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# ON A VARIABLE EXPONENT MODULAR HARDY-TYPE INEQUALITY

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Abstract. We show that a certain modular variable exponent Hardy-type inequality (of order $\alpha = 1$) in $\mathbb{R}^n$ with precise constant may be derived from the divergence theorem.

**რეზიუმე.** ნაჩვენებია, რომ ე. წ. განშლადობის თეორემიდან შეიძლება გამოყვანილი იქნას $n$-განზომილებიან სივრცეში მოდულარული $\alpha = 1$ რიგის ცვლადმაჩვენებლიანი ჰარდის ტიპის უტოლობა ზუსტი მუდმივით.

## 1. Introduction

In this short note we make use of the trick suggested in the paper [3] in the case of constant exponents $p$, to derive a certain modular Hardy-type inequality in $\mathbb{R}^n$ with variable exponent $p(x)$. Norm Hardy inequalities of such a type with variable exponents were studied in [2]. For the variable exponent analysis we refer to the book [1] and references therein.

In [4] it was shown that the modular inequalities with variable exponents cannot be valid for integral operators with a wide variety of kernels (called *proper* kernels in [4]). In particular, the kernels $|x-y|^{\alpha-n}$ are proper, so that we cannot have the variable modular Hardy inequality with this kernel. In the theorem below, in the case $\alpha = 1$ we show that a kind of a substitution for the variable exponent modular Hardy inequality is possible, see (2.2). In the case of constant exponent, the usual Hardy inequality immediately follows from (2.2) by the application of the Holder inequality.

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## 2. Modular Hardy-Type Inequality

**Theorem 2.1.** *Let $p \in C^1(\mathbb{R}^n \backslash \{0\})$ be a function with values in $(0, \infty)$. Then the following modular inequality with precise constant holds*

$$\left| \int\limits_{\mathbb{R}^n} \left( \frac{|u|}{|x|} \right)^{p(x)} \left[ (x \cdot \nabla p) \ln \frac{|u|}{|x|} + n - p(x) \right] dx \right| \leq$$

$$\leq \int\limits_{\mathbb{R}^n} p(x) \left( \frac{|u|}{|x|} \right)^{p(x)-1} |\nabla u| \, dx, \tag{2.1}$$

*where $u \in C_0^1(\mathbb{R}^n)$. In the case $p$ is homogeneous of degree $0$, $p = p\left(\frac{x}{|x|}\right)$, this takes the form*

$$\left| \int\limits_{\mathbb{R}^n} (n - p(x)) \left( \frac{|u|}{|x|} \right)^{p(x)} dx \right| \leq \int\limits_{\mathbb{R}^n} p(x) \left( \frac{|u|}{|x|} \right)^{p(x)-1} |\nabla u| \, dx. \tag{2.2}$$

*Proof.* In fact, the following identity

$$\int\limits_{\mathbb{R}^n} \left( \frac{|u|}{|x|} \right)^{p(x)} \left[ (x \cdot \nabla p) \ln \frac{|u|}{|x|} + n - p(x) \right] dx =$$

$$= - \int\limits_{\mathbb{R}^n} \frac{p(x)|u|^{p(x)-2}u}{|x|^{p(x)}} (x \cdot \nabla u) \, dx \tag{2.3}$$

holds from which (2.1) immediately follows. We make use of the trick from [3], where the Hardy inequality with constant exponent was derived from the divergence theorem

$$\int\limits_{\mathbb{R}^n} \operatorname{div} \overrightarrow{f} \, dx = 0$$

under the appropriate choice of the vector field $\overrightarrow{f}$. When $\overrightarrow{f} = a(x)\overrightarrow{g}$, where $a(x)$ is a scalar function, via the formula $\operatorname{div}(a\overrightarrow{g}) = \nabla a \cdot \overrightarrow{g} + a \operatorname{div} \overrightarrow{g}$ in the case $\overrightarrow{g} = x$ we have

$$\int\limits_{\mathbb{R}^n} x \cdot \nabla a \, dx = -n \int\limits_{\mathbb{R}^n} a \, dx. \tag{2.4}$$

We choose now $a(x) = \frac{|u|^{p(x)}}{\varepsilon + |x|^{p(x)}}$, $\varepsilon > 0$. Direct calculations yield

$$\nabla a = \frac{p(x)|u|^{p(x)-2}u\nabla u + |u|^{p(x)} \ln |u| \nabla p}{\varepsilon + |x|^{p(x)}} -$$

$$-|u|^{p(x)} \frac{p(x)|x|^{p(x)-2}x + |x|^{p(x)} \ln |x| \nabla p}{(\varepsilon + |x|^{p(x)})^2}$$

so that

$$x \cdot \nabla a = \frac{p(x)|u|^{p(x)-2}u\,(x \cdot \nabla u) + |u|^{p(x)} \ln |u|(x \cdot \nabla p)}{\varepsilon + |x|^{p(x)}} - \\ -|u|^{p(x)} \frac{p(x)|x|^{p(x)} + |x|^{p(x)} \ln |x|(x \cdot \nabla p)}{(\varepsilon + |x|^{p(x)})^2}$$

and (2.4) turns into

$$\int_{\mathbb{R}^n} \frac{|u|^{p(x)}}{\varepsilon + |x|^{p(x)}} \left[ (x \cdot \nabla p) \ln |u| + |x|^{p(x)} \frac{n\varepsilon + n - p(x) - \ln |x|(x \cdot \nabla p)}{\varepsilon + |x|^{p(x)}} \right] dx = \\ = -\int_{\mathbb{R}^n} \frac{p(x)|u|^{p(x)-2}u}{\varepsilon + |x|^{p(x)}} (x \cdot \nabla u)\, dx.$$

Passing to the limit as $\varepsilon \to 0$, we arrive at (2).

In the case $p$ is homogeneous of degree 0, it suffices to note that $(x \cdot \nabla p) = 0$. □

*Remark* 2.2. In the case of constant $p \in (1, n)$, the classical Hardy norm inequality

$$\left\| \frac{u}{|x|} \right\|_p \leq \frac{p}{n-p} \|\nabla u\|_p \tag{2.5}$$

with the best constant $\frac{p}{n-p}$ is immediately derived from (2.2) by the application of the Hölder inequality on the right-hand side of (2.2). If we proceed similarly in the case of variable homogeneous exponents, supposing that $1 \leq p(x) \leq p_+ =: \sup_{x \in \mathbb{R}^n} p(x)$, we obtain

$$\int_{\mathbb{R}^n} \left( \frac{|u|}{|x|} \right)^{p(x)} dx \cdot \left\| \left( \frac{|u|}{|x|} \right)^{p(x)-1} \right\|_q^{-1} \leq \frac{kp_+}{n - p_+} \|\nabla u\|_p, \quad k = \frac{1}{p_-} + \frac{1}{p'_-} \leq 2,$$

which does not provide an inequality of form (2.5) with variable exponent.

*Remark* 2.3. Note that (2.1) contains $\ln |u|$. Thus the admission of variable exponents (with variable radial part) under the choice of $p(x)$ such that $(x \cdot \nabla p(x)) \geq 0$, allows to have the above substitution (2.1) for the modular Hardy-type inequality with an additional growing factor $\ln |u|$ on the left-hand side.

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