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{FirstPag

# On the Regularization of a Multidimensional Integral Equation in Lebesgue Spaces with Variable Exponent

**S. G. Samko[1*] and S. M. Umarkhadzhiev[2**]**

[1]*University of Algarve, Portugal*

[2]*Chechen State University, Groznyi*

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**Abstract**—A multidimensional integral equation of the first kind with potential-type kernel of small order in spaces of summable functions of variable order is reduced to an equation of the second kind.

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## 1. INTRODUCTION

{sec1:v57

Consider the following multidimensional integral equation of the first kind:

$$\mathbf{M}^\alpha\varphi := \int_{\mathbb{R}^n} \frac{c(x,y)}{|x-y|^{n-\alpha}}\,\varphi(y)\,dy = f(x), \qquad x\in\mathbb{R}^n, \tag{1.1}$$ {eq1.1:v5

where $0<\alpha<1$ and it is assumed that the function $c(x,y)$ satisfies the following conditions:

1) we have

$$c(x,y)\in C(\mathbb{R}^n\times\mathbb{R}^n), \quad c(x,x)\in L^\infty(\mathbb{R}^n), \qquad \inf_{x\in\mathbb{R}^n}|c(x,x)|>0; \tag{1.2}$$ {eq1.2:v5

2) the function $c(x,y)$ obeys Hölder's condition with respect to the first variable (in $\mathbb{R}^n$, compactified by one point at infinity) and uniformly with respect to the second variable, i.e.,

$$|c(x,y)-c(z,y)| \le \frac{C|x-z|^\lambda}{(1+|x|)^\lambda(1+|z|)^\lambda}, \qquad \alpha<\lambda\le 1, \tag{1.3}$$ {eq1.3:v5

where $C>0$ is independent of $x$, $y$, $z$;

Note that relations (1.2), (1.3) imply that $c(x,y)\in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$.

It is known (see [1], [2], [3, Chap. 10]) that the hypersingular operator

$$\mathbf{R}f = \frac{\mu}{c(x,x)} \lim_{\substack{\varepsilon\to 0\\(L^p)}} \int_{\mathbb{R}^n} \frac{f(x)-f(x-y)}{|y|^{n+\alpha}}\,dy, \tag{1.4}$$ {eq1.4:v5

*E-mail: ssamko@ualg.pt
**E-mail: usmu@mail.ru

where

$$\mu = \frac{\alpha}{2\pi^{n+1}} \sin\frac{\alpha\pi}{2} \Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right),$$

while the limit is understood in the sense of strong convergence in $L^p(\mathbb{R}^n)$ and is the regularizer for Eq. (1.1) in the space $L^p(\mathbb{R}^n)$, i.e.,

$$\mathbf{R}\mathbf{M}^{\alpha}\varphi = \varphi + \mathbf{A}\varphi, \tag{1.5}$$ {eq1.5:v5

where $\mathbf{A}$ is an operator compact in the space $L^p(\mathbb{R}^n)$, $1 < p < n/\alpha$.

Regularization of the form (1.5) of Eq. (1.1) in the space $L^p(\mathbb{R}^n)$ was proved in [2] also in the case $1 \le \alpha < n$. The study of this case involves hypersingular integrals of higher order; see [3] with regard to hypersingular integrals of arbitrary order.

In this paper, we consider the case $0 < \alpha < 1$ and extend this result concerning regularization to the case of Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, taking into account recent progress in the theory of such spaces; in the case $1 \le \alpha < n$, a similar generalization involves significant technical difficulties and is not touched upon in this paper.

The first paper specially dealing with Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent was the paper [4] (in the one-dimensional case), and a subsequent development, including the multidimensional case, was given in [5]. The generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent have turned out to be a convenient apparatus in the study of models with so-called nonstandard growth (for example, in elasticity theory, hydromechanics, differential equations (in particular, equations with $p(x)$-Laplacian), and variational problems; see, for example, [6]–[8]).

These applications stimulated rapid progress in the theory of the spaces $L^{p(\cdot)}(\Omega)$; see the surveys [9]–[12], where further references can be found.

Although the spaces $L^{p(\cdot)}(\Omega)$ possess a number of undesirable properties (the spaces $L^{p(\cdot)}(\Omega)$ are not translation-invariant, functions in these spaces are not $p(x)$-continuous in the mean, Young's theorem for convolutions does not hold) there is a significant progress in their study and in the construction of harmonic analysis in these spaces stimulated by the applications pointed out above. This progress became possible for continuous exponents $p(x)$ satisfying the Dini logarithmic condition.

The greatest progress in the development of harmonic analysis and operator theory in the spaces $L^{p(\cdot)}(\Omega)$ involving the boundedness of the maximal operator was attained under conditions (2.5) and (2.6) (plus condition (2.7) in the case of unbounded sets $\Omega$).

In Sec. 2, we give necessary preliminaries dealing with the Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent and some auxiliary statements. Section 3 contains the statement of the main result. In Sec. 4, we give an estimate of the kernel of an integral operator $\mathbf{A}$ in (1.5) in connection with the fact that the well-known estimate of this kernel given in [3, Chap. 10] is insufficient for working in spaces with variable exponent. Using this estimate, we prove the main statement of this paper in Sec. 5.

By $C, c, c_1, \dots$ we denote absolute positive constants that can take different values even in the same row.

## 2. PRELIMINARIES {sec2:v57

### *2.1. On Lebesgue Spaces with Variable Exponent* {ssec2.1:

Let us present definitions and some main facts for the spaces $L^{p(\cdot)}(\Omega)$; for more details, we refer the reader to [13], [5], [14]. Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open set, $p\colon \Omega \to [1,\infty)$ is a measurable function on $\Omega$, and

$$p_- = \operatorname{ess\,inf}_{x\in\Omega} p(x), \qquad p_+ = \operatorname{ess\,sup}_{x\in\Omega} p(x).$$

By $L^{p(\cdot)}(\Omega)$ we denote the set of all measurable (on $\Omega$) functions $f$ for which the modular expression

$$I^p(f) := \int_\Omega |f(x)|^{p(x)}\,dx < \infty$$

is finite. Under the condition $1 \le p(x) \le p_+ < \infty$ on $\Omega$, we have a Banach space with respect to the norm

$$\|f\|_{p(\,\cdot\,)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$ {eq2.1:v5

The weighted space $L^{p(\,\cdot\,)}(\Omega, \varrho)$ is introduced in the ordinary way:

$$L^{p(\,\cdot\,)}(\Omega, \varrho) = \{f : \varrho f \in L^{p(\,\cdot\,)}(\Omega)\}, \qquad \|f\|_{L^{p(\,\cdot\,)}(\Omega,\varrho)} = \|\varrho f\|_{p(\,\cdot\,)}.$$

Let $p'(x)$ denote the conjugate exponent: $1/p(x) + 1/p'(x) = 1$. Hölder's inequality holds in the following form:

$$\left|\int_\Omega f(x)g(x)\,dx\right| \le k\|f\|_{p(\,\cdot\,)}\|g\|_{p'(\,\cdot\,)}, \qquad k = \frac{1}{p_-} + \frac{1}{p'_-}. \tag{2.2}$$ {eq2.2:v5

It is well known that, under the condition $1 \le p_- \le p(x) \le p_+ < \infty$, modular boundedness (convergence) is equivalent to boundedness (convergence) in the norm:

$$c_1 \le \|f\|_{p(\,\cdot\,)} \le c_2 \qquad \Longrightarrow \qquad c_3 \le I^p(f) \le c_4, \tag{2.3}$$ {eq2.3:v5

$$C_1 \le I^p(f) \le C_2 \qquad \Longrightarrow \qquad C_3 \le \|f\|_{p(\,\cdot\,)} \le C_4, \tag{2.4}$$ {eq2.4:v5

where

$$c_3 = \min(c_1^{p_-}, c_1^{p^+}), \qquad c_4 = \max(c_2^{p_-}, c_2^{p^+}),$$
$$C_3 = \min(C_1^{1/p_-}, C_1^{1/p^+}), \qquad C_4 = \max(C_2^{1/p_-}, C_2^{1/p^+}).$$

In what follows, we assume that the following condition holds:

$$1 < p_- \le p(x) \le p_+ < \infty, \qquad x \in \Omega. \tag{2.5}$$ {eq2.5:v5

It is well known that, in the theory of the spaces $L^{p(\,\cdot\,)}(\Omega)$, an important role is played by the condition

$$|p(x) - p(y)| \le \frac{C}{\ln(1/|x-y|)} \qquad \text{for all} \quad x, y \in \Omega, \quad |x-y| \le \frac{1}{2}, \tag{2.6}$$ {eq2.6:v5

where $C$ is independent of $x$ and $y$; this condition is known as the *local* log-*condition* or the *weak Lipschitz condition*, or the *Dini–Lipschitz condition* and, in the case where the set $\Omega$ is unbounded, it is also known as a *condition linking* $p(x)$ *with a constant*: there exists a constant $p_\infty$ also denoted by $p(\infty)$ such that

$$|p(x) - p_\infty| \le \frac{C}{\ln(2+|x|)}, \qquad x \in \Omega. \tag{2.7}$$ {eq2.7:v5

Let $\mathcal{P}(\Omega)$ denote the set of measurable functions $p\colon \Omega \to [1, \infty)$ satisfying condition (2.5) and (if $\Omega$ is unbounded) also condition (2.7); let $\mathbb{P}(\mathbb{R}^n)$ be the set of functions $p\colon \mathbb{R}^n \to [1,\infty)$ satisfying conditions (2.5), (2.6) and (if $\Omega$ is unbounded) also condition (2.7). Obviously, $\mathbb{P}(\Omega) \subset \mathcal{P}(\Omega)$.

### *2.2. On the Interpolation of Compactness in the Case of a Variable* $p(x)$ {ssec2.2:

In 1960, Krasnosel'skii [15] showed the possibility of "one-sided" interpolation of the compactness property in the spaces $L^p$ with constant $p$. The proof in [15] assumed that the sets $\Omega$ are bounded. However, such interpolation is also possible in the case of unbounded sets and in more general interpolation settings within the framework of interpolating pairs of general Banach spaces. For results obtained under various assumptions on interpolating pairs, see, for example, [16]–[20].

For the spaces $L^{p(\,\cdot\,)}(\Omega)$, the interpolation space is $L^{p_\theta(\,\cdot\,)}(\Omega)$, where

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}, \qquad \theta \in [0,1]. \tag{2.8}$$ {eq2.8:v5

On the basis of results from [20], such a one-sided interpolation of compactness was established in [21] in the following form.

{th1:v575

**Theorem 1.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let the variable exponents $p_j\colon \Omega \to (1,\infty)$, $j = 1, 2$, satisfy conditions* (2.5)–(2.7). *Let the linear operator $A$ defined on $L^{p_1(\,\cdot\,)}(\Omega) \cup L^{p_2(\,\cdot\,)}(\Omega)$ be bounded in the spaces $L^{p_j(\,\cdot\,)}(\Omega)$, $j = 1, 2$. If it is compact in the space $L^{p_1(\,\cdot\,)}(\Omega)$, then it is also compact in the space $L^{p_\theta(\,\cdot\,)}(\Omega)$ for any $\theta \in (0, 1]$.*

In applications, it is convenient to use the following statement obtained from Theorem 1 (see [22]).

{th2:v575

**Theorem 2.** *Suppose that $\Omega \subseteq \mathbb{R}^n$, the function $p\colon \Omega \to [1,\infty)$, $p(x)$ satisfies the inequalities*

$$1 \le p_- \le p(x) \le p_+ < \infty,$$

*and the number $p_0$ belongs to $(1,\infty)$. Then there exists a function $q\colon \Omega \to [1,\infty)$ with the same property*

$$1 < q_- \le q(x) \le q_+ < \infty$$

*and a number $\theta \in [0, 1)$ such that $L^{p(\,\cdot\,)}(\Omega)$ is an interpolation space between $L^{p_0}(\Omega)$ and $L^{q(\,\cdot\,)}(\Omega)$ corresponding to the interpolation parameter $\theta$. The number $\theta$ can take any value in the interval $\theta \in (0, \theta_0)$, where*

$$\theta_0 = \min\biggl\{1, \frac{p_0}{p_+}, \frac{p_0'}{p_-'}\biggr\},$$

*and then*

$$q(x) = \frac{p_0(1-\theta)p(x)}{p_0 - \theta p(x)}\,.$$

The effectiveness of the last statement for applications is obvious: it allows us to immediately obtain compactness in the space $L^{p(\,\cdot\,)}$ with variable exponent with the only requirement of boundedness for variable exponents, and with the knowledge of compactness only for constant exponents.

## 3. MAIN STATEMENT

{sec3:v57

Let

$$p^\sharp(x) = \frac{np(x)}{n - \alpha p(x)}$$

denote the Sobolev exponent. If $p \in \mathbb{P}(\mathbb{R}^n)$, then $p^\sharp \in \mathbb{P}(\mathbb{R}^n)$ in the case $\alpha p_+ < n$.

In the following theorem, the hypersingular operator $\mathbf{R}$ is expressed as

$$\mathbf{R}f = \frac{\mu}{c(x,x)} \lim_{\substack{\varepsilon\to 0\\ (L^{p(\,\cdot\,)})}} \int_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+\alpha}}\,dy, \tag{3.1}$$

{eq3.1:v5

where the limit is understood in the sense of strong convergence in $L^{p(\,\cdot\,)}(\mathbb{R}^n)$.

{th3:v575

**Theorem 3.** *Suppose that $0 < \alpha < 1$, the function $c(x, z)$ satisfies conditions* (1.2), (1.3), *and $p \in \mathbb{P}(\mathbb{R}^n)$ and $\alpha p_+ < n$. Then representation* (1.5) *is valid; in it, the operator $\mathbf{A}$ is bounded from the space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ to the space $L^{p(\,\cdot\,)}(\mathbb{R}^n) \cap L^{p^\sharp(\,\cdot\,)}(\mathbb{R}^n)$ and is compact in the space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$.*

## 4. AUXILIARY STATEMENTS

{sec4:v57

### *4.1. On the Operator* $\mathbf{A}$

{ssec4.1:

For the composition $\mathbf{RM}^\alpha$, it is well known that the operator $\mathbf{A}$ in the representation (1.1) is of the form

$$(\mathbf{A}\varphi)(x) := \int_{\mathbb{R}^n} \mathcal{K}(x, x-y)\varphi(y)\,dy, \tag{4.1}$$

{eq4.1:v5

where

$$\mathcal{K}(x,y) = \frac{\mu}{c(x,x)} \int_{\mathbb{R}^n} \frac{c(x,y)-c(\xi,y)}{|x-\xi|^{n+\alpha}|y-\xi|^{n-\alpha}}\,dy. \tag{4.2}$$

{eq4.2:v5

Obviously, in the case where the function $c(x,y)$ satisfies condition (1.3), the kernel $\mathcal{K}(x,y)$ admits the estimate

$$|\mathcal{K}(x,y)| \le \frac{C}{(1+|x|)^\lambda} \int_{\mathbb{R}^n} \frac{d\xi}{|\xi|^{n+\alpha-\lambda}|y-\xi|^{n-\alpha}(1+|x-\xi|)^\lambda}, \tag{4.3}$$

{eq4.3:v5

where the integral converges for $0<\alpha<\lambda$.

### *4.2. Estimate of the Kernel*

{lemt4v27

**Lemma 1.** *Suppose that* $0<\alpha<n$ *and* $\alpha<\lambda<n+\alpha$. *Then the integral in* (4.3) *admits the estimate*

$$\int_{\mathbb{R}^n} \frac{d\xi}{|\xi|^{n+\alpha-\lambda}|y-\xi|^{n-\alpha}(1+|x-\xi|)^\lambda} \le \frac{c}{|y|^{n-\alpha}}, \qquad x,y\in\mathbb{R}^n. \tag{4.4}$$

{eq4.4:v5

**Proof.** Let $I(x,y)$ denote the integral on the left-hand side of (4.4). Let us split this integral as follows:

$$\begin{aligned} I(x,y) = &\int_{|\xi|<|y|/2} \frac{d\xi}{|\xi|^{n+\alpha-\lambda}|y-\xi|^{n-\alpha}(1+|x-\xi|)^\lambda} \\ &+ \int_{|\xi|>|y|/2} \frac{d\xi}{|\xi|^{n+\alpha-\lambda}|y-\xi|^{n-\alpha}(1+|x-\xi|)^\lambda} =: B_1 + B_2. \end{aligned}$$

For $B_1$, let us use the fact the inequalities

$$|y-\xi| \ge |y|-|\xi| \ge \frac{|y|}{2}.$$

Then

$$B_1 \le \frac{C}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{d\xi}{|\xi|^{n+\alpha-\lambda}(1+|x-\xi|)^\lambda}$$

and it remains to use the fact that the last integral is bounded as a function of $x$; see, for example, [3, Lemma 1.38]).

To estimate $B_2$, let us use Hölder's inequality with exponent $r>1$, which will be defined below. We have

$$\begin{aligned} B_2 &\le \left\{\int_{|\xi|>|y|/2} \frac{d\xi}{(1+|x-\xi|)^{\lambda r}}\right\}^{1/r} \left\{\int_{|\xi|>|y|/2} \frac{d\xi}{|\xi|^{(n+\alpha-\lambda)r'}|y-\xi|^{(n-\alpha)r'}}\right\}^{1/r'} \\ &:= J_1^{1/r} J_2^{1/r'}, \end{aligned} \tag{4.5}$$

{eq4.5:v5

where $1/r+1/r'=1$. Choose $r>1$ so that the two integrals $J_1$ and $J_2$ converge:

$$\lambda r > n, \qquad (n-\alpha)r' < n, \qquad (2n-\lambda)r' > n,$$

which implies

$$\max\Bigl(0, \frac{\lambda - n}{n}\Bigr) < \frac{1}{r} < \min\Bigl(1, \frac{\alpha}{n}\Bigr). \tag{4.6}$$

{eq4.6:v5

This interval is not empty for $\lambda < n + \alpha$. We assume that $r$ has been chosen from (4.6).

For $|y| \le 1$, we have $J_1 \le \text{const}$, $x \in \mathbb{R}^n$. For $|y| > 1$, we obtain

$$J_1 \le c \int_{|\xi|>|y|/2} \frac{d\xi}{|\xi|^{\lambda r}} = c|y|^{n-\lambda r},$$

so that

$$J_1 \le c \begin{cases} 1, & |y| \le 1, \\ |y|^{n-\lambda r}, & |y| > 1. \end{cases} \tag{4.7}$$

{eq4.7:v5

Performing a similarity transformation $\xi = |y|\tau$ in the integral $J_2$, we obtain

$$J_2 = c|y|^{n-(2n-\lambda)r'} \int_{|\tau|>1/2} \frac{d\tau}{|\tau|^{(n+\alpha-\lambda)r'} |y/|y| - \tau|^{(n-\alpha)r'}}\,.$$

Hence

$$J_2 = c|y|^{n-(2n-\lambda)r'}, \qquad x, y \in \mathbb{R}^n. \tag{4.8}$$

{eq4.8:v5

Then, using (4.5) and taking into account estimates (4.7) and (4.8), for $x, y \in \mathbb{R}^n$ we obtain

$$B_2 \le C \begin{cases} |y|^{-n-n/r+\lambda}, & |y| \le 1, \\ |y|^{-n}, & |y| > 1. \end{cases} \tag{4.9}$$

{eq4.9:v5

The required inequality $B_2 \le c|y|^{\alpha-n}$ is valid under the appropriate choice of $r$ from the condition

$$-n - \frac{n}{r} + \lambda \ge \alpha - n, \qquad \text{i.e.,} \quad \frac{1}{r} \le \frac{\lambda - \alpha}{n}\,.$$

Such a choice is compatible with condition (4.6) for $\alpha < n$, which proves the lemma. □

**Corollary.** *Let* $0 < \alpha < 1$, *and let the function* $c(x, y)$ *satisfy condition* (1.3) *with* $\alpha < \lambda \le 1$. *Then the kernel* $\mathcal{K}(x, x-y)$ *of the operator* (4.1) *admits the estimate*

$$|\mathcal{K}(x, x-y)| \le \frac{C}{(1+|x|)^\lambda |x-y|^{n-\alpha}}\,. \tag{4.10}$$

{eq4.10:v

In order to prove this statement, it suffices to refer to inequality (4.3) and estimate (4.4).

### *4.3. The Hardy–Stein–Weiss Inequality with a Variable Exponent* $p(x)$

{ssec4.3:

In [23], the following Hardy–Stein–Weiss inequality with a variable exponent $p(x)$ was proved (see Theorem A in [23]):

$$\biggl\| |x|^{\beta-\alpha} \int_\Omega \frac{f(y)\,dy}{|y|^\beta |x-y|^{n-\alpha}} \biggr\|_{p(\cdot)} \le c\|f\|_{p(\cdot)}, \qquad 0 \in \overline{\Omega}, \tag{4.11}$$

{eq4.11:v

under the assumption that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $p \in \mathbb{P}(\Omega)$, and

$$0 < \alpha < n, \qquad \alpha - \frac{n}{p(0)} < \beta < \frac{n}{p'(0)}\,.$$

In the following lemma, we justify the validity of a similar inequality for the case $\Omega = \mathbb{R}^n$.

{lem2:v57

**Lemma 2.** *Inequality* (4.11) *holds also in the case of an unbounded domain* $\Omega$ *if* $p \in \mathbb{P}(\Omega)$ *and*

$$0 < \alpha < n, \qquad \alpha - \frac{n}{p(0)} < \beta < \frac{n}{p'(0)}, \qquad \alpha - \frac{n}{p(\infty)} < \beta < \frac{n}{p'(\infty)}. \tag{4.12}$$

{eq4.12:v

**Proof.** Assume that the function $f$ is nonnegative and is extended by zero beyond $\Omega$. Following arguments from [23], we obtain

$$|x|^{\beta-\alpha} \int_\Omega \frac{f(y)\,dy}{|y|^\beta |x-y|^{n-\alpha}} = |x|^{\beta-\alpha} \int_{|y-x|\le 2|x|} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} + |x|^{\beta-\alpha} \int_{|y-x|\ge 2|x|} \frac{f(y)\,dy}{|y|^\beta |x-y|^{n-\alpha}}$$
$$:= \mathcal{A}_{\alpha,\beta} f(x) + \mathcal{B}_{\alpha,\beta} f(x).$$

As shown in [23], $\mathcal{A}_{\alpha,\beta} f(x)$ satisfies the pointwise estimate

$$\mathcal{A}_{\alpha,\beta} f(x) \le c M^\beta f(x), \qquad \text{where} \quad M^\beta f = \sup_{r>0} \frac{|x|^\beta}{r^n} \int_{|y-x|<r} \frac{|f(y)|}{|y|^\beta}\,dy, \tag{4.13}$$

{eq4.13:v

is the Hardy–Littlewood maximal weight function (the proof of this fact in [23] did not use the fact of the fact that $\Omega$ is a bounded set). Therefore, the boundedness of the operator $\mathcal{A}_{\alpha,\beta}$ in $L^{p(\cdot)}(\Omega)$ under conditions (4.12) is a consequence of well-known results on the weighted boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\Omega)$; see [24, Theorem C].

As to the operator $\mathcal{B}_{\alpha,\beta}$, note that it follows from the inequality $|x-y| \ge 2|x|$ that

$$|y| \ge |x-y| - |x| \ge |x-y| - \frac{|x-y|}{2} = \frac{|x-y|}{2},$$

i.e.,

$$\{y : |x-y| > 2|x|\} \subset \{y : |x-y| < 2|y|\}, \tag{4.14}$$

{eq4.14:v

and then

$$\mathcal{B}_{\alpha,\beta} f(x) \le |x|^{\beta-\alpha} \int_{|y-x|\le 2|y|} \frac{f(y)\,dy}{|y|^\beta |x-y|^{n-\alpha}} := \mathcal{B}_{\alpha,\beta} f(x),$$

so that it suffices that the operator $\mathcal{B}_{\alpha,\beta}$ be bounded. It remains to note that the operator adjoint to $B_{\alpha,\beta}$ is

$$\mathcal{B}^*_{\alpha,\beta} = \mathcal{A}_{\alpha,\alpha-\beta}.$$

In view of estimate (4.13), the boundedness conditions for the operator $\mathcal{B}^*_{\alpha,\beta}$ in the dual space $L^{p'(\cdot)}(\Omega)$ coincide with conditions (4.12), which proves Lemma 2. □

**Remark.** For another version of the Stein–Weiss weighted inequality with a variable exponent $p(x)$ from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p_\alpha(\cdot)}(\mathbb{R}^n)$, see [25, Theorem A].

## 5. PROOF OF THEOREM 3

{sec5:v57

Let us prove the following theorem, from which Theorem 3 readily follows.

{th4:v575

**Theorem 4.** *Suppose that* $0 < \alpha < n$, $\lambda \ge \alpha$, *and* $p \in \mathbb{P}(\mathbb{R}^n)$. *Under the condition* $\alpha p_+ < n$, *the operator*

$$(K\varphi)(x) := \frac{1}{(1+|x|)^\lambda} \int_{\mathbb{R}^n} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}$$

*is bounded as an operator from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to*

$$L^{p(\cdot)}(\mathbb{R}^n) \cap L^{p^\sharp(\cdot)}(\mathbb{R}^n, \varrho), \qquad \varrho(x) = (1+|x|)^\lambda, \quad p^\sharp(x) = \frac{np(x)}{n-\alpha p(x)}.$$

**Proof.** The boundedness of the operator $K$ from $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ to $L^{p_\alpha(\,\cdot\,)}(\mathbb{R}^n,\varrho)$ is a trivial consequence of Sobolev's theorem for the Riesz potential

$$g(x) := \int_{\mathbb{R}^n} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}$$

for variable exponents $p \in \mathbb{P}(\mathbb{R}^n)$ and the whole space $\mathbb{R}^n$; this fact was proved in [26] under the condition $\alpha p_+ < n$ (see Corollary 2.12 in [26]).

Under the assumptions (4.12), the boundedness of the operator $K$ in $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ follows from Lemma 2 (with $\beta = 0$). Note that, in the case $\lambda > \alpha$, the boundedness in $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ under the condition $\alpha p_+ < n$ can easily be obtained from Hölder's inequality without using such tools as the Hardy–Stein–Weiss inequality. Indeed, in view of the linearity of the operator $K$ and relations (2.3) and (2.4), it suffices to show that

$$|I^p(K\varphi)| \le c \qquad \text{for all} \quad \varphi(x), \quad \|\varphi\|_{p(\,\cdot\,)} \le 1.$$

We have

$$I^p(K\varphi)(x) = \int_{\mathbb{R}^n} [a(x)g(x)]^{p(x)}\,dx, \qquad \text{where} \quad a(x) = (1+|x|)^{-\lambda}.$$

Applying Hölder's inequality (2.2) with variable exponent $r(x) = p^\sharp(x)/p(x)$, $1 < r_- \le r_+ < \infty$, we obtain

$$I^p(K\varphi) \le k\big\||g(\,\cdot\,)|^{p(\,\cdot\,)}\big\|_{r(\,\cdot\,)}\big\|[a(\,\cdot\,)]^{p(\,\cdot\,)}\big\|_{r'(\,\cdot\,)}.$$

In view of relations (2.4) and the Sobolev's theorem for variable exponents (mentioned above), we have

$$\big\||g(\,\cdot\,)|^{p(\,\cdot\,)}\big\|_{r(\,\cdot\,)} \le C \qquad \text{for} \quad \|\varphi\|_{p(\,\cdot\,)} \le 1,$$

while, in view of the same relations (2.4), the second multiplier $\|[a(\,\cdot\,)]^{p(\,\cdot\,)}\|_{r'(\,\cdot\,)}$ is a finite constant, because $\lambda p(x) r'(x) = \lambda n/\alpha > n$. □

**Proof of Theorem 3.** To show the validity of the representation (1.5) with the operator $\mathbf{A}$ in the form (4.1) on the functions $\varphi \in L^{p(\,\cdot\,)}(\mathbb{R}^n)$ and the operator $\mathbf{R}$ expressed in the form (3.1), in view of the boundedness of the operator $\mathbf{A}$ (proved below), it suffices to note that the composition $\mathbf{RM}$ with the operator $\mathbf{R}$ expressed as (4.6), is bounded in $L^{p(\,\cdot\,)}(\mathbb{R}^n)$. This follows from the fact that the operator $\mathbf{M}$ is bounded from $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ to the space of Riesz potentials $I^\alpha[L^{p(\,\cdot\,)}(\mathbb{R}^n)]$, which can be shown in the same way as for a constant exponent $p$ in [3, Chap. 10] if we take into account the estimates of the truncated hypersingular integrals obtained for the space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ in [27], [28], after which it remains to use the boundedness of $\mathbf{R}\colon L^{p(\,\cdot\,)}(\mathbb{R}^n) \to I^\alpha[L^{p(\,\cdot\,)}(\mathbb{R}^n)]$, which was proved in [27].

In view of Lemma 1, the boundedness of the operator $\mathbf{A}$ follows from Theorem 4.

To prove the compactness of the operator $\mathbf{A}$ in the space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$, we proceed as follows. Using Theorem 2, for the space $L^{p(\,\cdot\,)}(\mathbb{R}^n)$ under consideration, we construct the interpolating pair of spaces $L^{p_0}(\mathbb{R}^n)$ and $L^{q(\,\cdot\,)}(\mathbb{R}^n)$; the first space has a constant exponent $p_0 = p_+$, while the second space has exponent

$$q(x) = \frac{p_+(1-\theta)p(x)}{p_+ - \theta p(x)}\,.$$

Since $p_+ < n/\alpha$, in view of results from [2], the operator $\mathbf{A}$ is compact in $L^{p_+}(\mathbb{R}^n)$. The boundedness of the operator $\mathbf{A}$ in the space $L^{q(\,\cdot\,)}(\mathbb{R}^n)$ can be obtained from the first part of Theorem 3. To this end, it is necessary to verify the validity of the conditions $q \in \mathbb{P}(\mathbb{R}^n)$ and $q_+ < n/\alpha$.

1) In view of Theorem 2, the inequalities $1 < q_- \le q(x) \le q_+ < \infty$ and $x \in \mathbb{R}^n$, are valid.

2) Let us prove the inequality

$$|q(x) - q(y)| \le \frac{A}{\ln(1/|x-y|)} \qquad \text{for all} \quad x, y \in \mathbb{R}^n, \quad |x-y| \le \frac{1}{2}\,.$$

We have

$$|q(x)-q(y)| = (1-\theta)\frac{|p(x)-p(y)|}{|1-\theta p(x)/p_+||1-\theta p(y)/p_+|}| \le \frac{A}{\ln(1/|x-y|)}\,. \tag{5.1}$$

{eq5.1:v5

3) The inequality

$$|q(x)-q_\infty| \le \frac{A}{\ln(2+|x|)}\,, \qquad x\in\mathbb{R}^n,$$

is proved just as above.

4) The inequality $q_+ < n/\alpha$ can also be verified directly:

$$\frac{1}{q(x)} = \frac{1}{1-\theta}\left(\frac{1}{p(x)}-\frac{\theta}{p_+}\right) \ge \frac{1}{1-\theta}\left(\frac{1}{p_+}-\frac{\theta}{p_+}\right) = \frac{1}{p_+}\,.$$

Thus, the operator $\mathbf{A}$ is compact in the space $L^{p_+}(\mathbb{R}^n)$ and is bounded in the space $L^{q(\cdot)}(\mathbb{R}^n)$. Therefore, by Theorem 1, it is compact in the interpolation space $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 3 is proved. $\square$

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