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VARIABLE EXPONENT CAMPANATO SPACES

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We study variable exponent Campanato spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ on spaces of homogeneous type. We prove an embedding result between variable exponent Campanato spaces. We also prove that these spaces are equivalent, up to norms, to variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(X)$ with $\lambda_+ < 1$ and variable exponent Hölder spaces $H^{\alpha(\cdot)}(X)$ with $\lambda_- > 1$. In the setting of an arbitrary quasimetric measure spaces, we introduce the log-Hölder condition for $p(x)$ with the distance $d(x,y)$ replaced by $\mu B(x,d(x,y))$, which provides a weaker restriction on $p(x)$ in the general setting and show that some basic facts for variable exponent Lebesgue spaces hold without the assumption that X is homogeneous or even Ahlfors lower or upper regular. However, the main results for Campanato spaces are proved in the setting of homogeneous spaces X. Bibliography: 34 titles.

1 Introduction

We study variable exponent Campanato spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$, where the underlying space X is a quasimetric measure space. In most of the results, we suppose that X is of homogeneous type. Campanato spaces are well known in the case where p and λ are constant and in the Euclidean setting (cf., for example, [1]–[3]). The investigation of variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ and the corresponding Sobolev spaces $W^{m,p(\cdot)}(X)$ was intensively studied during the last decade. For the progress in this field and related topics of harmonic analysis and operator theory we refer to the surveys [4]–[7]. The investigation of variable exponent Morrey spaces was recently started in [8]–[10], where the boundedness of maximal, singular, and potential type operators in such spaces was studied.

In this paper, we consider variable exponent Campanato spaces over spaces of homogeneous type. We start with an embedding statement for such spaces. Making use of a modified Diening

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inequality, we prove the coincidence of variable exponent Campanato space, up to equivalence of norms, with variable exponent Morrey spaces (when $\sup_x \lambda(x) < 1$) and with variable exponent Hölder spaces (when $\inf_x \lambda(x) > 1$).

The assumptions on the exponents are standard within the framework of variable exponent spaces, namely, p and λ are assumed to be log-Hölder continuous. Note that we introduce the log-condition in a form weaker than usual (cf. (2.18)), which allows us to formulate some facts for inhomogeneous spaces.

Note that a similar coincidence of variable exponent Campanato spaces with Morrey or Hölder spaces within the framework of Euclidean spaces with the Lebesgue measure was recently shown in [11], where there was introduced the concept of $p(\cdot)$-average of a function f in a set E, which is an extension of the average function. This $p(\cdot)$-average was fundamental in proving one of the coincidences. Our approach is different and covers the setting of quasimetric measure spaces.

In the case of constant p, there are also well known generalized Morrey–Campanato spaces (cf., for example, [12]–[15]), where $r^{-\lambda}$ is replaced by a function $\varphi(r)$, or more generally, by $\varphi(x,r)$ subject to some assumptions. In the case of variable $p(x)$, such generalized Morrey spaces were studied in [9, 16]. We do not touch such a kind of generalization for variable exponent Campanato spaces in this paper.

We introduce the notation:

d_X denotes the diameter of a set X,

$A \sim B$ for positive A and B means that there exists $c > 0$ such that $c^{-1}A \leqslant B \leqslant cA$;

$B(x,r) = \{y \in X : d(x,y) < r\}$,

$\displaystyle\fint_E f(y)\,\mathrm{d}\mu(y) = f_E$ denotes the integral average of f, viz., $\displaystyle\frac{1}{\mu(E)}\int_E f(y)\,\mathrm{d}\mu(y)$,

c, C denote various absolute positive constants which may be different even in the same line.

2 Preliminaries on Variable Exponent Spaces

2.1 Spaces of homogeneous type

Given a set X, a *quasimetric* is a function $d : X \times X \to [0,\infty)$ which satisfies the usual metric axioms with the triangle inequality replaced by the *quasitriangle inequality*

$$d(x,y) \leqslant \mathscr{Q}[d(x,z) + d(z,y)], \quad \mathscr{Q} \geqslant 1 \tag{2.1}$$

where $x, y, z \in X$ and $\mathscr{Q}$ is often called the *quasitriangular constant of* d. We assume that

$$d(x,y) = d(y,x).$$

Two quasimetrics d and d' on X are *equivalent* if $d(x,y) \sim d'(x,y)$ for all $x, y \in X$.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x,r)$. Everywhere in the sequel, we suppose that all the balls have finite measure, i.e.,

$\mu B(x,r) < \infty$ for all $x \in X$ and $r > 0$, and the space of compactly supported continuous functions is dense in $L^1(X,\mu)$.

We say that a measure μ is *lower Ahlfors α-regular* if

$$\mu B(x,r) \geqslant cr^\alpha \tag{2.2}$$

and *upper Ahlfors β-regular* if

$$\mu B(x,r) \leqslant cr^\beta, \tag{2.3}$$

where $\alpha, \beta, c > 0$ are independent of x and r. If $\alpha = \beta$, the measure μ is called *Ahlfors α-regular.*

The condition

$$\mu B(x,2r) \leqslant \mathscr{D}\mu B(x,r), \quad \mathscr{D} > 1, \tag{2.4}$$

on the measure μ with $\mathscr{D}$ independent of $x \in X$ and $0 < r < d_X$, is known as the *doubling condition.*

The triplet (X,d,μ), with μ satisfying the doubling condition, is called a *space of homogeneous type.* This notion sometimes is introduced in a slightly more general way (cf., for example, [17]). Balls in a general space of homogeneous type are not necessarily open, although there exists a continuous quasimetric d' equivalent to d for which every ball is open.

Iterating the inequality (2.4), we obtain

$$\frac{\mu B(x,R)}{\mu B(y,r)} \leqslant \mathscr{D}\left(\frac{R}{r}\right)^N, \quad 0 < r \leqslant R, \tag{2.5}$$

for all d-balls $B(x,R)$ and $B(y,r)$ with $B(y,r) \subset B(x,R)$, where $N = \log_2 \mathscr{D}$ is called the *doubling order of μ.* We will also mention (2.5) as the doubling condition.

From (2.5) it follows that every homogeneous type space (X,d,μ) with a finite measure is lower Ahlfors N-regular.

In some results, we need the following condition.

$$\mu\left(B(x,R)\backslash B(x,r)\right) > 0 \tag{2.6}$$

for all $x \in X$ and r, R with $0 < r < R < d_X$.

The validity of the reverse doubling condition, following from the doubling condition under certain restrictions, is well known (cf., for example, [18, p. 269]). Since there are various formulations of this result, we give the one more suited for our purposes (cf., for example, [19, p. 13]).

Lemma 2.1. *Let (X,d,μ) be a space of homogeneous type. If* (2.6) *is valid, then the measure μ satisfies the reverse doubling condition*

$$\frac{\mu B(x,r)}{\mu B(x,R)} \leqslant C\left(\frac{r}{R}\right)^\gamma \tag{2.7}$$

for all $x \in X$ and $0 < r \leqslant R < d_X$, where $C, \gamma > 0$.

Remark 2.2. Note that the condition $B(x,R)\backslash B(x,r) \neq \emptyset$, valid under the assumption (2.6), is also fulfilled under the assumption that X is connected and $B(x,R)^c \neq \varnothing$, as shown in [19, Proposition 3.3].

Note that the reverse doubling condition (2.7), together with $\mu X < \infty$, implies that $d_X < \infty$ and the measure μ, is upper Ahlfors γ-regular.

2.2 Variable exponent Lebesgue spaces

2.2.1 Definitions

Let p be a μ-measurable function on X. In main, we assume that

$$1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty, \tag{2.8}$$

where $p_- := \operatorname*{ess\,inf}_{x\in X} p(x)$ and $p_+ := \operatorname*{ess\,sup}_{x\in X} p(x)$, but sometimes admit the range

$$1 \leqslant p(x) \leqslant \infty. \tag{2.9}$$

In the case $1 \leqslant p(x) < \infty$, we denote by $L^{p(\cdot)}(X)$ the space of all μ-measurable functions f on X such that

$$I^{p(\cdot)}\left(\frac{f}{\lambda}\right) := \int\limits_X \left|\frac{f(y)}{\lambda}\right|^{p(y)} \mathrm{d}\mu(y) < \infty \tag{2.10}$$

for some $\lambda = \lambda(f) > 0$. Endowed with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I^{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}, \tag{2.11}$$

this space is a Banach function space (cf. [20, 21]). We have the following relation between the modular (2.10) and the norm (2.11):

$$\|f\|^{\theta}_{p(\cdot)} \leqslant I^{p(\cdot)}(f) \leqslant \|f\|^{\sigma}_{p(\cdot)}, \tag{2.12}$$

where

$$\theta = \begin{cases} p_+ & \text{if } \|f\|_{p(\cdot)} \leqslant 1, \\ p_- & \text{if } \|f\|_{p(\cdot)} \geqslant 1, \end{cases} \quad \text{and} \quad \begin{cases} p_- & \text{if } \|f\|_{p(\cdot)} \leqslant 1, \\ p_+ & \text{if } \|f\|_{p(\cdot)} \geqslant 1. \end{cases} \tag{2.13}$$

We denote by $p'(\cdot)$ the conjugate exponent

$$p'(x) = \frac{p(x)}{p(x) - 1},$$

with the usual convention that $p' = \infty$ when $p = 1$. The Hölder inequality is valid in the form

$$\int\limits_X |f(x)\varphi(x)|\,\mathrm{d}\mu(x) \leqslant \left(\frac{1}{p_-} + \frac{1}{p_+}\right) \|f\|_{p(\cdot)}\|\varphi\|_{p'(\cdot)}. \tag{2.14}$$

If $\mu X < \infty$ and $1 \leqslant p(x) \leqslant q(x) \leqslant q^+ < \infty$, we have the embedding

$$L^{q(\cdot)}(X) \hookrightarrow L^{p(\cdot)}(X). \tag{2.15}$$

In the case of an unbounded exponent $p(\cdot)$, the norm in the variable exponent Lebesgue space is introduced as follows:

$$\|f\|_{L^{p(\cdot)}(X)} := \|f\|_{L^{p(\cdot)}(X\setminus X_\infty)} + \|f\|_{L^\infty(X_\infty)}, \tag{2.16}$$

where $X_\infty := \{x \in X : p(x) = \infty\}$.

For the basics of the theory of variable exponent Lebesgue spaces see [21, 22].

2.2.2 On log-condition on quasimetric measure spaces

The standard *local logarithmic condition* on quasimetric measure spaces (X, d, μ) is usually introduced in the form

$$|p(x) - p(y)| \leqslant \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \leqslant \frac{1}{2}, \quad x, y \in X, \tag{2.17}$$

where $C_p > 0$ is independent of x and y (cf., for example, [23, 20]). The condition (2.17) is also known as *Dini-Lipschitz condition*, *weak-Lipschitz condition* or even log-*Hölder continuity condition*.

We denote by $\mathscr{P}^{\log}(X)$ the set of all μ-measurable functions satisfying the log-Hölder continuity condition (2.17).

In the context of general quasimetric measure spaces, we find it more natural to introduce also such a log-type condition in terms of measures of balls than in terms of distances. Namely, we introduce the class $\mathscr{P}^{\log}_\mu(X)$ of functions $p : X \to [1, \infty)$ satisfying the condition

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln \mu B(x, d(x,y))} \tag{2.18}$$

for all $x, y \in X$ such that $\mu B(x, d(x,y)) < 1/2$. This is a weaker assumption on $p(x)$ than (2.17), as can be seen from Lemma 2.3. Note that from the assumption $d(x,y) = d(y,x)$ it follows that (2.18) is equivalent to its symmetrical form

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln \mu B(x, d(x,y))} + \frac{A}{-\ln \mu B(y, d(x,y))}. \tag{2.19}$$

The log-condition in form (2.18), coinciding with (2.17) in the Euclidean space, is more suitable in the context of general quasimetric measure spaces, not only because it is weaker than (2.3), but also because in some statements it simultaneously allows us to put less restrictions on the (X, d, μ), not requiring its homogeneity or even the lower Ahlfors condition (cf., for example, Lemmas 2.4–2.6).

Lemma 2.3. *If (X, d, μ) has the lower Ahlfors property, then*

$$\mathscr{P}^{\log}(X) \subseteq \mathscr{P}^{\log}_\mu(X). \tag{2.20}$$

Proof. Let $r = d(x,y)$ and $\mu B(x,r) \leqslant 1/2$. Then from the lower Ahlfors condition we obtain

$$\frac{1}{\ln \frac{1}{c^{\frac{1}{\alpha}} r}} \leqslant \frac{\alpha}{-\ln \mu B(x,r)},$$

where $c > 0$ and $\alpha > 0$ are constants from (2.2), from which the statement in (2.20) follows. $\square$

Lemma 2.4. *Let (X, d, μ) be a quasimetric measure space with finite measure, and let $p \in \mathscr{P}^{\log}_{\mu}(X)$. Then*

$$C^{-1}\mu B(x,r) \leqslant (\mu B(x,r))^{\frac{p(x)}{p(y)}} \leqslant C\mu B(x,r) \tag{2.21}$$

for all $x, y \in X$ such that $y \in B(x,r)$, with the constant $C \geqslant 1$ independent of x, y, r.

Proof. By the fact that X has finite measure and p is bounded from below, it suffices to check (2.21) only for small r, for example, for r such that $\mu B(x,r) < 1/2$. The relation (2.21) is equivalent to the inequality

$$\left| \frac{p(x) - p(y)}{p(y)} \ln\left[\mu B(x,r)\right] \right| \leqslant \ln C, \tag{2.22}$$

which immediately follows from the condition $p \in \mathscr{P}^{\log}_{\mu}(X)$ for all $r > d(x,y)$. □

We will also need the following estimate.

Lemma 2.5. *Let (X, d, μ) be a quasimetric measure space with finite measure, and let $p \in \mathscr{P}^{log}_{\mu}(X)$. Then*

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)} \sim (\mu B(x,r))^{\frac{1}{p(x)}}. \tag{2.23}$$

Proof. The inequality

$$\|\chi_{B(x,r)}\|_{p(\cdot)} \leqslant C(\mu B(x,r))^{\frac{1}{p(x)}}$$

was proved in [23] and [8] under the conditions that (X, d, μ) has the lower Ahlfors property and $p \in \mathscr{P}^{\log}(X)$. The analysis of the proof in [8] shows that it is also valid under the only assumption that $p \in \mathscr{P}^{\log}_{\mu}(X)$. Note that this inequality holds even if $p(x)$ may be unbounded.

To prove the inverse inequality, by the definition of the norm, we just need to show that there exists $A > 0$ such that

$$I^{p(\cdot)}\left(\frac{\chi_{B(x,r)}(\cdot)}{A(\mu B(x,r))^{\frac{1}{p(x)}}}\right) \geqslant 1, \tag{2.24}$$

which easily follows from (2.21) by integrating over $X \backslash X_\infty$ with obvious estimation over X_∞. □

2.3 Variable exponent Morrey spaces

Let λ be a μ-measurable function on X with the range in $[0,1]$. Following [8, 10, 24], we define the *variable exponent Morrey space* $L^{p(\cdot),\lambda(\cdot)}(X)$ as the set of all integrable functions f on X such that

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} |f(y)|^{p(y)}\, \mathrm{d}\mu(y) < \infty. \tag{2.25}$$

We recollect here some basic facts from [8]. Note that the presentation in [8] was given in the Euclidean setting, however the basics we present here are valid within the framework of general quasimetric measure spaces.

The norm in the space $L^{p(\cdot),\lambda(\cdot)}(X)$ can be introduced in two forms

$$\|f\|_1 = \inf\left\{\lambda > 0 : I^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\} \tag{2.26}$$

and

$$\|f\|_2 = \sup_{x\in X, r>0} \left\|(\mu B(x,r))^{-\frac{\lambda(x)}{p(\cdot)}} f\chi_{B(x,r)}\right\|_{p(\cdot)} \tag{2.27}$$

which coincide for every $f \in L^{p(\cdot),\lambda(\cdot)}(X)$ (cf. details in [8] in the case of Euclidean space together with the Lebesgue measure; the proof remains the same for an arbitrary quasimetric measure space). Therefore, we can define the norm in the variable exponent Morrey space as

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} := \|f\|_1 = \|f\|_2.$$

As in the case of variable exponent Lebesgue spaces, we have some relation between the norms (2.26), (2.27), and the modular (2.25); namely,

$$\|f\|_j^\theta \leqslant I^{p(\cdot),\lambda(\cdot)}(f) \leqslant \|f\|_j^\sigma, \quad j = 1, 2, \tag{2.28}$$

where θ and σ are defined in (2.13).

Let $p \in \mathscr{P}_\mu^{\log}(X)$. Then the above norms are also equivalent to the norm

$$\|f\|^*_{L^{p(\cdot),\lambda(\cdot)}(X)} = \sup_{x\in X, r>0} (\mu B(x,r))^{-\frac{\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot)}(B(x,r))}. \tag{2.29}$$

The following embedding was proved in [8, Lemma 7] in the case of Euclidean space endowed with Lebesgue measure and in [24, Proposition 1.3] within the framework of spaces of homogeneous type, but the analysis of the proof in [8] and [24] shows that this embedding holds under the assumptions of Lemma 2.6.

Lemma 2.6. *Suppose that (X, d, μ) is a quasimetric measure space with finite measure, $p, q \in \mathscr{P}_\mu^{\log}(X)$, $0 \leqslant \lambda(x) \leqslant 1$, and $0 \leqslant \nu(x) \leqslant 1$. Let also $1 \leqslant p(x) \leqslant q(x)$ and*

$$\frac{1-\lambda(x)}{p(x)} \geqslant \frac{1-\nu(x)}{q(x)}. \tag{2.30}$$

Then

$$L^{q(\cdot),\nu(\cdot)}(X) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(X). \tag{2.31}$$

In the theory of variable exponent Lebesgue spaces, the *Diening inequality*

$$\left(\fint_{B(x,r)} |f(y)|\, \mathrm{d}y\right)^{p(x)} \leqslant C\left(1 + \fint_{B(x,r)} |f(y)|^{p(y)}\, \mathrm{d}y\right) \tag{2.32}$$

plays a role of the Jensen integral inequality. It is valid whenever the exponent p satisfies the log-Hölder continuity condition (2.17) and

$$\int_{B(x,r)} |f(y)|^{p(y)}\, \mathrm{d}y \leqslant 1.$$

In the following lemma, we observe that this inequality holds in a more general form and within the framework of quasimetric measure spaces.

Lemma 2.7. *Suppose that (X,d,μ) is a quasimetric measure space with finite measure satisfying the condition* (2.6), $0 \leqslant \lambda(x) \leqslant 1$, *and* $p \in \mathscr{P}^{\log}_{\mu}(X)$ *satisfies* (2.8). *Then*

$$\Bigg(\fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y)\Bigg)^{p(z)} \leqslant C\Bigg(1+\fint_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}\mu(y)\Bigg) \tag{2.33}$$

for all $z \in B(x,r)$ provided that

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \leqslant 1.$$

Proof. The estimate (2.33) is obvious when $\mu B(x,r) \geqslant \delta > 0$. Therefore, we consider only those x and r for which $\mu B(x,r) \leqslant \delta < 1$, where δ will be chosen sufficiently small later. Let

$$p_r = p_r(x) = \inf_{t\in B(x,r)} p(t).$$

We apply the Hölder inequality with the exponent p_r and get

$$\Bigg(\fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y)\Bigg)^{p(z)} \leqslant \Bigg(\fint_{B(x,r)} |f(y)|^{p_r}\,\mathrm{d}\mu(y)\Bigg)^{\frac{p(z)}{p_r}}.$$

Hence

$$\Bigg(\fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y)\Bigg)^{p(z)} \leqslant \frac{1}{(\mu B(x,r))^{\frac{p(z)}{p_r}}}\Bigg(\mu B(x,r) + \int\limits_{\substack{y\in B(x,r)\\ |f(y)|\geqslant 1}} |f(y)|^{p_r}\,\mathrm{d}\mu(y)\Bigg)^{\frac{p(z)}{p_r}}.$$

Since $p_r \leqslant p(y)$ for all $y \in B(x,r)$ and p is a bounded function, we obtain

$$\Bigg(\fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y)\Bigg)^{p(z)} \leqslant C(\mu B(x,r))^{\frac{p(z)}{p_r}(\lambda(x)-1)}\Bigg(\frac{(\mu B(x,r))^{1-\lambda(x)}}{2} + \frac{1}{2(\mu B(x,r))^{\lambda(x)}}\int_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}\mu(y)\Bigg)^{\frac{p(z)}{p_r}}.$$

The expression in the parenthesis on the right-hand side is less than 1 for sufficiently small δ, which implies

$$\Bigg(\fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y)\Bigg)^{p(z)} \leqslant C\Bigg(1+\fint_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}\mu(y)\Bigg)(\mu B(x,r))^{(\lambda(x)-1)\left(\frac{p(z)}{p_r}-1\right)}.$$

It remains to show that

$$(\mu B(x,r))^{p_r-p(z)} \leqslant C < \infty,$$

which is easily obtained in a similar way as in the proof of Lemma 2.4, because there always exists a $\xi_r \in B(x,r)$ such that

$$|p(\xi_r) - p_r| < \frac{1}{|\ln \mu B(x,r)|}.$$

We then have

$$|p_r - p(z)||\ln \mu B(x,r)| \leqslant (|p_r - p(\xi_r)| + |p(\xi_r) - p(x)| + |p(x) - p(z)|)\,|\ln \mu B(x,r)| \leqslant C.$$

The proof is complete. □

2.4 Variable exponent Hölder spaces

We also deal with variable exponent Hölder spaces $H^{\alpha(\cdot)}(X)$ (for more details on $H^{\alpha(\cdot)}(X)$ in the Euclidean setting cf., for example, [25]–[29], and for Hölder spaces H^α within the framework of metric measure spaces (X, d, μ) with constant exponent α cf., for example, [30]–[33] and also [34] for the case where X is a compact manifold).

Let $\alpha(x)$ be a μ-measurable real-valued nonnegative function on X. We say that a bounded function f belongs to $H^{\alpha(\cdot)}(X)$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leqslant C \cdot d(x,y)^{\max\{\alpha(x),\alpha(y)\}}$$

for every $x, y \in X$. This space is a Banach space with respect to the norm

$$\|f\|_{H^{\alpha(\cdot)}(X)} = \|f\|_{L^\infty} + \lfloor f \rfloor_{\alpha(\cdot)},$$

where

$$\lfloor f \rfloor_{\alpha(\cdot)} := \sup_{x,y\in X} \frac{|f(x) - f(y)|}{d(x,y)^{\max\{\alpha(x),\alpha(y)\}}}.$$

3 Variable Exponent Campanato Spaces

3.1 On equivalent (semi)norms

Let λ be a μ-measurable function on X with the range in $[0,\infty)$. We define the *variable exponent Campanato space* $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ as the set of all integrable functions f on X such that

$$\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^{p(y)} \mathrm{d}\mu(y) < \infty, \tag{3.1}$$

where $f_{B(x,r)}$ is the integral average of a function f on $B(x,r)$.

We endow the variable Campanato space with the seminorms

$$_1\lfloor f \rfloor^{p(\cdot)}_{\lambda(\cdot)} := \inf\left\{\eta > 0 : \mathscr{I}^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \leqslant 1\right\} \tag{3.2}$$

and

$$ {}_2\lfloor f \rfloor_{\lambda(\cdot)}^{p(\cdot)} := \sup_{x\in X; r>0} \left\| (\mu B(x,r))^{-\frac{\lambda(x)}{p(\cdot)}} \left(f - f_{B(x,r)}\right) \chi_{B(x,r)} \right\|_{L^{p(\cdot)}(X)} . \tag{3.3} $$

When no confusion arise, we simply write $\lfloor f \rfloor_1$ and $\lfloor f \rfloor_2$ to avoid, as much as possible, cumbersome notation.

As in the case of variable Lebesgue space, we have some relation between the functional (3.1) and seminorms (3.2) and (3.3). We follow [8] in the proof of the following lemmas.

Lemma 3.1. *Let (X, d, μ) be a quasimetric measure space. For every function $f \in \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$*

$$ \lfloor f \rfloor_i^{\theta} \leqslant \mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) \leqslant \lfloor f \rfloor_i^{\sigma}, \quad i = 1, 2, \tag{3.4} $$

where

$$ \theta = \begin{cases} p_+ & \text{if } \lfloor f \rfloor_i \leqslant 1, \\ p_- & \text{if } \lfloor f \rfloor_i \geqslant 1, \end{cases} \qquad \text{and} \qquad \sigma = \begin{cases} p_- & \text{if } \lfloor f \rfloor_i \leqslant 1, \\ p_+ & \text{if } \lfloor f \rfloor_i \geqslant 1, \end{cases} $$

Proof. Let

$$ F_{x,r}(\eta) = \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} \left| \frac{f(y) - f_{B(x,r)}}{\eta} \right|^{p(y)} \mathrm{d}\mu(y). \tag{3.5} $$

We note that for fixed $(x, r) \in X \times (0, d_X)$ the function $F_{x,r}(\eta)$ is continuous and decreasing in $\eta \in (0, \infty)$. We have

$$ \sup_{x\in X, r>0} F_{x,r}(1) = \mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) . \tag{3.6} $$

By the definition of seminorm $\lfloor f \rfloor_1$,

$$ \sup_{x\in X, r>0} F_{x,r}\left(\lfloor f \rfloor_1\right) = 1. \tag{3.7} $$

The relation (3.4) with $i = 1$ follows from (3.6), (3.7), and the monotonicity of $F_{x,r}(\eta)$.

In the case $i = 2$, we define

$$ \phi_{x,r}(\cdot) := (\mu B(x,r))^{-\frac{\lambda(x)}{p(\cdot)}} \left(f(\cdot) - f_{\widetilde{B}(x,r)} \right) \chi_{\widetilde{B}(x,r)}(\cdot). $$

By (2.12) and (2.13), we have

$$ \|\phi_{x,r}(\cdot)\|_{p(\cdot)}^{p_-} \leqslant I^{p(\cdot)}(\phi_{x,r}(\cdot)) \leqslant \|\phi_{x,r}(\cdot)\|_{p(\cdot)}^{p_+} $$

if

$$ \|\phi_{x,r}(\cdot)\|_{p(\cdot)} \geqslant 1 $$

and similarly in the case $\|\phi_{x,r}(\cdot)\|_{p(\cdot)} \leqslant 1$. Taking the supremum with regard to x and r, we obtain the desired result. □

Lemma 3.2. *Let (X, d, μ) be a quasimetric measure space. Then for $f \in \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ we have*

$$ \lfloor f \rfloor_1 = \lfloor f \rfloor_2. $$

Proof. We have

$$\lfloor f\rfloor_2 = \sup_{x\in X; r>0} \{\mu_{x,r} > 0 : F_{x,r}(\mu_{x,r}) = 1\},$$

where $F_{x,r}(\cdot)$ is the function defined in (3.5). By $F_{x,r}(\mu_{x,r}) = 1$ and the inequality $F_{x,r}(\lfloor f\rfloor_1) \leqslant 1$ which follows from (3.7) we obtain

$$\lfloor f\rfloor_2 \leqslant \lfloor f\rfloor_1$$

in view of the monotonicity of $F_{x,r}(\eta)$ in η.

The other inequality follows since, by (3.4), we have

$$\lfloor f\rfloor_1 \leqslant \begin{cases} \lfloor f\rfloor_2^{\frac{p_-}{p_+}} & \text{if } \lfloor f\rfloor_1 \leqslant 1,\\ \lfloor f\rfloor_2 & \text{if } \lfloor f\rfloor_1 \geqslant 1, \lfloor f\rfloor_2 \leqslant 1,\\ \lfloor f\rfloor_2^{\frac{p_+}{p_-}} & \text{if } \lfloor f\rfloor_1 \geqslant 1, \lfloor f\rfloor_2 \geqslant 1. \end{cases}$$

Substituting f by $\dfrac{f}{\lfloor f\rfloor_2}$, we find $\left\lfloor \dfrac{f}{\lfloor f\rfloor_2}\right\rfloor_1 \leqslant 1$, which gives the opposite inequality. The lemma is proved. ☐

We now define the variable exponent Campanato norm.

Definition 3.3. The variable exponent Campanato space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ is endowed with the norm

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} := {}_1\lfloor f\rfloor^{p(\cdot)}_{\lambda(\cdot)} + \|f\|_{L^1(X)}. \tag{3.8}$$

Since $\lfloor\cdot\rfloor_1$ and $\lfloor\cdot\rfloor_2$ coincide, we can take either $\lfloor f\rfloor_2$ or $\lfloor f\rfloor_1$ in (3.8).

We can also introduce the Campanato seminorm in the form

$${}_*\lfloor f\rfloor^{p(\cdot)}_{\lambda(\cdot)} = \sup_{x\in X; r>0} (\mu B(x,r))^{-\frac{\lambda(x)}{p(x)}} \left\|f - f_{B(x,r)}\right\|_{L^{p(\cdot)}(B(x,r))}. \tag{3.9}$$

The seminorms ${}_*\lfloor f\rfloor$ and $\lfloor f\rfloor_2$ are equivalent for $p \in \mathscr{P}^{\log}_\mu(X)$. Correspondingly, we can also deal with the norm

$$\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} := {}_*\lfloor f\rfloor^{p(\cdot)}_{\lambda(\cdot)} + \|f\|_{L^1(X)}, \tag{3.10}$$

not distinguishing between (3.8) and (3.10) when $p \in \mathscr{P}^{\log}_\mu(X)$.

3.2 Imbedding theorem

Theorem 3.4. *Let (X, d, μ) be a quasimetric measure space with finite measure, and let λ and ν be nonnegative bounded functions. If $p, q \in \mathscr{P}^{\log}_\mu(X)$, $1 \leqslant p(x) \leqslant q(x) \leqslant q_+ < \infty$, and*

$$\frac{1-\lambda(x)}{p(x)} \geqslant \frac{1-\nu(x)}{q(x)}, \tag{3.11}$$

then

$$\mathscr{L}^{q(\cdot),\nu(\cdot)}(X) \hookrightarrow \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X). \tag{3.12}$$

Proof. We only need to show that $\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) \leqslant C$ with some $C > 0$ independent of f whenever $\mathscr{I}^{q(\cdot),\nu(\cdot)}(f) \leqslant 1$, or equivalently ${}_1\lfloor f \rfloor^{q(\cdot)}_{\nu(\cdot)} \leqslant 1$.

By the Hölder inequality with exponent $p_1(\cdot) = q(\cdot)/p(\cdot)$,

$$\int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|^{p(y)}\, \mathrm{d}\mu(y) \leqslant C\|\chi_{B(x,r)}\|_{p_1'(\cdot)} \underbrace{\|(f(\cdot) - f_{B(x,r)})^{p(\cdot)}\chi_{B(x,r)}\|_{p_1(\cdot)}}_{\Psi_{x,r}(f)}. \tag{3.13}$$

By Lemma 2.5,

$$\|\chi_{B(x,r)}\|_{p_1'(\cdot)} \leqslant C(\mu B(x,r))^{\left(1-\frac{p(x)}{q(x)}\right)}. \tag{3.14}$$

For the other norm, we have the estimate

$$\begin{aligned} \Psi_{x,r}(f) &= \inf\left\{\eta > 0 : \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|^{q(y)}\eta^{-\frac{q(y)}{p(y)}}\, \mathrm{d}\mu(y) \leqslant 1\right\} \\ &\leqslant A^{p_+}(\mu B(x,r))^{\nu(x)\frac{p(x)}{q(x)}}, \end{aligned} \tag{3.15}$$

where $A \geqslant 1$ is the constant (independent of x, y, r) from the inequalities

$$A^{-1}(\mu B(x,r))^{\frac{\nu(x)}{q(y)}} \leqslant (\mu B(x,r))^{\frac{\nu(x)p(x)}{q(x)p(y)}} \leqslant A(\mu B(x,r))^{\frac{\nu(x)}{q(y)}} \tag{3.16}$$

obtained in a similar way as (2.21).

Using (3.14) and (3.15) in (3.13), we obtain

$$(\mu B(x,r))^{-\lambda(x)} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|^{p(y)}\, \mathrm{d}\mu(y) \leqslant C(\mu B(x,r))^{\frac{p(x)}{q(x)}(\nu(x)-1)+1-\lambda(x)}$$

which implies (3.12) in view of the inequality (3.11). □

3.3 Coincidence of variable exponent Campanato spaces with variable exponent Morrey spaces in the case $\lambda_+ < 1$

We start with the following lemma. Recall that the modulars $\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f)$ and $I^{p(\cdot),\lambda(\cdot)}(f)$ were introduced in (3.1) and (2.25).

Lemma 3.5. *Suppose that (X, d, μ) is a quasimetric measure space with finite measure, $p \in \mathscr{P}^{\log}_\mu(X)$ satisfies (2.8), and $0 \leqslant \lambda(x) \leqslant 1$. For $f \in L^{p(\cdot),\lambda(\cdot)}(X)$ such that $I^{p(\cdot),\lambda(\cdot)}(f) \leqslant 1$*

$$\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) \leqslant C\big[I^{p(\cdot),\lambda(\cdot)}(f) + \sup_{x\in X, r>0}(\mu B(x,r))^{1-\lambda(x)}\big], \tag{3.17}$$

where C is independent of f and x.

Proof. We have

$$\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) = \sup_{x\in X, r>0} \frac{1}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} \big| f(y) - f_{B(x,r)}\big|^{p(y)}\, \mathrm{d}\mu(y).$$

By the inequality

$$(a+b)^p \leqslant 2^{p-1}(a^p+b^p),$$

we have

$$\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) \leqslant \sup_{x\in X, r>0} \frac{2^{p_+-1}}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} \left\{ |f(y)|^{p(y)} + \left| \fint\limits_{B(x,r)} |f(\xi)|\,\mathrm{d}\mu(\xi) \right|^{p(y)} \right\} \mathrm{d}\mu(y).$$

In the second term, we use the Diening inequality (2.33), which is possible since

$$I^{p(\cdot),\lambda(\cdot)}(f) \leqslant 1$$

yields

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \leqslant 1.$$

As a result, we obtain

$$\mathscr{I}^{p(\cdot),\lambda(\cdot)}(f) \leqslant C \sup_{x\in X, r>0} \left(\frac{1}{(\mu B(x,r))^{\lambda(x)}} \int\limits_{B(x,r)} |f(y)|^{p(y)}\,\mathrm{d}\mu(y) + (\mu B(x,r))^{1-\lambda(x)} \right).$$

Hence (3.17) follows. □

Corollary 3.6. *Suppose that* (X,d,μ) *is a quasimetric measure space with finite measure,* $p \in \mathscr{P}^{\log}_\mu(X)$ *satisfies* (2.8), *and* $0 \leqslant \lambda(x) \leqslant 1$. *Then*

$$L^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X). \tag{3.18}$$

Proof. It suffices to show that

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} := \|f\|_{L^1} + \lfloor f \rfloor_1 \leqslant C$$

for all f such that $\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \leqslant 1$. By (2.31), the inequality $\|f\|_{L^1} \leqslant C\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)}$ follows from (2.31) and the estimate of the seminorm $\lfloor f \rfloor_1$ follows from (3.17). □

To prove the other embedding, we need some auxiliary assertions.

Lemma 3.7. *Let* (X,d,μ) *be a quasimetric measure space with finite measure. Then there exists a constant* C *such that*

$$|f_{B(x,\rho)} - f_{B(x,\sigma)}| \leqslant C \left(\frac{(\mu B(x,\rho))^{\lambda(x)} + (\mu B(x,\sigma))^{\lambda(x)}}{\mu B(x,\sigma)} \right)^{\frac{1}{p(x)}} \|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} \tag{3.19}$$

for all $x \in X$ *and* $0 < \sigma < \rho < d_X$.

Proof. By Lemma 2.5,

$$\|f_{B(x,\rho)} - f_{B(x,\sigma)}\|_{L^{p(\cdot)}(B(x,\sigma))} = \left|f_{B(x,\rho)} - f_{B(x,\sigma)}\right| \|\chi_{B(x,\sigma)}\|_{L^{p(\cdot)}(X)}$$
$$\geqslant c\left|f_{B(x,\rho)} - f_{B(x,\sigma)}\right| (\mu B(x,\sigma))^{\frac{1}{p(x)}}.$$

Hence

$$\left|f_{B(x,\rho)} - f_{B(x,\sigma)}\right| \leqslant \frac{c}{(\mu B(x,\sigma))^{\frac{1}{p(x)}}} \|f_{B(x,\rho)} - f_{B(x,\sigma)}\|_{L^{p(\cdot)}(B(x,\sigma))}$$
$$\leqslant \frac{c}{(\mu B(x,\sigma))^{\frac{1}{p(x)}}} \Big(\|f - f_{B(x,\sigma)}\|_{L^{p(\cdot)}(B(x,\sigma))} + \|f - f_{B(x,\rho)}\|_{L^{p(\cdot)}(B(x,\rho))}\Big).$$

By (3.9), we arrive at (3.19). □

Lemma 3.8. *Let (X, d, μ) be a space of homogeneous type with finite measure, and let λ be a nonnegative real-valued function with $\lambda_+ < 1$. Then there exists a constant $C = C(p, \lambda, \mathscr{D})$ such that*

$$|f_{B(x,r)} - f_{B(x,r/2^m)}| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \sum_{k=0}^{m-1} \mathscr{D}^{kN\frac{1-\lambda(x)}{p(x)}} \tag{3.20}$$

for all $(x,r) \in X \times (0, d_X)$, where $\mathscr{D}$ is the constant from the doubling condition (2.4) *and $N = \log_2 \mathscr{D}$ is the exponent from* (2.5)*.*

Proof. By Lemma 3.7,

$$|f_{B(x,r/2^{k+1})} - f_{B(x,r/2^k)}| \leqslant C\Theta_{r,k,x,\lambda}^{\frac{1}{p(x)}} \cdot \|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)}, \tag{3.21}$$

where

$$\Theta_{r,k,x,\lambda} := \frac{(\mu B(x, r/2^{k+1}))^{\lambda(x)} + (\mu B(x, r/2^k))^{\lambda(x)}}{\mu B(x, r/2^{k+1})}. \tag{3.22}$$

This expression may be estimated as follows:

$$\Theta_{r,k,x,\lambda}^{\frac{1}{p(x)}} \leqslant (\mu B(x, r/2^{k+1}))^{\frac{\lambda(x)-1}{p(x)}} \left(1 + \mathscr{D}^{\lambda(x)}\right)^{\frac{1}{p(x)}}$$
$$\leqslant 2^{\frac{1}{p_-}} \mathscr{D}^{\frac{\lambda(x)}{p(x)}} \mathscr{D}^{\frac{1-\lambda(x)}{p(x)}} 2^{kN\frac{1-\lambda(x)}{p(x)}} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}}$$
$$\leqslant c \cdot \mathscr{D}^{kN\frac{1-\lambda(x)}{p(x)}} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \tag{3.23}$$

where the first inequality comes from the doubling condition (2.4) and the second one is obtained from (2.5) with $N = \log_2 \mathscr{D}$.

Then from (3.21) and (3.23), for $k = 1, 2, \dots$ we obtain

$$|f_{B(x,r/2^{k+1})} - f_{B(x,r/2^k)}| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} \mathscr{D}^{kN\frac{1-\lambda(x)}{p(x)}} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \tag{3.24}$$

where C is independent of k. Summing over $k = 0, 1, \dots, m-1$, we obtain (3.20). □

Lemma 3.9. *Suppose that (X,d,μ) is a space of homogeneous type with finite measure, the condition (2.6) is satisfied, and λ is a nonnegative real-valued function with $\lambda_+ < 1$. Then there exists a constant $C = C(\mathscr{D},p,\lambda) > 0$ such that for any $f \in \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ and all $(x,\rho) \in X \times (0,d_X)$*

$$|f_{B(x,\rho)}| \leqslant |f_X| + C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)}(\mu B(x,\rho))^{\lambda(x)-1/p(x)}. \tag{3.25}$$

Proof. Fix $f \in \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ and $\rho \in (0,d_X)$. For a constant $a \geqslant 2$, which will be determined later, we choose $m \in \mathbb{N}_0$ such that

$$\frac{d_X}{a^{m+1}} \leqslant \rho < \frac{d_X}{a^m}.$$

We have

$$|f_{B(x,\rho)}| \leqslant |f_X| + |f_X - f_{B(x,d_X/a^m)}| + |f_{B(x,d_X/a^m)} - f_{B(x,\rho)}|.$$

By Lemma 3.7 applied to $|f_{B(x,d_X/a^m)} - f_{B(x,\rho)}|$, we obtain

$$|f_{B(x,d_X/a^m)} - f_{B(x,\rho)}| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)}(\mu B(x,\rho))^{\frac{\lambda(x)-1}{p(x)}}\left(1+\left(\frac{\mu B(x,d_X/a^m)}{\mu B(x,\rho)}\right)^{\lambda(x)}\right)^{\frac{1}{p(x)}} \tag{3.26}$$

and the quotient of the measures in (3.26) is uniformly bounded by the doubling condition (2.5) and the relation between ρ and d_X/a^m.

Since $f_X = f_{B(x,d_X)}$ for all $x \in X$, we can apply Lemma 3.8 to $|f_{B(x,d_X/a^m)} - f_X|$, which yields

$$|f_{B(x,d_X/a^m)} - f_X| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)}(\mu B(x,d_X))^{\frac{\lambda(x)-1}{p(x)}} \underbrace{\frac{\mathscr{D}^{mN\frac{1-\lambda(x)}{p(x)}}-1}{\mathscr{D}^{N\frac{1-\lambda(x)}{p(x)}}-1}}_{\sigma}$$

$$\leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)}(a^{m\gamma}\mu B(x,\rho))^{\frac{\lambda(x)-1}{p(x)}} \cdot \sigma$$

where the second inequality comes from the reverse doubling condition (2.7). Taking $a^\gamma \geqslant \mathscr{D}^N$, we see that $\sigma a^{m\gamma\frac{\lambda(x)-1}{p(x)}}$ is bounded, which gives (3.25). ☐

Lemma 3.10. *Let (X,d,μ) be a space of homogeneous type with finite measure. For all $f \in \mathscr{L}^{1,\lambda(\cdot)}(X)$ and all $x,y \in \overline{X}$ there exists a constant $C = C(\mathscr{D},\lambda)$ such that for $r = 2\mathscr{Q}d(x,y)$ (where $\mathscr{Q}$ is the constant from (2.1))*

$$|f_{B(x,r)} - f_{B(y,r)}| \leqslant C\|f\|^*_{\mathscr{L}^{1,\lambda(\cdot)}(X)}\big[(\mu B(x,r))^{\lambda(x)-1} + (\mu B(y,r))^{\lambda(y)-1}\big].$$

Proof. Let $\Upsilon_r := B(x,r) \cap B(y,r)$. We have

$$B\left(x,\frac{r}{2\mathscr{Q}}\right) \subset \Upsilon_r \subset B(x,r) \tag{3.27}$$

which implies

$$\mu(\Upsilon_r) \geqslant \mu\Big(B\Big(x,\frac{r}{2\mathscr{Q}}\Big)\Big) \geqslant C\mu(B(x,r)),$$

where the last inequality comes from the doubling condition (2.5). It is obvious that

$$\mu(\Upsilon_r) \geqslant C\mu(B(y,r)).$$

For $t \in \Upsilon_r$ we have

$$|f_{B(x,r)} - f_{B(y,r)}| \leqslant |f_{B(x,r)} - f(t)| + |f(t) - f_{B(y,r)}|. \tag{3.28}$$

Integrating (3.28) with respect to the variable t over Υ_r, we obtain

$$|f_{B(x,r)} - f_{B(y,r)}|\mu(\Upsilon_r) \leqslant \|f\|^*_{\mathscr{L}^{1,\lambda(\cdot)}(X)}\big[(\mu B(x,r))^{\lambda(x)} + (\mu B(y,r))^{\lambda(y)}\big]$$

which implies the required assertion. □

We are now in position to prove the coincidence of the spaces $L^{p(\cdot),\lambda(\cdot)}(X)$ and $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$, up to the equivalence of norms, in the case where $\lambda(x)$ does not attain the value 1.

Theorem 3.11. *Suppose that (X,d,μ) is a space of homogeneous type with finite measure, $p \in \mathscr{P}^{\log}_\mu(X)$ satisfies (2.8), and λ is a nonnegative real-valued function with $\lambda_+ < 1$. Then*

$$\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X) \cong L^{p(\cdot),\lambda(\cdot)}(X).$$

Proof. We need to prove that

$$c_1\|f\|^*_{L^{p(\cdot),\lambda(\cdot)}(X)} \leqslant \|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} \leqslant c_2\|f\|^*_{L^{p(\cdot),\lambda(\cdot)}(X)}. \tag{3.29}$$

In view of Corollary 3.6, it suffices to prove the left inequality in (3.29). By the definition of the norm in (2.29), we need to estimate $\|f\|_{L^{p(\cdot)}(B(x,r))}$. We have

$$\|f\|_{L^{p(\cdot)}(B(x,r))} \leqslant \|f - f_{B(x,r)}\|_{L^{p(\cdot)}(B(x,r))} + |f_{B(x,r)}| \cdot \|\chi_{B(x,r)}\|_{p(\cdot)}.$$

Using the estimate (2.23) and the inequality (3.25) in the last term, we obtain

$$\begin{aligned}&(\mu B(x,r))^{-\frac{\lambda(x)}{p(x)}}\|f\|_{L^{p(\cdot)}(B(x,r))}\\ &\qquad\leqslant \|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} + C(\mu B(x,r))^{\frac{1-\lambda(x)}{p(x)}}\Big(|f_X| + (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}}\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)}\Big)\end{aligned}$$

which implies the left inequality in (3.29). □

3.4 Coincidence of variable exponent Campanato space with variable exponent Hölder spaces in the case $\lambda_- > 1$

Note that the assumption $\lambda_- > 1$ in Theorems 3.14– 3.16 below yields $\inf\limits_{x\in X}\alpha(x) > 0$ for the resulting exponent $\alpha(x)$.

We need the following two lemmas. The first lemma is a counterpart to Lemma 3.8.

Lemma 3.12. *Let (X,d,μ) be a quasimetric measure space satisfying the reverse doubling condition* (2.7) (*by Lemma 2.1, it holds if (X,d,μ) is homogenous, X has finite measure and* *the condition (2.6) is fulfilled), and let λ be a bounded real-valued function with $\lambda_- > 1$. Then there exists a constant $C = C(p, \lambda, \mathscr{D})$ such that*

$$|f_{B(x,r)} - f_{B(x,r/2^m)}| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \sum_{k=0}^{m-1} 2^{k\gamma \frac{1-\lambda(x)}{p(x)}} \tag{3.30}$$

for all $(x,r) \in X \times (0, d_X)$, where γ is the exponent from the reverse doubling condition (2.7).

Proof. Let $\Theta_{r,k,x,\lambda}$ be as in (3.22). We have

$$\Theta_{r,k,x,\lambda}^{\frac{1}{p(x)}} \leqslant \big(\mu B(x, r/2^{k+1})\big)^{\frac{\lambda(x)-1}{p(x)}} \big(1 + \mathscr{D}^{\lambda(x)}\big)^{\frac{1}{p(x)}} \leqslant C 2^{k\gamma \frac{1-\lambda(x)}{p(x)}} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \tag{3.31}$$

where the second inequality comes from the reverse doubling condition (2.7). By (3.21) and (3.31), for $k = 1, 2, \dots$ we get

$$|f_{B(x,r/2^{k+1})} - f_{B(x,r/2^k)}| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} 2^{k\gamma \frac{1-\lambda(x)}{p(x)}} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}} \tag{3.32}$$

where C is independent of k. Summing over $k = 0, 1, \dots, m-1$, we obtain (3.30). $\square$

In the following lemma, we use the notation

$$S(\lambda) := \{x \in X : \lambda(x) = 1\},$$
$$X_\delta(\lambda) = \{x \in X : \lambda(x) \geqslant 1 + \delta\}, \quad \delta > 0.$$

Lemma 3.13. *Let (X, d, μ) be a quasimetric measure space satisfying the reverse doubling condition (2.7), and let λ be a real-valued bounded function with values in $[1, \infty)$. If $\mu S(\lambda) = 0$, then for every $f \in \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ there exists a function $\widetilde{f}$ defined on $\overline{X}$ such that f is equal to $\widetilde{f}$ almost everywhere on X and*

$$\lim_{r \to 0+} f_{B(x,r)} = \widetilde{f}(x)$$

for all $x \in \overline{X} \backslash S(\lambda)$, where the convergence is uniform on every bounded subset of $X_\delta(\lambda)$ for every fixed $\delta > 0$.

Proof. The fact that

$$\lim_{r \to 0+} f_{B(x,r)} = f(x)$$

almost everywhere in X, is the content of the well-known *Lebesgue differentiation theorem* (cf. [17, p. 17]).

Let us now prove the uniform convergence of $f_{B(x,r)}$ with respect to $x \in X_\delta$. We fix $r \in (0, d_X)$. By Lemma 3.12 and the reverse doubling condition (2.7), for $x \notin S(\lambda)$ we have

$$\begin{aligned} |f_{B(x,r/2^m)} - f_{B(x,r/2^{m+q})}| &\leqslant \frac{C}{1 - 2^{\gamma \frac{1-\lambda(x)}{p(x)}}} \|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} (\mu B(x, r/2^m))^{\frac{\lambda(x)-1}{p(x)}} \\ &\leqslant \frac{C}{\lambda(x) - 1} \|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} \underbrace{2^{m\gamma\left(\frac{1-\lambda(x)}{p(x)}\right)} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}}}_{I_1} \end{aligned} \tag{3.33}$$

where the constant C is independent of x, m, q. From (3.33) we see that the sequence $\{f_{B(x,r/2^m)}\}_{m=1}^{\infty}$ is a Cauchy sequence uniformly in x on every set $\overline{X_\delta}$. Let

$$\widetilde{f}(x) := \lim_{m\to\infty} f_{B(x,r/2^m)}, \quad x \in \overline{X}.$$

The function $\widetilde{f}$ is well defined in the sense that it is independent of r since for any $0 < s \neq r < d_X$

$$\begin{aligned}|f_{B(x,s/2^m)} - \widetilde{f}(x)| &\leqslant |f_{B(x,r/2^m)} - \widetilde{f}(x)| + |f_{B(x,r/2^m)} - f_{B(x,s/2^m)}|\\ &\leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} I_1 + |f_{B(x,r/2^m)} - f_{B(x,s/2^m)}|\\ &\leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} (I_1 + I_2),\end{aligned} \tag{3.34}$$

where I_1 comes from (3.33) and I_2 is given by

$$\begin{aligned}I_2 \leqslant{}& C(\mu B(x,r/2^m))^{\frac{\lambda(x)-1}{p(x)}} \left(\frac{\mu B(x,r/2^m)}{\min\{\mu B(x,r/2^m), \mu B(x,s/2^m)\}}\right)^{\frac{1}{p(x)}}\\ &+ C(\mu B(x,s/2^m))^{\frac{\lambda(x)-1}{p(x)}} \left(\frac{\mu B(x,s/2^m)}{\min\{\mu B(x,r/2^m), \mu B(x,s/2^m)\}}\right)^{\frac{1}{p(x)}}\end{aligned}$$

due to Lemma 3.7 and the monotonicity of measure.

Since I_2 tends to 0 as $m \to \infty$ because of the reverse doubling condition (2.7), (3.34) tends to 0 as $m \to \infty$ uniformly in $x \in \overline{X_\delta}$. Therefore, $\widetilde{f}$ is the uniform limit of any sequence of type $\{f_{B(x,s/2^m)}\}_{m=1}^{\infty}$, uniform in $x \in \overline{X_\delta}$, where s is an arbitrary real number in $(0, d_X)$.

Letting $m \to \infty$ in (3.30), we obtain

$$|f_{B(x,r)} - \widetilde{f}(x)| \leqslant C\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} (\mu B(x,r))^{\frac{\lambda(x)-1}{p(x)}}, \quad x \in \overline{X_\delta} \tag{3.35}$$

which uniformly tends to zero as $r \to 0$ by the reverse doubling condition. $\square$

In the following assertions, we prove the equivalence between variable exponent Campanato spaces and variable exponent Hölder spaces whenever the exponents p and λ are log-Hölder continuous.

Theorem 3.14. *Suppose that (X, d, μ) is a space of homogeneous type with finite measure, $p \in \mathscr{P}^{\log}(X)$, and λ is a bounded real-valued function with $\lambda_- > 1$. Then*

$$H^{\alpha(\cdot)}(X) \hookrightarrow \mathscr{L}^{p(\cdot),\lambda(\cdot)}(X) \tag{3.36}$$

if α satisfies the log-*Hölder continuity condition* (2.17) *and* $\alpha(x) \geqslant N\dfrac{\lambda(x)-1}{p(x)}$.

Proof. Let $f \in H^{\alpha}(\cdot)(X)$ be such that $\|f\|_{H^{\alpha}(\cdot)}(X) \leqslant 1$. We only need to prove that

$$\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)} \leqslant C. \tag{3.37}$$

It suffices to check that $\lfloor f \rfloor^* \leqslant C$ since

$$\|f\|_{L^1(X)} \leqslant \mu X \|f\|_{L^\infty(X)} \leqslant C.$$

Since the case $r \geqslant 1/2$ is obvious, we only consider the case $r < 1/2$. We have

$$\|f - f_{B(x,r)}\|_{L^{p(\cdot)}(B(x,r))} \leqslant \Big\| \fint_{B(x,r)} |f(\cdot) - f(t)| \,\mathrm{d}\mu(t) \Big\|_{L^{p(\cdot)}(B(x,r))}$$

$$\leqslant \Big\| \fint_{B(x,r)} d(\cdot, t)^{\max\{\alpha(\cdot),\alpha(t)\}} \lfloor f \rfloor_{\alpha(\cdot)} \,\mathrm{d}\mu(t) \Big\|_{L^{p(\cdot)}(B(x,r))}. \qquad (3.38)$$

Since

$$\|f\|_{H^{\alpha(\cdot)}(X)} \leqslant 1,$$

from the log-Hölder continuity of α, Lemma 2.5, and the lower Ahlfors condition we get

$$\|f - f_{B(x,r)}\|_{L^{p(\cdot)}(B(x,r))} \leqslant Cr^{\alpha(x)} \|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)}$$

$$\leqslant Cr^{\alpha(x)} (\mu B(x,r))^{\frac{1}{p(x)}} \leqslant C(\mu B(x,r))^{\frac{\alpha(x)}{N} + \frac{1}{p(x)}}, \qquad (3.39)$$

which proves (3.37). □

Theorem 3.15. *Suppose that (X, d, μ) is a space of homogeneous type with finite measure, the condition* (2.6) *is satisfied, $p \in \mathscr{P}^{\log}(X)$ satisfies* (2.8)*, and λ is a bounded real-valued function with $\lambda_- > 1$. Then*

$$\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow H^{\alpha(\cdot)}(X) \qquad (3.40)$$

if α satisfies the log-*Hölder continuity condition* (2.17) *and* $\alpha(x) \leqslant \gamma \dfrac{\lambda(x) - 1}{p(x)}$.

Proof. It suffices to prove that

$$\mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X) \hookrightarrow H^{\alpha(\cdot)}(X) \qquad (3.41)$$

since

$$\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow \mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X)$$

by the embedding (3.12).

Let $f \in \mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X)$. To prove the estimate

$$\lfloor f \rfloor_{\alpha(\cdot)} \leqslant C\|f\|^*_{\mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X)}, \qquad (3.42)$$

we proceed as follows. By Lemma 3.13, when $1 < \lambda_-$, we can take $\widetilde{f}$ instead of f. Assuming that $d(x, y)$ is small and $r = 2\mathscr{Q}d(x, y)$, we then have

$$|\widetilde{f}(x) - \widetilde{f}(y)| \leqslant |\widetilde{f}(x) - f_{B(x,r)}| + |f_{B(x,r)} - f_{B(y,r)}| + |\widetilde{f}(y) - f_{B(y,r)}|$$

$$\leqslant C\|f\|^*_{\mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X)} \Big[(\mu B(x,r))^{\frac{\alpha(x)}{\gamma}} + (\mu B(y,r))^{\frac{\alpha(y)}{\gamma}}\Big]$$

$$\leqslant C\|f\|^*_{\mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X)} \big(r^{\alpha(x)} + r^{\alpha(y)}\big)$$

$$\leqslant C\|f\|^*_{\mathscr{L}^{1, \frac{\alpha(\cdot)}{\gamma} + 1}(X)} r^{\max\{\alpha(x),\alpha(y)\}},$$

where the second inequality comes from (3.35) and Lemma 3.10, the third inequality comes from the fact that μ is upper Ahlfors γ-regular, and the last inequality is obtained from the fact that α is log-Hölder continuous.

To estimate the essential supremum norm on X, we observe that for any fixed $0 < r \leqslant d_X$

$$\|f\|_{L^\infty(B(x,r))} \leqslant \|f - f_{B(x,r)}\|_{L^\infty(B(x,r))} + \|f_{B(x,r)}\|_{L^\infty(B(x,r))}. \tag{3.43}$$

The estimate

$$\|f_{B(x,r)}\|_{L^\infty(B(x,r))} \leqslant \|f\|_{L^1(X)}(\mu B(x,r))^{-1}$$

is obvious. For the other norm we have

$$\begin{aligned}\left\|f - f_{B(x,r)}\right\|_{L^\infty(B(x,r))} &\leqslant \left\| \fint_{B(x,r)} |f(\cdot) - f(t)| \,\mathrm{d}\mu(t) \right\|_{L^\infty(B(x,r))} \\ &\leqslant \left\| \fint_{B(x,r)} d(\cdot, t)^{\max\{\alpha(\cdot),\alpha(t)\}} \lfloor f \rfloor_{\alpha(\cdot)} \,\mathrm{d}\mu(t) \right\|_{L^\infty(B(x,r))} \leqslant C \lfloor f \rfloor_{\alpha(\cdot)},\end{aligned}$$

where C is independent of r, but depends only on d_X and α_+. Collecting (3.43) and the estimates obtained for $\|\cdot\|_{L^\infty}$, and taking $r = d_X$, we find

$$\|f\|_{L^\infty(X)} \leqslant C \left(\|f\|_{L^1(X)} + \lfloor f \rfloor_{\alpha(\cdot)}\right)$$

which, together with (3.42), gives us (3.41). □

Theorem 3.16. *Suppose that (X, d, μ) is a space of homogeneous type with finite measure, $p \in \mathscr{P}^{\log}(X)$, λ is a real-valued function with $\lambda_- > 1$, μ is Ahlfors Q-regular, and λ satisfies the* log*-Hölder continuity condition* (2.17)*. Then*

$$\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X) \cong H^{\alpha(\cdot)}(X), \tag{3.44}$$

where $\alpha(x) = Q\dfrac{\lambda(x) - 1}{p(x)}$.

Proof. The assertion follows from the embeddings obtained in Theorems 3.14 and 3.15. □

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