible if

$$\liminf_{r \to 0} \inf_{\xi \in \mathbb{R}} \left| \det \tilde{\sigma}(A_\Gamma^{MN})(r, \xi) \right| > 0,$$

for $t \in \mathcal{F}$, and $\tilde{\sigma}\,{}_+^t(A_\Gamma^{MN}) \neq 0$ for $t \in \Gamma \backslash \mathcal{F}$.

Theorem 48 and the Simonenko local principle imply the following result.

Theorem 50. *The operator A_Γ^{MN}, where Γ and w satisfy the assumptions of Theorem 42, is a Fredholm operator in $L^{p(\cdot)}(\Gamma, w)$, if and only if the local symbol $\tilde{\sigma}^t(A_\Gamma^{MN})$ is invertible for every point $t \in \Gamma$.*

Remark 51. The statement of Theorem 50 can be extended on operators in the Banach algebra obtained by the closure of operators A_Γ^{MN} in $\mathcal{B}(L^{p(\cdot)}(\Gamma, w))$. We are going to do it in a forthcoming paper.

4.7.1. Index Formula. Let $A = aI + bS_\Gamma$, where $a, b \in PSO(\Gamma)$ and $\Gamma \in \mathcal{L}_{sl}$. Let A be a Fredholm operator in $L^{p(\cdot)}(\Gamma, w)$, where $w \in \mathcal{A}_{p(\cdot)}^{sl}(\Gamma)$. Then the Fredholm index of $A : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is given by the formula

$$\begin{aligned} \textit{index}\, A = &- \sum_{j=1}^{K} (2\pi)^{-1} \left[\arg \frac{a(t) + b(t)}{a(t) - b(t)} \right]_{t \in \Gamma_j} \\ &- \sum_{j=1}^{L} (2\pi)^{-1} \lim_{r \to 0} \left[\arg \det \tilde{\sigma}(A^{t_j})(r, \xi) \right]_{\xi = -\infty}^{\infty}. \end{aligned} \tag{4.35}$$

In this formula, K is the number of the oriented and rectifiable simple smooth arcs generating the composed curve Γ, and L is the number of nodes of the curve Γ.

The index formula (4.35) is proved by the method of separation of singularities, and this proof is similar to that for the constant p (see for instance [3,4,36]).

Remark 52. All the results of the paper remain valid if we replace the classes $\mathcal{C}^\infty(0,\varepsilon), \tilde{\mathcal{C}}^\infty(0,\varepsilon), \mathcal{C}^\infty_{sl}(0,\varepsilon), \tilde{\mathcal{C}}^\infty_{sl}(0,\varepsilon)$ in the assumptions on the curve Γ and the weights near nodes by the classes $\mathcal{C}^m(0,\varepsilon), \tilde{\mathcal{C}}^m(0,\varepsilon), \mathcal{C}^m_{sl}(0,\varepsilon), \tilde{\mathcal{C}}^m_{sl}(0,\varepsilon)$ where m is sufficiently large.

In relation to Remark 52, see also Definition 55 and Lemma 56 in the next section.

5. On Comparison of the Used Class of Oscillating Weights with the Bary–Stechkin Type Weights

We wish to compare the class of weights w used in this paper with the class of oscillating weights known as Bary–Stechkin class which was used in various papers, see for instance [23,27]. In the proofs in this section we follow some ideas of paper [43].

We call two non-negative functions f and g equivalent, if

$$c_1 f(x) \le g(x) \le c_2 f(x), \quad c_1 > 0, \;\; c_2 > 0.$$

Note that the weighted variable exponent spaces obviously does not change if we replace the weight by an equivalent weight; for us it is also important to observe that the Bary-Stechkin class, defined below, is also closed with respect to the equivalence of functions.

We need some definitions. Recall that a non-negative function f on $[0,\ell], 0 < \ell < \infty$, is called almost increasing (almost decreasing), if there exists a constant $C(\ge 1)$ such that $f(x) \le Cf(y)$ for all $x \le y$ ($x \ge y$, respectively). Equivalently, a function f is almost increasing (almost decreasing), if it is equivalent to an increasing (decreasing, resp.) function g.

5.1. Bary–Stechkin class Φ

Definition 53. Let $0 < \ell < \infty$. 1) By $W = W([0,\ell])$ we denote the class of functions φ continuous and positive on $(0,\ell]$ such that there exists the finite limit $\lim_{x\to 0}\varphi(x)$; 2) by $W_0 = W_0([0,\ell])$ we denote the class of functions $\varphi \in W$ almost increasing on $(0,\ell)$; 3) by $\widetilde{W} = \widetilde{W}([0,\ell])$ we denote the class of functions $w \in W$ such that $x^a w(x) \in W_0$ for some $a = a(w) \in \mathbb{R}^1$; 4) by $\underline{W} = \underline{W}([0,\ell])$ we denote the class of functions $w \in W$ such that there exists a number $b \in \mathbb{R}^1$ such that $\frac{f(t)}{t^b}$ is almost decreasing.

The classes $\widetilde{W}_0$, $\underline{W}_0$ are known to be characterized in terms of the Matuszewska-Orlicz indices $m(w)$ and $M(w)$ of w:

$$w \in \widetilde{W}_0 \Longleftrightarrow -\infty < m(w) \le \infty, \tag{5.1}$$

$$w \in \underline{W}_0 \Longleftrightarrow -\infty \le m(w) < \infty; \tag{5.2}$$

We refer to [31,32] for the Matuszewska-Orlicz indices and to [16] and [42] for statements (5.1) and (5.2).

Definition 54. We introduce the Bary–Stechkin class Φ as the class of functions in W with finite Matuszewska-Orlicz indices, that is,

$$\Phi = \widetilde{W} \cap \underline{W}. \tag{5.3}$$

Note that the Bary–Stechkin class is usually introduced as the two-parameter class Φ_β^α of functions $w \in \widetilde{W}$ satisfying the conditions

$$\int_0^x \frac{w(t)}{t^{1+\alpha}} dt \leq C\, \frac{w(x)}{x^\alpha}, \qquad \int_x^\ell \frac{w(t)}{t^{1+\beta}} dt \leq C\, \frac{w(x)}{x^\beta}, \tag{5.4}$$

non-empty if and only if $\alpha < \beta$; we have

$$\Phi = \bigcup_{-\infty<\alpha<\beta<+\infty} \Phi_\beta^\alpha,$$

which follows from the fact that $w \in \Phi_\beta^\alpha$, if and only if $\alpha < m(w) \leq M(w) < \beta$, see [16,42].

5.2. Simonenko Type Class $\mathbb{S}^2$

Let $0 < \ell < \infty$. The indices

$$p(w) = \inf_{0<x\leq\ell} \frac{xw'(x)}{w(x)}, \quad q(w) = \sup_{0<x\leq\ell} \frac{xw'(x)}{w(x)} \tag{5.5}$$

which appeared in (4.3) and (4.4) are known as Simonenko indices, see [45], and it is known that

$$p(w) \leq m(w) \leq M(w) \leq q(w), \tag{5.6}$$

see [32, Theorem 11.11]. The class of functions on $(0, \ell)$ with finite Simonenko indices may be called Simonenko class. We introduce a slight generalization of this notion as inspired by conditions (3.9) and (3.11).

Definition 55. We say that a weight function $w = e^{v(x)}$ is in the Simonenko type class $\mathbb{S}^N, N = 1, 2, 3, \ldots,$ if

$$\sup_{x\in(0,\ell)} \left| \left(x\frac{d}{dx} \right)^k v(x) \right| < \infty, \quad k = 1, 2, \ldots, N \tag{5.7}$$

and

$$\lim_{x\to 0} \left(x\frac{d}{dx} \right)^2 v(x) = 0. \tag{5.8}$$

Obviously, $\mathbb{S}^{N+1} \subset \mathbb{S}^N$. We are mainly interested in the case $N = 2$. A connection of this class with Simonenko indices becomes clear, if we observe that conditions (5.7) and (5.8) with $N = 2$ in terms of the weight w itself have the form

$$\sup_{x\in(0,\ell)} \left| \frac{xw'(x)}{w(x)} \right| < \infty, \quad \sup_{x\in(0,\ell)} \left| x\frac{d}{dx} \frac{xw'(x)}{w(x)} \right| < \infty \tag{5.9}$$

and

$$\lim_{x\to 0} x\frac{d}{dx}\left(\frac{xw'(x)}{w(x)}\right) = 0.$$

Lemma 56. *Given a function* $w \in \widetilde{W}_0 \cap \underline{W}_0$*, for every* $N = 1, 2, 3, \ldots.$ *there exists a function*

$$w_N \in C^N([0,\ell]) \cap \left(\widetilde{W}_0 \cap \underline{W}_0\right)$$

equivalent to w*, and such that* $v(x) = \log w_N(x)$ *satisfies conditions* (5.7)*. It may be chosen as*

$$w_N(x) = x^\alpha \int_0^x \frac{w(t)\left(\ln \frac{x}{t}\right)^{N-1}}{t^{1+\alpha}} dt \tag{5.10}$$

with any α *such that* $\alpha < m(w)$*.*

Proof. Let first $N = 1$. The proof of the equivalence $w_1(x) \sim w(x)$ is direct, via the usage of the properties

$$m(w) = \sup\left\{\mu > 0 : \ \frac{w(x)}{x^\mu} \quad \text{is almost increasing}\right\}, \tag{5.11}$$

$$M(w) = \inf\left\{\nu > 0 : \ \frac{w(x)}{x^\nu} \quad \text{is almost decreasing}\right\}, \tag{5.12}$$

see [16, Theorem 3.6], for the proof of (5.11) and (5.12). By direct differentiation of $w_1(x)$ we obtain

$$x\frac{d}{dx}w_1(x) = \alpha w_1(x) + w(x). \tag{5.13}$$

Then the first inequality in (5.9), corresponding to the case $N = 1$, holds because $w_1 \sim w$. Note that $w \in \widetilde{W}_0 \cap \underline{W}_0 \implies w_1 \in \widetilde{W}_0 \cap \underline{W}_0$ and w and w_1 as equivalent functions have equal Matuszewska-Orlich indices, see [32, Theorem 11.4].

For $N > 1$ the statement is obtained by iteration of the procedure. Indeed, by the already proved equivalence $w_1 \sim w$, we have

$$w(x) \sim w_1(x) \sim x^\alpha \int_0^x \frac{w_1(t)}{t^{1+\alpha}} dt = x^\alpha \int_0^x \frac{w(s)ds}{s^{1+\alpha}} \int_s^x \frac{dt}{t} = w_2(x).$$

By direct differentiation we obtain

$$x\frac{d}{dx}w_2(x) = \alpha w_2(x) + w_1(x). \tag{5.14}$$

Consequently

$$\frac{xw_2'(x)}{w_2(x)} = \alpha + \frac{w_1(x)}{w_2(x)}, \tag{5.15}$$

whence the first inequality in (5.9) follows in view of the equivalence $w_1 \sim w_2$.

Furthermore, differentiating (5.15), by (5.14) and (5.15) we obtain

$$x\frac{d}{dx}\frac{xw_2'(x)}{w_2(x)} = \frac{w}{w_2} - \left(\frac{w_1}{w_2}\right)^2$$

whence the second inequality in (5.9) follows in view of the equivalence $w \sim w_1 \sim w_2$.

For $N > 2$ the statement is obtained by induction following (5.13) and (5.15). □

Remark 57. It is known that the interval defined by Matuszewska-Orlicz indices in general is narrower than that defined by Simonenko indices, namely

$$[m(w), M(w)] \subseteq [p(w), q(w)], \tag{5.16}$$

see [32, Theorem 11.11]. Therefore, any function having finite Simonenko indices, also has finite Matuszewska-Orlicz indices and consequently belongs to the Bary–Steckin class Φ.

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