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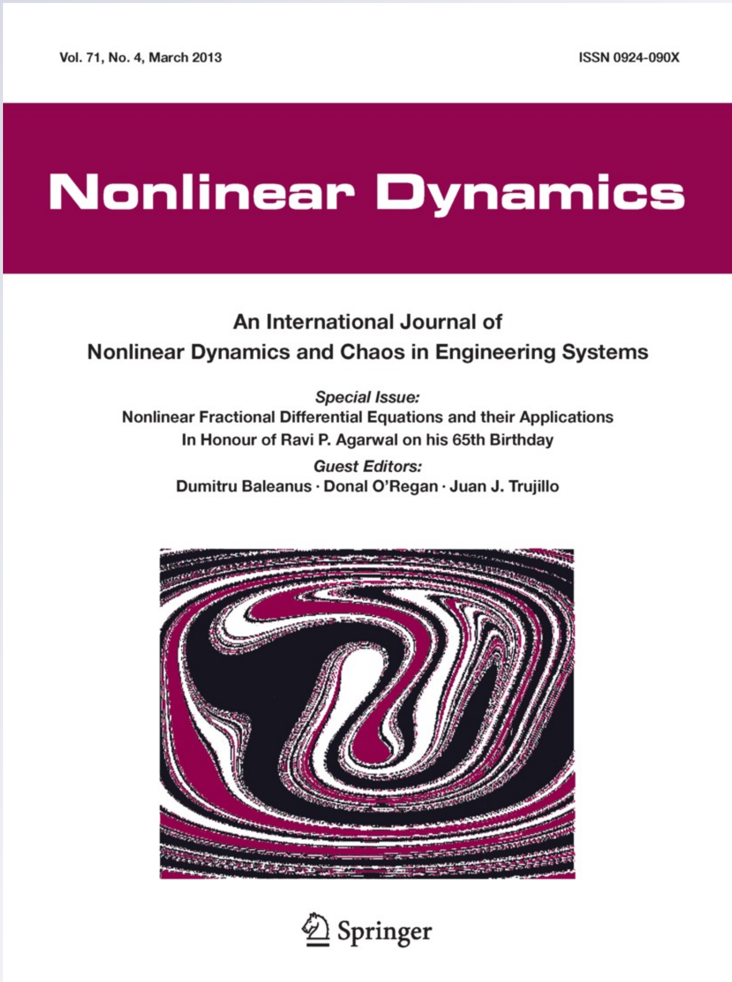
Nonlinear Dynamics
An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems

ISSN 0924-090X
Volume 71
Number 4

Nonlinear Dyn (2013) 71:653-662
DOI 10.1007/s11071-012-0485-0



Springer

Nonlinear Dyn (2013) 71:653–662
DOI 10.1007/s11071-012-0485-0

REVIEW

Fractional integration and differentiation of variable order: an overview

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Received: 30 March 2012 / Accepted: 25 May 2012 / Published online: 14 June 2012

Abstract We give an overview of a selection of studies on fractional operations of integration and differentiation of variable order, when this order may vary from point to point. We touch on both the Euclidean setting and also the general setting within the framework of quasimetric measure spaces.

Keywords Fractional integrals · Riesz potentials · Laplacians · Variable orders · Quasimetric measure spaces

Notation

$\mathbb{R}^n$ is the n-dimensional Euclidean space, $|x|=\sqrt{x_1^2+\cdots+x_n^2}$;
$\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ centered at the origin, $|\mathbb{S}^{n-1}|$ is its surface area;
Δ is the Laplace operator;
(X,ϱ,μ) denotes a quasimetric measure space with quasidistance ϱ and measure μ;
$\delta(x,\Omega)=\inf_{y\in\Omega}\varrho(x,y)$ is the distance of a point $x\in X$ to a set $\Omega\subset X$.

1 Introduction: recalling about fractional Laplacians of constant order

Last decade there was an increase of interest to studies of various operators and function spaces in the "variable exponent setting". The latter means that the parameters defining the operator and/or the space (which usually are constant), may vary from point to point.

In this overview we touch on several trends and results in this topic related to fractional Laplacian and other forms of fractional integro-differentiation. Fractional calculus, not a mainstream in mathematics till about the middle of the 20th century, nowadays is a very wide area with many applications, where fractional differential operations play an important role. We refer to [52] for a detailed historical account and books Diethelm [9], Hilfer (Ed.) [17], Kilbas, Srivastava, and Trujillo [20], Kiryakova [21], McBride [33], Miller and Ross [35], Oldham and Spanier [36], Podlubny [37], Rubin [41], Samko [50], Samko, Kilbas and Marichev [52] for fractional calculus and fractional differential equations in general.

This overview concerns studies rather in functional analysis and harmonic analysis than various applications of fractional operators. The overview in no way pretends to be complete.

We also refer to the paper [51], where the reader can find an overview more related to variable function spaces than to variable order operators. Note that nowadays there exists a vast field of research known as "variable exponent analysis", we refer to the book [6] and the surveying papers [7, 24, 28, 51]. We almost do not touch on the studies related to variability of the parameters of the space.

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We start below by recalling basic notions of Fractional Calculus of constant order.

1.1 The case of the whole space $\mathbb{R}^n$

As is well known, one may properly define fractional powers of various operators in these or other way, for instance, via spectral decompositions, or formulas of the type of Balakrishnan [3]. We refer to the books [32, 61] for such general functional analysis approaches. In applications the common preference is to deal with more direct constructions, in the intrinsic terms of the given setting. In the case where one works with fractional powers of an operator A considered on functions defined on the whole Euclidean space $\mathbb{R}^n, n \geq 1$, and the operator A is translation invariant, then a certain direct construction is suggested by the Fourier analysis approach. Indeed, any translation invariant operator is a convolution with some function a, in general a distribution (Hörmander's theorem). Then a natural approach is to define the fractional power $A^\alpha, \alpha \in \mathbb{R}^1$, via

$$A^\alpha f = F^{-1}[\hat{a}]^\alpha F f,$$

where F is the Fourier transform, $\hat{a}$ is the Fourier transform of the distribution a and $[\hat{a}]^\alpha$ stands for the operation of multiplication. In this one assumes that $[\hat{a}]^\alpha$ is well defined (and in general, it should be a multiplier in the corresponding test function space). In particular, in this way the fractional powers $[P(D)]^\alpha$ of a differential operator $P(D), D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, where P is a polynomial, are standardly defined by

$$\big[P(D)\big]^\alpha f = F^{-1}\big[\widehat{P(-i\xi)}\big]^\alpha F f,$$

which suits well for elliptic operators. In particular,

$$(-\Delta)^{\frac{\alpha}{2}} f = F^{-1}|\xi|^\alpha F f$$

for the Laplace operator. However, this may not be the end of the story, because one is interested in the direct expression for $(-\Delta)^{\frac{\alpha}{2}} f$ in terms of the function f itself, not via its Fourier transform. Since so defined operator $(-\Delta)^{\frac{\alpha}{2}}$ is again an operator invariant with respect to translations, we are aware that, in terms of the function f itself, it is a convolution of f with some distribution. Thus one has to explicitly calculate it. The case of negative exponents $\alpha = -\beta$ with $0 < \beta < n$ is easier in a sense: in this case, the distribution $|\xi|^\alpha$ is an ordinary locally integrable with also locally integrable Fourier transform and $(-\Delta)^{-\frac{\beta}{2}} f$ has the form

$$(-\Delta)^{-\frac{\beta}{2}} f(x) = \frac{1}{\gamma_n(\beta)} \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\beta}} =: I^\beta f(x), \tag{1.1}$$

where $\gamma_n(\beta) = \frac{2^\beta \pi^{\frac{n}{2}} \Gamma(\frac{\beta}{2})}{\Gamma(\frac{n-\beta}{2})}$, which was first realized by Riesz [39]. The case $\beta \notin (0, n)$ may be treated by the method of analytic continuation with respect to α, or by Hadamard's method of finite parts, which is the same in this case (see details with respect to both the methods in [52], Sect. 25). Clearly, the analytic continuation of (1.1) into the strip $\Re\alpha \in (0, 2)$ is

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \frac{1}{\gamma_n(-\alpha)} \int_{\mathbb{R}^n} \frac{f(x-y) - f(x)}{|y|^{n+\alpha}} =: \mathbb{D}^\alpha f(x), \tag{1.2}$$

where $0 < \Re\alpha < 2$. In case of sufficiently nice functions, this integral exists and is absolutely convergent when $0 < \Re\alpha < 1$ and in the principal value sense when $1 \leq \alpha < 2$. This construction is known as *hypersingular integral* and often also called *Riesz fractional derivative*. In the theory of fractionally differentiable functions it was first used by Stein [56] to characterize fractional order Sobolev spaces (Bessel potential spaces). There are two approaches to regularize the integral in the case $\alpha > 2$: either subtract the Taylor polynomial in the nominator in (1.2) in the neighborhood $|y| < 1$ of the origin, or to make use of the finite differences of higher order. We do not go into details, this is not our goal in this overview, but just refer for details to the books [50, 52].

Without danger of confusing we may also use the same notation both for positive and negative exponents: $\mathbb{D}^\alpha = I^{-\alpha}$, when $\Re\alpha < 0$ and assume that $\mathbb{D}^0 = I$ (the identity operator). In the case of sufficiently nice functions f, the *semigroup property*

$$\mathbb{D}^\alpha \mathbb{D}^\beta = \mathbb{D}^{\alpha+\beta} \tag{1.3}$$

holds for all complex α and β.

Finally, observe that in the one-dimensional case of $\mathbb{R}^1$, besides the above constructions which in the case $n = 1$ formally are $(-\frac{d^2}{dx^2})^{\frac{\alpha}{2}}$, the unilateral constructions $(\pm\frac{d}{dx})^\alpha$ are more often used, known as *Liouville fractional derivatives*. The corresponding hypersingular form for the latter is known as the Marchaud fractional derivative. It goes back to the paper

[31] and has the form

$$\mathbb{D}^{\alpha} f(x)=\frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x)-f(x-t)}{t^{1+\alpha}}\, dt, \quad x \in \mathbb{R}^1. \tag{1.4}$$

In the case of the half-axis $\mathbb{R}^1_+$, more important for applications, the *Riemann–Liouville fractional derivatives are* used. The corresponding difference form (Marchaud form) in this case is

$$D^{\alpha} f(x)=\frac{f(x)}{\Gamma(1-\alpha)x^{\alpha}}+\frac{\alpha}{\Gamma(1-\alpha)} \times \int_0^{x} \frac{f(x)-f(x-t)}{t^{1+\alpha}}\, dt, \quad x>0. \tag{1.5}$$

1.2 What about domains different from the whole space?

There exist approaches to fractional powers of the Laplace operators suited for domains different from $\mathbb{R}^n$, with boundary conditions taken into account, which we do not touch on in this overview, but mention a Marchaud type formula corresponding to the case of domains in $\mathbb{R}^n$, introduced in [38]. It is defined via direct application of the hypersingular integral (1.2) to a function continued as identical zero beyond the domain.

Let Ω be an arbitrary domain in $\mathbb{R}^n$ and $\partial\Omega$ its boundary. No smoothness or regularity of the boundary is assumed. For a function $f(x)$ defined on Ω, we put

$$\tilde{f}(x)=\begin{cases} f(x), & x\in\Omega,\\ 0, & x\notin\Omega.\end{cases}$$

Then we define the Riesz type fractional derivative of the function f, related to the domain Ω as

$$\mathbb{D}^{\alpha}_{\Omega} f(x)=\mathbb{D}^{\alpha}\tilde{f}(x), \quad x\in\Omega, \tag{1.6}$$

dealing with the result only for $x\in\Omega$. This leads to the following formula (compare with the Marchaud formula (1.5)):

$$\mathbb{D}^{\alpha}_{\Omega} f(x) = c_{\Omega}(x)\frac{f(x)}{[\delta(x,\partial\Omega)]^{\alpha}} + \lambda_n(\alpha)\int_{\Omega}\frac{f(x)-f(y)}{|x-y|^{n+\alpha}}\, dy, \quad x\in\Omega, \tag{1.7}$$

where $0<\alpha<1$, $\lambda_n(\alpha)=\frac{2^{\alpha}\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{n+\alpha}{2})\sin\frac{\alpha\pi}{2}}{\pi^{1+\frac{n}{2}}}$, $\delta(x,\partial\Omega)$ is the distance to the boundary, and $c_{\Omega}(x)$ is a certain bounded function (if Ω has no outer cusps, or more precisely, if $\mathbb{R}^n\backslash\Omega$ satisfies the so called cone property, then also $\inf_{x\in\Omega} c_{\Omega}(x)>0$).

The advantage of the definition (1.6)–(1.7) is that it gives the result in the intrinsic terms of the domain Ω, but the disadvantage is that the semigroup property $\mathbb{D}^{\alpha}_{\Omega}\mathbb{D}^{\beta}_{\Omega}=\mathbb{D}^{\alpha+\beta}_{\Omega}$ fails except for the trivial case $\Omega=\mathbb{R}^n$. In particular, the inversion formula $\mathbb{D}^{\alpha}_{\Omega} I^{\alpha}_{\Omega}=I$ for the corresponding potential operator I^{α}_{Ω} over Ω, which holds for the case $\Omega=\mathbb{R}^n$, is no more valid.

Having given the above basics known in Fractional Calculus, we pass to the main topic of this overview.

2 Fractional operators of variable order in the Euclidean setting

2.1 One-dimensional case

One-dimensional Riemann–Liouville fractional integration may be considered with variable order:

$$I^{\alpha(\cdot)}_{a+} f(x)=\frac{1}{\Gamma[\alpha(x)]}\int_a^x f(y)(x-y)^{\alpha(x)-1}\, dy, \quad \alpha(x)>0,$$

where $a\geq-\infty$ and $x>a$. Similarly, the fractional derivative may be introduced, of variable order, if one replaces α, for instance in (1.5), by $\alpha(x)$. Such variable order fractional integrals and derivatives were introduced in [40, 46, 47, 53], where it was dealt with just from the point of view of mathematical curiosity. However, later there appeared papers where such operators were given a certain physical meaning and they were used in applications in physics and signal processing, see for instance, [4, 5, 22, 23, 60].

Obviously, one should forget about the semigroup property in the case of variable orders.

For one-dimensional fractional integrals of variable order (of Riesz type) along curves in the complex plane we refer to [29].

From the function-theoretic point of view, one of the main questions in the study of such operators is: what are mapping properties of the operator $I^{\alpha(x)}$ in these or other spaces of functions, first of all in the popular Hölder and Lebesgue spaces H^{λ} and L^p.

Since these mapping properties depend on the value the exponent $\alpha(x)$ at the point x, an immediate observation is that a natural setting of the problem should include the spaces with variable characteristics as well. Hölder spaces $H^{\lambda(\cdot)}[a,b]$ of variable order, introduced in [18, 19, 40], are naturally defined as the set of functions continuous on $[a,b]$, $-\infty < a < b < \infty$, such that

$$\left|f(x+h)-f(x)\right| \le c|h|^{\lambda(x)} \tag{2.1}$$

for all $x, x+h \in [a,b]$, where $0 < \lambda(x) \le 1$. This is a Banach space with respect to the norm $\|f\|_{C[a,b]} + H(f)$, where

$$H(f) = \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^{\lambda(x)}} = \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^{\lambda(y)}}. \tag{2.2}$$

The cases where $a=-\infty$ and/or $b=\infty$, may be also admitted, with a certain modification if the infinite point is admitted as the point of Hölder continuity. As a generalization of the classical result of Hardy and Littlewood, in [40, 49] it was proved that the operator $I_{a+}^{\alpha(\cdot)}$ maps functions $f \in H^{\lambda(\cdot)}$ vanishing at the point $x=a$, into the space $H^{\lambda(\cdot)+\alpha(\cdot)}$ assuming that $\inf \alpha(x) > 0$, $\sup \lambda(x) + \alpha(x) < 1$ and $\alpha(x)$ and $\lambda(x)$ satisfy the so called log-condition. The limiting case $\alpha(x)+\lambda(x)=1$ is also admitted with the Lipschitz class modified by the logarithmic factor in this case.

An interesting case is where the exponent $\alpha(x)$ may vanish at some points, which would mean that the fractional integral behaves at these points as the identity operator. We touch on the problem of mapping properties in such cases later in a more general setting of arbitrary metric measure spaces and general fractional operator.

We mention also what happens with the inversion formula $D_+^\alpha I_+^\alpha f \equiv f$ for the Liouville fractional integral I_+^α and fractional derivative D_+^α, valid for the constant exponent α, when α is variable. We may now write $D_+^{\alpha(\cdot)} I_+^{\alpha(\cdot)} = I + K$, where I is the identity operator and K is some integral operator. The reader can find a precise expression for the kernel of this operator and its estimation in [47]. One of the main questions is: what should be supposed about $\alpha(x)$ to guarantee that the additional term K is sufficiently nice, say, a compact operator. As was shown in [47], the following conditions are sufficient for the compactness of K in the space $L^p(-\infty,b), b \le \infty : \alpha(x)$ is locally differentiable, $|\alpha(x) - |\alpha(x-h)| \le c|h|(1+|x|)^{-1} \times (1+|x-h|)^{-1}, 0 < \inf \alpha(x), \sup \alpha(x) < 1$, and $\alpha(-\infty) < \frac{1}{p}$.

2.2 Multi-dimensional case

The Riesz fractional integral and the hypersingular integral The expression

$$I^{\alpha(\cdot)} f(x) = \frac{1}{\gamma_n[\alpha(x)]} \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}, \quad \alpha(x) > 0, \tag{2.3}$$

for the Riesz potential of variable order assumes that $\alpha(x)$ nowhere vanishes. If $\inf \alpha(x) > 0$ and $\sup \alpha(x) < n$, the factor $\frac{1}{\gamma_n[\alpha(x)]}$ is inessential for the study of the mapping properties of the operator; recall that in the case of constant α the presence of this factor was important for the validity of the semigroup property $I^\alpha I^\beta = I^{\alpha+\beta}$), which is no more expected for variable orders.

An interesting question relates to the admission of the order $\alpha(x)$ which may be degenerate at some points. Then we have to study mapping properties of $I^{\alpha(\cdot)}$ in these or other function spaces, taking into account the degeneracy of the order $\alpha(x)$. Note that $\frac{1}{\gamma_n(\alpha)} \to 0$ as $\alpha \to 0$, so that the presence of normalizing factor $\frac{1}{\gamma_n[\alpha(x)]}$, equivalent to $\frac{\alpha(x)}{|\mathbb{S}^{n-1}|}$ as $\alpha(x) \to 0$, is of importance when we admit a possibility for $\alpha(x)$ to degenerate. Then we expect that the operator with this normalizing factor will behave as the identity operator at the points of degeneracy.

Similarly, the corresponding variable order hypersingular integral (written for the case $0 < \alpha(x) < 1$) is

$$\mathbb{D}^{\alpha(\cdot)} f(x) = \int_{\mathbb{R}^n} \frac{f(x)-f(x-y)}{|y|^{n+\alpha(x)}}\,dy,$$

where for simplicity we omit the normalizing factor.

The spherical potentials and hypersingular integrals Now let $x, \sigma \in \mathbb{S}^{n-1}$ and $f(\sigma)$ be a function defined on $\mathbb{S}^{n-1}$. To introduce the spherical fractional integral of Riesz type, we may try just to copy the construction (2.3) and introduce the spherical potential operator of variable order of the function f directly as

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \frac{1}{\gamma_{n-1}[\alpha(x)]} \int_{\mathbb{S}^{n-1}} \frac{f(\sigma)\,d\sigma}{|x-\sigma|^{n-1-\alpha(x)}}, \quad x \in \mathbb{S}^{n-1}, \tag{2.4}$$

where $d\sigma$ stands for the surface measure on $\mathbb{S}^{n-1}$ and we assume that $0 < \alpha(x) < n-1$.

It is known that the space $\mathbb{R}^n$ may be one-to-one transformed onto the n-dimensional sphere via the stereographic projection. Under this projection the spatial potential over $\mathbb{R}^n$ transforms into exactly the spherical potential over $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, up to some weight function. The stereographic projection maps the sphere $\mathbb{S}^n$ onto the space $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ via the change of variables in $\mathbb{R}^{n+1}$: $\xi = s(x) = \{s_1(x), s_2(x), \dots, s_{n+1}(x)\}$ where

$$s_k(x) = \frac{2x_k}{1+|x|^2}, \quad k = 1, 2, \dots, n \quad \text{and}$$

$$s_{n+1}(x) = \frac{|x|^2-1}{|x|^2+1},$$

$x \in \mathbb{R}^{n+1}$, $|x| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$ (see [34]). The formulas $|x-y| = \frac{2|\sigma-\xi|}{|\sigma-e_{n+1}|\cdot|\xi-e_{n+1}|}$, $dy = \frac{2^n}{|\sigma-e_{n+1}|^{2n}}\,d\sigma$, where $e_{n+1} = (0, 0, \dots, 0, 1)$, hold, which imply the relation

$$\int_{\mathbb{R}^n} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha(x)}} = 2^{\tilde{\alpha}(\xi)}|\xi - e_{n+1}|^{n-\tilde{\alpha}(\xi)} \int_{\mathbb{S}^n} \frac{\varphi_*(\sigma)\,d\sigma}{|\xi-\sigma|^{n-\tilde{\alpha}(\xi)}}, \tag{2.5}$$

where $\varphi_*(\sigma) = \frac{\varphi[s^{-1}(\sigma)]}{|\sigma-e_{n+1}|^{n+\tilde{\alpha}(\xi)}}$ and $\tilde{\alpha}(\xi) = \alpha[s^{-1}(\xi)]$.

Therefore, via the stereographic projection we can transfer many properties of spatial fractional integrals to the case of similar spherical integrals.

2.3 About mapping properties of the fractional operator

2.3.1 *The case* $\inf \alpha(x) > 0, \sup \alpha(x) < n$

The first step was done in [48] for fractional integrals

$$I_\Omega^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}$$

over bounded domains in $\mathbb{R}^n$ and it was to show how this integral improves integrability properties of a the function f, in terms of the Sobolev theorem: $I_\Omega^{\alpha(\cdot)} : L^{p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$ within the framework of variable exponent Lebesgue spaces, where $q(x)$ is the Sobolev exponent defined by $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$. We do not give here the definition of such spaces, but refer the reader to the book [6], see also the surveying papers [7, 24, 28, 51]. Such a variable exponent Sobolev theorem holds when the exponents satisfy the so called log condition (continuous with modulus of continuity of logarithmic type).

For the corresponding fractional differentiation operator

$$\mathcal{D}_\Omega^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x)-f(y)}{|x-y|^{n+\alpha(x)}}\,dy, \quad x \in \Omega, \tag{2.6}$$

in [2] it was shown that it transforms Sobolev $W^{1,p(\cdot)}(\Omega)$-functions into $L^{q(\cdot)}$-integrable functions under some natural relation between $p(x)$ and $q(x)$.

For further results on the integrals $I_\Omega^{\alpha(\cdot)} f(x)$ in variable exponent Lebesgue spaces, including the case of unbounded domains and weighted versions we refer for instance to [26, 42, 43, 54, 55], see also references in [7, 24, 28, 51]. We also refer to the study of variable order fractional integrals in variable exponent Morrey spaces in [1, 15, 16, 27].

2.3.2 *The case where* $\alpha(x)$ *may degenerate*

Problems arising in the case where $\alpha(x) = 0$ on some set (of measure zero), were first resolved for the spherical fractional operator (2.4) in [45] in the setting of variable exponent Hölder spaces $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$. The case of non-vanishing orders $\alpha(x)$ was earlier studied in [57–59] (where also the case of generalized Hölder spaces was studied).

In [45] complex values of $\alpha(x)$ were also admitted. This more general setting together with degeneracy of $\alpha(x)$ led to a certain exclusion of purely imaginary orders $\alpha(x) = i\theta(x)$:

$$\max_{x\in\mathbb{S}^{n-1}} \left|\arg \alpha(x)\right| < \frac{\pi}{2} - \varepsilon \quad \text{for some } \varepsilon > 0. \tag{2.7}$$

Under this assumption in [45] was proved, in particular, the statement given in Theorem 2.1, below. In that theorem, the operator $\alpha(x)\mathfrak{I}^{\alpha(\cdot)}$ at the points x_0 where $\alpha(x_0) = 0$ is interpreted as the limit $\alpha(x_0)\mathfrak{I}^{\alpha(x_0)} = \lim_{\beta\to 0} \beta\mathfrak{I}^{\beta}$. As is well known, such a limit in the case of spatial fractional integrals is the identity operator, up to a constant factor. The same holds in the case of spherical integrals.

Theorem 2.1 *Let* $\alpha \in \mathrm{Lip}(\mathbb{S}^{n-1})$ *and the set* $\{x \in \mathbb{S}^{n-1} : \Re\alpha(x) = 0\}$ *have measure zero. The operator* $\alpha(x)\mathfrak{I}^{\alpha(\cdot)}$ *acts boundedly from the space* $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$

into the space $H^{\lambda(\cdot)+\alpha(\cdot)}(\mathbb{S}^{n-1},\alpha)$, *if* $\sup_{x\in\mathbb{S}^{n-1}}[\lambda(x)+\Re\alpha(x)]<1$.

The above mentioned tendency to the identity operator is obviously reflected in this theorem: at the points where $\alpha(x)=0$ there is stated only the conservation of the smoothness properties of the function f, but in general when $\Re\alpha(x)\to 0$, the limiting operator, under condition (2.7), is a singular integral operator of Calderon–Zygmund type, also preserving the smoothness properties, in general).

It is worthwhile noticing that for the corresponding spherical fractional differentiation operator

$$\mathfrak{D}^{\alpha(\cdot)}f(x)=\lim_{\varepsilon\to 0}\int_{\substack{\mathbb{S}^{n-1}\\ |x-\sigma|\geq\varepsilon}}\frac{f(\sigma)-f(x)}{|x-\sigma|^{n-1+\alpha(x)}}\,d\sigma,\quad x\in\mathbb{S}^{n-1}, \tag{2.8}$$

where $0<\Re\alpha(x)<1$, a symmetrical statement holds on mapping of $H^{\lambda(\cdot)}(\mathbb{S}^{n-1})$ into $H^{\lambda(\cdot)-\alpha(\cdot)}(\mathbb{S}^{n-1})$ under the assumption that

$$\min_{x\in\mathbb{S}^{n-1}}\Re\alpha(x)>0,\qquad \max_{x\in\mathbb{S}^{n-1}}\Re\alpha(x)<1,$$

and $\min_{x\in\mathbb{S}^{n-1}}\Re[\lambda(x)-\alpha(x)]>0$, see Theorem 3.13 in [45] (for simplicity, we do not touch on the degeneracy cases in this result).

In fact, in [45] there was obtained a more general statement on mapping properties within the framework of generalized Hölder spaces, defined by a prescribed dominant of the continuity modulus, see details in [45].

From the above statement for spherical fractional integrals, one can derive corresponding results for spatial fractional operators via relations of type (2.5).

In the above statements we saw a possibility of *pointwise variable* improvement, by fractional integration, and worsening, by fractional differentiation, of smoothness properties of functions. Does this possibility depend on the geometrical structure of the underlying set? Fractional integrals of variable order may be considered on arbitrary domains in $\mathbb{R}^n$, surfaces, manifolds, fractal sets, and in general, in the setting of quasimetric measure spaces. In the next section we show that such a pointwise variable estimation is possible within the framework of quasimetric measure spaces, satisfying some natural assumptions.

3 Fractional operators over quasimetric measure spaces

Let (X,d,μ) denote a measure space with quasimetric ϱ and a non-negative measure μ. By $B(x,r)=B_X(x,r)=\{y\in X:\varrho(x,y)<r\}$ we denote a ball in X. We restrict the choice of the space (X,ϱ,μ) by the natural condition that all the balls $B(x,r)$ are measurable and $\mu S(x,r)=0$ for all the spheres $S(x,r)=\{y\in X:\varrho(x,y)=r\}, x\in X, r\geq 0$.

Fractional type integral operators in such a general context may be defined in different ways:

$$\mathfrak{I}^{\alpha(\cdot)}f(x)=\int_\Omega\frac{[\varrho(x,y)]^{\alpha(x)}}{\mu B(x,\varrho(x,y))}f(y)\,d\mu(y), \tag{3.1}$$

$$\mathcal{I}^{\alpha}f(x)=\int_\Omega\frac{f(y)\,d\mu(y)}{\mu B(x,\varrho(x,y))^{1-\alpha(x)}}, \tag{3.2}$$

$$I^{\alpha(\cdot)}f(x)=\int_\Omega\frac{f(y)\,d\mu(y)}{[\varrho(x,y)]^{N-\alpha(x)}}, \tag{3.3}$$

where $\Omega\subseteq X, \alpha(x)>0$ and N should be thought as a kind of dimension of X, see for instance [10, 24, 25]. These forms are equivalent in the Euclidean case (with $\alpha(x)$ replaced by $\frac{\alpha(x)}{N}$ in (3.2)), but not equivalent in general. An arbitrary (X,ϱ,μ) may have no "dimension", but has the so called lower and upper dimensions, which in their turn may depend on the point x. In the case where the measure satisfies the growth condition

$$\mu B(x,r)\leq Kr^N\quad\text{as}\quad r\to 0,\quad K>0, \tag{3.4}$$

the exponent N may be used to define $I^{\alpha(\cdot)}f(x)$ (the exponent N is not necessarily an integer).

We mention results related to the fractional integral (3.3), obtained in [44], which show how this operator improves the properties of functions in terms of their Hölder behavior, in a general setting of quasimetric measure spaces, satisfying the growth condition (3.4).

Within the framework of the Hölder spaces $H^{\lambda(\cdot)}(\Omega)$ with a variable exponent, we consider the fractional integral (3.3) and also the corresponding fractional differentiation operator (hypersingular integral)

$$D^{\alpha}f(x)=\lim_{\varepsilon\to 0}\int_{y\in\Omega:\varrho(x,y)>\varepsilon}\frac{f(y)-f(x)}{\varrho(x,y)^{N+\alpha(x)}}\,d\mu(y),\quad x\in\Omega, \tag{3.5}$$

of real-valued variable order $\alpha(x)$ with values in the interval $0 \le \alpha(x) < 1$. We assume that Ω is an open bounded set (the case of infinite sets also may be admitted, where we have to specially study the behavior at infinity, this study depending on what meaning we give to Hölder behavior at infinity; we do not touch on this issue here).

In the case of constant α a study of such operators in the general setting of quasimetric measure spaces (X, ϱ, μ) with growth condition, was made in [11–14].

The estimates given in the sequel reveal the nature of mapping properties of the operators I^α and D^α in dependence of local values of $\alpha(x)$ and $\lambda(x)$. We denote

$$\Pi_\alpha = \{x \in \Omega : \alpha(x) = 0\}$$

and suppose that $\mu(\Pi_\alpha) = 0$.

As shown in [30], every quasimetric $\varrho(x, y)$ on a quasimetric space (X, ϱ) admits an equivalent quasimetric ϱ_1 for which there exists an exponent $\theta \in (0, 1]$ such that the property

$$\begin{aligned} &\left|\varrho_1(x,z) - \varrho_1(y,z)\right| \\ &\quad \le M\varrho_1^\theta(x,y)\left\{\varrho_1(x,z) + \varrho_1(y,z)\right\}^{1-\theta} \end{aligned} \tag{3.6}$$

holds. When ϱ is a metric, then ϱ satisfies (3.6) with $\theta = 1$ and $M = 1$. Everywhere in the sequel we suppose that ϱ is already chosen as equal to ϱ_1.

For fixed $x \in \Omega$ we consider the local continuity modulus

$$\omega(f, x, h) = \sup_{\substack{z\in\Omega:\\ \varrho(x,z)\le h}} \left|f(x) - f(z)\right| \tag{3.7}$$

of a function f at the point x. Below we suppose that $|h| < 1$ (assuming that $\operatorname{diam} X \ge 1$).

For a function $\lambda(x)$ defined on Ω we suppose that $\lambda_- := \inf_{x\in X} \lambda(x) > 0$, and $\lambda_+ := \sup_{x\in X} \lambda(x) < 1$, and by $H^{\lambda(\cdot)}(\Omega)$ denote the space of functions $f \in C(\Omega)$ such that $\omega(f, x, h) \le Ch^{\lambda(x)}$. Equipped with the norm

$$\|f\|_{H^{\lambda(\cdot)}(\Omega)} = \|f\|_{C(\Omega)} + \sup_{x\in\Omega} \sup_{h\in(0,1)} \frac{\omega(f,x,h)}{h^{\lambda(x)}},$$

this is a Banach space.

In Hölder norm estimations of functions $I^\alpha f$, the case

$$\mathfrak{I}_\alpha(x) := I^\alpha(1)(x) = \int_\Omega \frac{d\mu(z)}{\varrho(x,z)^{N-\alpha(x)}} \tag{3.8}$$

of the fractional integral of a constant function $f \equiv 1$ plays an important role. Since Ω is bounded, it is well defined. In the case of constant order, $\alpha(x) = \alpha = \text{const}$, the following notion is known: a set Ω is said to satisfy *the cancellation property with respect to the fractional integral*, if $\int_\Omega [\frac{1}{\varrho(x,z)^{N-\alpha}} - \frac{1}{\varrho(y,z)^{N-\alpha(x)}}]\, d\mu(z)$ for all $x, y \in \Omega$, i.e. the fractional integral of a constant is also a constant (when $\mathfrak{I}_\alpha(x)$ is well defined).

The cancellation property holds for $\Omega = X = \mathbb{R}^N$. Among bounded sets the sphere $X = \mathbb{S}^{n-1}$ is another example. Note that for variable orders $\alpha(x)$ the cancellation property fails even in these examples (see for instance [11] about the importance of the cancellation property for mapping properties of fractional integrals of constant order in Hölder spaces on quasimetric measure spaces.

When we consider the whole space $H^{\lambda(\cdot)}(\Omega)$, i.e. we do not consider for instance, a subspace of functions vanishing on the boundary, the condition

$$\mathfrak{I}_\alpha(1) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

is necessary for the mapping $I^\alpha : H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$ to hold. However, this condition seldom holds, because $\mathfrak{I}_\alpha(1)(x)$ has a worse behavior when x approaches the boundary: $\mathfrak{I}_\alpha(1) \sim c[\delta(x, \partial\Omega)]^{\alpha(x)}$ in general, where $\delta(x, \partial\Omega)$ is the distance to the boundary (for constant α and the Euclidean case see [8]). therefore, a natural setting for mapping in Hölder spaces should involve functions f vanishing on the boundary. We do not touch on such a setting in this overview, but impose another condition, see (3.13), which does not require that functions f vanishes on the boundary, but instead the order $\alpha(x)$ does.

What we need is based on the following weighted estimate of the local continuity modulus of the fractional integral $I^\alpha f$ via that of the function f itself, where the used weight is exactly the function $\alpha(x)$.

Theorem 3.1 I. *Let Ω be a bounded open set in X, $d = \operatorname{diam}\Omega$, let α satisfy the log-condition and $0 \le \inf_{x\in\Omega} \alpha(x) \le \sup_{x\in\Omega} \alpha(x) < \min(\theta, N)$. Then for all $x \in \Omega\backslash\Pi_\alpha$ the following Zygmund type estimate is valid*:

$$\begin{aligned} \omega\big(\alpha I^\alpha f, x, h\big) &\le Ch^{\alpha(x)}\omega(f,x,h) \\ &\quad + Ch^\theta \int_h^d \frac{\omega(f,x,t)\,dt}{t^{1+\theta-\alpha(x)}} \end{aligned}$$

$$+ C\omega(\alpha, x, h) \int_h^d \frac{\omega(f, x, t)\, dt}{t^{2-\alpha(x)}} + C\omega(\alpha\Im_\alpha, x, h)\|f\|_{C(\Omega)}. \quad (3.9)$$

II. *Let* $\alpha(x)$ *satisfy the log-condition and* $0 \le \inf_{x\in\Omega}\alpha(x) \le \max_{x\in\Omega}\alpha(x) < 1$. *If* $f \in C(\Omega)$, *then for all* $x, y \in \Omega$ *with* $\varrho(x, y) < h$ *such that* $\alpha(x) \neq 0$ *and* $\alpha(y) \neq 0$, *the following estimate is valid*:

$$\begin{aligned} &\left|(D^\alpha f)(x) - (D^\alpha f)(y)\right| \\ &\quad \le \frac{C}{\min(\alpha(x), \alpha(y))} \\ &\qquad \times \int_0^h \left[\frac{\omega(f, x, t)}{t^{1+\alpha(x)}} + \frac{\omega(f, y, t)}{t^{1+\alpha(y)}}\right] dt \\ &\qquad + C \int_h^2 \left[\omega(\alpha, x, h) + h^\theta t^{1-\theta}\right] \frac{\omega(f, x, t)\, dt}{t^{2+\alpha(x)}}. \end{aligned} \quad (3.10)$$

From these estimates one can easily derive the following statements on the mapping properties, where $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha) = \{f : \alpha(x)f(x) \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)\}$.

Theorem 3.2 *Let*

$$\alpha(x) \ge 0, \quad \max_{x\in\Omega}\alpha(x) < \min(\theta, N), \qquad \alpha(x) \in \mathrm{Lip}(\Omega), \quad (3.11)$$

and

$$\sup_{x\in\Omega}\left[\lambda(x) + \alpha(x)\right] < \theta. \quad (3.12)$$

If

$$\alpha\Im_\alpha \in H^{\lambda(\cdot)+\alpha(\cdot)}, \quad (3.13)$$

then the operator $I^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the weighted space* $H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega, \alpha)$.

Theorem 3.3 *Under the condition* (3.11), *the operator* $D^{\alpha(\cdot)}$ *is bounded from the space* $H^{\lambda(\cdot)}(\Omega)$ *into the space* $H^{\lambda(\cdot)-\alpha(\cdot)}(\Omega)$, *if*

$$0 < \inf_{x\in\Omega}\left\{\lambda(x) - \alpha(x)\right\}, \qquad \sup_{x\in\Omega}\lambda(x) < 1.$$

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