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Morrey-Campanato Spaces: an Overview

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Dedicated to the 70th anniversary of Professor Vladimir Rabinovich

Abstract. In this paper we overview known and recently obtained results on Morrey-Campanato spaces with respect to the properties of the spaces themselves, that is, we do not touch the study of operators in these spaces. In particular, we overview equivalent definitions of various versions of the spaces, the so-called φ- and θ-generalizations, structure of the spaces, embeddings, dual spaces, etc.

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Contents

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1. Introduction

We started our studies of various operators in Morrey and Campanato-type spaces several years ago, mainly in the case of maximal, singular and potential operators in such spaces with variable exponents and Hardy operators in Morrey spaces with constant exponents. We discovered that there existed a vast bibliography on the subject counting many hundreds of publications, especially on applications to differential equations. They include in particular the books A. Kufner, O. John and S. Fučík [63] (1977) and M. Giaquinta [40] (1983). We refer also to Section 27 of the book O.V. Besov, V.P. Il′in and S.M. Nikol′skiĭ [13] (1996) (see also the English translation [14, 15] of the first Russian edition of [13]) where an important overview on anisotropic Morrey type spaces may be found.

The earliest overview on Morrey-Campanato spaces seems to be first given in the paper J. Peetre [86] (1969). Probably the next one was M.H. Taibleson and G. Weiss [104] (1979).

During the last several decades there was a kind of a boom in studies in Morrey-Campanato-type spaces and their usage in applications, both enriching each other. Many of them, as well as various old results, were not covered in the existing surveys or books, but were of interest.

In the study of this topic and search of references, also in the historical retrospective, we made many notes in our notebooks. Our personal overview of those notes led us to the idea to collect and edit them, and publish it as a survey which may be useful for others involved into research around the Morrey-Campanato-type spaces.

This however led us to a manuscript exceeding one hundred pages, which is not well suited for a paper. About 4/5 of that overview was naturally related to the study of various operators, mainly classical operators of harmonic analysis, in Morrey-Campanato-type spaces, and about 1/5 of it was connected with the spaces themselves, i.e., proper definitions of various versions of the spaces, study of the structure of the spaces, preduals, etc. We made a decision to restrict ourselves to this first portion. It is presented in this paper. We hope to submit the remaining part for publication elsewhere. Note that in this paper we do not touch Sobolev-Morrey and Besov-Morrey type spaces as well as other generalizations of such a kind and refer a potential reader to Section 27 of the above cited books [13–15] and the recent book [114] (2010) titled "*Morrey and Campanato Meet Besov, Lizorkin and Triebel*".

The subjects we touch in this overview may be seen from Contents. Inside every Subsection we mainly follow the chronological order which more or less corresponds to a natural way of generalization from the simple to more advanced.

We could have lost some references. Anyway, we tried to do our best through a vast search in MathSciNet, MathNetRu and other sources. In the case the overview occasionally proves to be not complete in this or other item, we will be grateful to the readers for the indication of possible omissions. To be clear, we emphasize once again that in this survey we do not touch mapping properties of operators, so that many important papers on the behaviour of the classical operators of harmonic analysis in Morrey and Campanato spaces remained beyond this overview. We are aware of the fact that sometimes such a separation is rather relative because any property of an operator in a space may be considered as a property of the space. Nevertheless we had to follow the choice we made. Otherwise we would exceed any reasonable limit for this paper.

2. Morrey spaces

2.1. Classical Morrey spaces

The spaces which bear the name of Morrey spaces were introduced in 1938 by C. Morrey [71] in relation to regularity problems of solutions to partial differential equations.

We start from the definition of these spaces. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We denote $\widetilde{B}(x,r) = B(x,r) \cap \Omega, x \in \Omega, r > 0$, and $|A|$ will stand for the Lebesgue measure of a measurable subset in $\mathbb{R}^n$.

Definition 2.1 (Morrey spaces). Let $1 \le p < \infty$ and $\lambda \ge 0$. The *Morrey space* $L^{p,\lambda}(\Omega)$ is defined as

$$L^{p,\lambda}(\Omega) = \left\{ f \in L^p(\Omega) : \sup_{x\in\Omega; r>0} \frac{1}{r^\lambda} \int_{\widetilde{B}(x,r)} |f(y)|^p \, \mathrm{d}y < \infty \right\}. \tag{1}$$

This is a Banach space with respect to the norm

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x\in\Omega; r>0} \left(\frac{1}{r^\lambda} \int_{\widetilde{B}(x,r)} |f(y)|^p \, \mathrm{d}y \right)^{1/p}. \tag{2}$$

The space $L^{p,\lambda}(\Omega)$ is trivial when $\lambda > n$ ($L^{p,\lambda}(\Omega) = \{0\}$) and $L^{p,0}(\Omega) \cong L^p(\Omega)$ and $L^{p,n}(\Omega) \cong L^\infty(\Omega)$. In the case $\lambda \in (0, n]$, the space $L^{p,\lambda}(\Omega)$ is non-separable.

Note that for these spaces sometimes another notation, $M^{p,q}$, is used. Apart from the choice of a different letter M, the second parameter is also introduced into the norm in a way different from (2), namely

$$\|f\|_{M^{p,q}(\Omega)} := \sup_{x\in\Omega; r>0} r^{\frac{n}{q}-\frac{n}{p}} \|f\|_{L^p(\widetilde{B}(x,r))}.$$

In this survey we mainly follow the notation in (1)–(2).

The local version of such spaces, with only one point $x = 0$ taken into account, has a connection with studies of N. Wiener [111] (1930), [112] (1932), who considered functions f for which

$$\frac{1}{T^{1-\alpha}} \int_0^T |f(x)|^p \, \mathrm{d}x, \quad \alpha \in (0,1), \quad p = 1 \text{ or } p = 2$$

is limited in $T > 0$ or tends to zero as $T \to \infty$. In the multidimensional case such local spaces defined by the norm

$$\|f\|_{B^p} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \tag{3}$$

appeared in A. Beurling [16] (1964) as the dual of the so-called *Beurling algebra*. He also considered similar spaces with $\sup_{r>1}$ instead of $\sup_{r>0}$. Similar local Morrey type spaces with the norm of type (3) where $\frac{1}{|B(0,r)|}$ is replaced by $\frac{1}{|B(0,r)|^\lambda}$ appeared in V.S. Guliev [47] (1994), see also [50] (1996), and in J. García-Cuerva and M.J.L. Herrero [38] (1994). In [38] and J. Alvarez, M. Guzmán-Partida and J. Lakey [8] (2000) there were introduced the function space $B^{q,\lambda}(\mathbb{R}^n)$ characterized by the norm

$$\|f\|_{B^{p,\lambda}} = \sup_{r>1} \left(\frac{1}{|B(0,r)|^{1+\frac{\lambda}{p}}} \int_{B(0,r)} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \tag{4}$$

(called inhomogeneous) and also its homogeneous version $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ of type (4) with the supremum taken over $r > 0$.

Morrey spaces are a particular case of Campanato spaces considered in Section 4 and we present many results for Morrey spaces in that section in the context of Campanato spaces. Nevertheless, in this section we dwell on some results just for Morrey spaces.

2.1.1. Embeddings in Morrey spaces. By application of the Hölder inequality to integrals over $\widetilde{B}(x,r)$ the embedding for Morrey spaces follows:

Theorem 2.2. *Let $1 \le p \le q < \infty$ and let λ, ν be non-negative numbers. Then*

$$L^{q,\nu}(\Omega) \hookrightarrow L^{p,\lambda}(\Omega) \tag{5}$$

under the condition

$$\frac{\lambda - n}{p} \le \frac{\nu - n}{q} \tag{6}$$

if $|\Omega|$ *is finite and the condition*

$$\frac{\lambda - n}{p} = \frac{\nu - n}{q} \tag{7}$$

if $|\Omega|$ *is infinite.*

Condition (6) is necessary and sufficient for embedding (5) in case of "nice" sets Ω, see L.C. Piccinini [89] (1969), where $\Omega = Q_0$ was a cube in $\mathbb{R}^n$, see also a similar result for a modification $L_r^{p,\lambda}, p, r \in [1, \infty)$ of Morrey spaces in Y. Furusho [36] (1980). This modification is introduced as follows: let $\overline{S}$ be the family of all systems $S = \{Q_j \colon \bigcup Q_j \subset Q_0\}$ consisting of a finite number of non-intersecting parallel subcubes Q_j, and let $\|u\|_{L^{(p,\lambda)}(Q_j)} = \sup_{Q \subset Q_j} |Q|^{\frac{\lambda - n}{np}} \|u\|_{L^p(Q)}$, and

$$\|u\|_{L_r^{p,\lambda}(Q_0)} = \sup_{S \in \overline{S}} \left\{ \sum_{Q_j \in S} \|u\|^r_{L^{(p,\lambda)} Q_j} \right\}^{1/r};$$

there is proved a necessary and sufficient condition for the validity of the embedding $L_r^{p,\lambda} \hookrightarrow L_s^{q,\mu}$ in the case of $n/r - \lambda/p \le 1$ and $n/s - \mu/q \le 1$.

See also embedding theorems for Campanato spaces in Subsection 4.1.

2.1.2. Hölder's inequality. For Morrey spaces the following Hölder type inequality holds (obtained by application of the usual Hölder inequality to integrals over $\widetilde{B}(x,r)$, see for instance Lemma 11 in P. Olsen [78] (1995)).

Theorem 2.3 (Hölder's inequality in Morrey spaces). *Let* $f \in L^{p,\lambda}(\Omega)$ *and* $g \in L^{q,\mu}(\Omega)$*. Then*

$$\|fg\|_{L^{r,\nu}(\Omega)} \le \|f\|_{L^{p,\lambda}(\Omega)} \|g\|_{L^{q,\mu}(\Omega)}, \tag{8}$$

where $1 \le p < \infty, 1 \le q < \infty, \ \frac{1}{p} + \frac{1}{q} \ge 1$ *and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{\nu}{r} = \frac{\lambda}{p} + \frac{\mu}{q}.$$

2.1.3. Weak Morrey spaces. Weak Morrey-Campanato spaces appeared already in the paper by S. Spanne [100] (1966), see also Subsection 4.3. Such weak-type Morrey spaces defined by the condition

$$\sup_{\substack{t>0 \\ x \in \Omega}} t^p |\{y \in \Omega : |f(y)| > t\} \cap B(x,r)| \le C r^{\lambda}$$

where $\Omega \subset \mathbb{R}^n$, were used by M. Ragusa [91] (1995). In the paper C. Miao and B. Yuan [70] (2007) weak Morrey spaces $M^*_{p,\lambda}$ were defined in a more general setting in terms of Lorentz spaces of functions initially defined on non-atomic measurable spaces. For the spaces $M^*_{p,\lambda} = \{f : \|f\|^*_{p,\lambda} < \infty\}$ introduced via the norm

$$\|f\|^*_{p,\lambda} = \sup_{x,r} r^{-\frac{\lambda}{p}} \sup_{t>0} t\mu\,\{y : |f(y)| > t, \quad y \in B(x,r)\}$$

there were proved an embedding theorem and a convexity property.

2.1.4. Interpolation. G. Stampacchia [101] (1964), [102] (1965) and S. Campanato & M. Murthy [21] (1965) proved interpolation properties of Morrey spaces (in fact they obtained the result for the more general space, now called Campanato space, see its definition in Section 4). Loosely speaking, they proved (in the spirit of Riesz-Thorin interpolation theorem) that if T is a bounded linear operator from L^{q_i} to L^{p_i,λ_i}, $i = 1, 2$, then T is bounded from L^q to $L^{p,\lambda}$ for the corresponding intermediate values of p, q and λ, see the precise formulation in Theorem 4.5 in the setting of Campanato spaces. The conclusion in the other direction is false, see the comments after Theorem 4.5.

2.1.5. Preduals. Recall that for a given normed space X, a normed space Y is called predual of X, if X is dual of Y.

Preduals of Morrey spaces were studied by some authors, namely by C. Zorko [115] (1986), D.A. Adams [3] (1988), E.A. Kalita [57] (1998) and D.R. Adams and J. Xiao [4] (2004). Following D.R. Adams and J. Xiao, we denote the preduals obtained in [115], [57] and [4] by $Z^{q,\lambda}$, $K^{q,\lambda}$ and $H^{q,\lambda}$, respectively, $q = \frac{p}{p-1}$. The first two spaces are defined by the following norms

$$\|f\|_{Z^{q,\lambda}} = \inf \left\{ \|\{c_k\}\|_{\ell^1} :\ f = \sum_k c_k a_k \right\}$$

where a_k is a (q,λ)-atom and the infimum is taken with respect to all possible atomic decompositions of f (a function a on $\mathbb{R}^n$ is called a (q,λ)-atom, if it is supported on a ball $B \subset \mathbb{R}^n$ and $\|a\|_q \le |B|^{-\frac{\lambda}{np}}$); note that in C. Zorko [115] the predual was introduced in a more general setting of generalized Morrey spaces;

$$\|f\|_{K^{q,\lambda}} = \inf_\sigma \left(\int_{\mathbb{R}^n} |f(x)|^q \omega_\sigma^{1-q}(x)\,\mathrm{d}x \right)^{1/q},$$

with

$$\omega_\sigma(x) = \int_{\mathbb{R}^{n+1}_+} r^{-\lambda} 1_{\mathbb{R}^1_+}(r - |x-y|)\,\mathrm{d}\sigma(y,r),$$

where the infimum is taken over all non-negative Radon measures $\sigma(y,r)$ on $\mathbb{R}^{n+1}_+$ with the normalization $\sigma(\mathbb{R}^{n+1}_+) = 1$;

$$\|f\|_{H^{q,\lambda}} = \inf_\omega \left(\int_{\mathbb{R}^n} |f(x)|^q \omega^{1-q}(x)\,\mathrm{d}x \right)^{1/q},$$

where the infimum is taken over all nonnegative functions on $\mathbb{R}^n$ satisfying the condition

$$\|\omega\|_{L^1(\Lambda_\lambda^{(\infty)})} \le 1, \tag{9}$$

with the λ-dimensional Hausdorff capacity $\Lambda_\lambda^{(\infty)}$, the introduction of the latter norm in [4] being based on the previous studies in [3].

As shown in [4], for $1 < p < \infty$, $0 < \lambda < n$,

$$Z^{q,\lambda} = K^{q,\lambda} = H^{q,\lambda} \quad \text{with} \quad \|f\|_{Z^{q,\lambda}} \sim \|f\|_{K^{q,\lambda}} \sim \|f\|_{H^{q,\lambda}}$$

and the Morrey space may be characterized in terms of its predual by the following theorem.

Theorem 2.4. *Let* $1 < p < \infty,\ 0 < \lambda < n$. *Then*

$$\|f\|_{L^{p,\lambda}} = \sup_{\omega} \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, \mathrm{d}x \right)^{1/p}$$

where the supremum is taken with respect to all nonnegative functions on $\mathbb{R}^n$ *satisfying the condition* (9).

An interested reader may be also referred to Sections 5–7 of [4] with respect to Morrey type capacities.

In the case of Campanato spaces, M.H. Taibleson and G. Weiss [105] (1980) proved that they are dual to some Hardy spaces.

2.1.6. Vanishing Morrey spaces $VL^{p,\lambda}$. Morrey space $L^{p,\lambda}$, as noted, is not separable in the case $\lambda > 0$. A version of Morrey space where it is possible to approximate by "nice functions" is the so-called vanishing Morrey space $VL^{p,\lambda}(\Omega)$ introduced by C. Vitanza [110] (1990). This is a subspace of functions in $L^{p,\lambda}(\Omega)$, which satisfy the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<\varrho<r}} \frac{1}{\varrho^\lambda} \int_{\widetilde{B}(x,\varrho)} |f(y)|^p \, \mathrm{d}y = 0. \tag{10}$$

2.1.7. Different underlying spaces. The spaces $L^{p,\lambda}$ may be introduced on sets of different nature, for instance, an n-dimensional compact manifold via local charts (see M. Geisler [39] (1988)) where the spaces introduced in this way were characterized in terms of geodesic distances and other quantities on the manifold. In Subsection 2.2 we touch a more general setting when the underlying space is a quasimetric measure space. Morrey spaces and their generalizations in the case where the underlying spaces is the Heisenberg group were studied in V. Gulyiev [50] (1996).

2.1.8. Anisotropic Morrey spaces. Morrey spaces corresponding to anisotropic distances appeared first in G. Barozzi [12] (1965) defined in the following way. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p \geq 1$ and $0 \leq \lambda \leq n$. Let $\overline{m} = (m_1, \dots, m_n)$ be an n-tuple of non-negative numbers, $m_j \geq 1$ and $m = \max(m_1, \dots, m_n)$. Let $B_{\overline{m}}(x,r) = \{y \in \Omega : d_{\overline{m}}(x,y) < r\}$ be an anisotropic ball defined by the distance

$$d_{\overline{m}}(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^{m_j} \right)^{1/m}.$$

Then the corresponding Morrey space is introduced by the condition

$$\sup_{x,r} \frac{1}{r^\lambda} \int_{B_{\overline{m}}(x,r)} |f(y)|^p \, \mathrm{d}y < \infty.$$

The corresponding anisotropic Sobolev spaces were also introduced in [12].

In a more general setting such anisotropic Morrey spaces were later studied by V.P. Il′in in [52] (1959), [53](1971), see the presentation of the latter results also in Section 27 of the book [13].

Morrey spaces with integral means over one-parametrical ellipsoids were introduced in L. Softova in [98] (2007) with the aim to study anisotropic singular integrals. Let $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a given vector with $\alpha_i \geq 1, i = 1, \dots, n$, and

$$\mathscr{E}_{\overline{\alpha}}(x,r) = \left\{ y \in \mathbb{R}^n \colon \sum_{k=1}^{n} \frac{(x_k - y_k)^2}{r^{2\alpha_k}} < 1 \right\} \tag{11}$$

be an ellipsoid centered at the point $x \in \mathbb{R}^n$. Then the anisotropic space $L^{p,\lambda}(\mathbb{R}^n)$ localized at the origin and corresponding to the given vector $\overline{\alpha}$, is defined by the norm

$$\|f\|_{p,\lambda} = \sup_{r>0} \left(\frac{1}{r^\lambda} \int_{\mathscr{E}_{\overline{\alpha}}(0,r)} |f(y)|^p \, \mathrm{d}y \right)^{1/p} < \infty. \tag{12}$$

See also Subsection 3.1 for the generalized anisotropic Morrey spaces of such a kind introduced in L. Softova [97] (2006).

Anisotropic Morrey spaces $L^{p,\overline{\lambda}}(\Omega)$, $\overline{\lambda} = (\lambda_1, \dots, \lambda_n)$ may be also introduced, with means taken over rectangles centered at the point x with independent lengths of sides. Such spaces $\mathcal{L}^{p,\lambda_1,\lambda_2}(\mathbb{R}^2_+)$ were introduced in L.-E. Persson and N. Samko [88] (2010) for the case $\Omega = \mathbb{R}^2_+$ by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda_1,\lambda_2}} = \sup_{\substack{x_1>0, x_2>0 \\ r_1>0, r_2>0}} \left(\frac{1}{r_1^{\lambda_1} r_2^{\lambda_2}} \int_{(x_1-r_1)_+}^{x_1+r_1} \int_{(x_2-r_2)_+}^{x_2+r_2} |f(y_1,y_2)|^p \, \mathrm{d}y_1 \, \mathrm{d}y_2 \right)^{1/p} \tag{13}$$

with the aim to study two-dimensional Hardy operators in such spaces.

2.1.9. Miscellaneous. As is well known, Morrey spaces have been generalized or modified in various ways in order to obtain existence and uniqueness of solutions to partial differential equations. One of such modifications, $L^{p,\lambda}(\Omega,t)$ introduced in M. Transirico et al. [108] (1995) (with $t=1$) and A. Canale et al. [22] (1998), is aimed to better reflect the local nature of solutions, first of all for unbounded domains, being defined by the norm

$$\|f\|_{L^{p,\lambda}(\Omega,t)} = \sup_{\substack{x\in\Omega \\ 0<r<t}} \left(\frac{1}{r^\lambda} \int_{\widetilde{B}(x,r)} |f(y)|^p \, \mathrm{d}y \right)^{1/p} ;$$

in [22] the corresponding Sobolev spaces were also dealt with.

In P. Cavaliere, G. Manzo and A. Vitolo [23](1996) Morrey spaces were intentionally studied on unbounded domains with the main emphasis on the connection between Morrey type and BMO spaces and embedding and density results involving the continuity of the translation operator.

Another modification of Morrey spaces is known under the name of Stummel class introduced in M.A. Ragusa and P. Zamboni [92] (2001) (with the goal to obtain a better version of the Sobolev type embedding). The Stummel class is defined, for $0<p<n$, as

$$S_p = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \lim_{r\to 0} \eta(r) = 0, \quad \eta(r) = \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-p}} \, \mathrm{d}y \right\},$$

which is the Stummel-Kato class in the case $p=2$. Note that
$\eta(r) \geq \sup\limits_{x\in\mathbb{R}^n} \frac{1}{r^{n-p}} \int\limits_{|x-y|<r} |f(y)| \, \mathrm{d}y$.

In general $L^{1,\lambda}$ is contained in S_p, if $\lambda > n-p$, and in the case $\eta(r) \sim r^\alpha$ the following equivalence holds:

$$f \in S_p \quad \Longleftrightarrow \quad f \in L^{1,n-p+\alpha},$$

see Lemma 1.1 in [92]. Some versions of Stummel classes with η different from powers are also studied there, which corresponds to the generalized Morrey spaces studied in Subsection 3.

S. Leonardi [64] (2002) introduced a similar version of such a space, defined by the norm

$$\|f\|_{N^{p,\lambda}(\Omega)} := \sup_{x\in\Omega} \left\{ \int_\Omega \frac{|f(y)|^p}{|x-y|^\lambda} \, \mathrm{d}y \right\}^{1/p}$$

and proved a certain version of the Miranda-Talenti inequality in terms of Sobolev type spaces related to the norms $\|f\|_{N^{p,\lambda}(\Omega)}$.

A more general hybrid of Morrey and Stummel type spaces, the space denoted by $M^{p,\lambda}_\beta(X,\mu)$, was introduced in Eridani, V. Kokilashvili and A. Meskhi [34] on a quasi-metric measure space (X, ρ, μ), with the norm defined by

$$\|f\|_{M^{p,\lambda}_\beta} := \sup_{\substack{x\in X\\ r>0}} \left(\frac{1}{r^\lambda} \int_{\rho(x,y)<r} |f(y)|^p \rho^\beta(x,y)\, \mathrm{d}\mu(y) \right)^{1/p}.$$

2.2. Morrey spaces over $\mathbb{R}^n$ in case of a general measure

Y. Sawano and H. Tanaka [95] (2005) introduced Morrey spaces in $\mathbb{R}^n$, but with a Radon measure μ as follows

$$\mathscr{M}^p_q(k,\mu) = \left\{ f\colon \sup_Q |\mu(kQ)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, \mathrm{d}\mu \right)^{1/q} < \infty \right\}, \tag{14}$$

where Q is a closed cube whose edges are parallel to the coordinate axes and it is supposed that the measure μ is not necessarily a doubling measure but satisfies the growth condition

$$\mu(B(x,r)) \le c_0\, r^\ell$$

for some fixed constants $c_0 > 0$ and $\ell \in (0,n]$, and $\mu(Q) > 0$. It is shown that the definition of the space does not depend on the choice of the parameter $k > 1$, that is,

$$\mathscr{M}^p_q(k_1,\mu) = \mathscr{M}^p_q(k_2,\mu) \tag{15}$$

for all $k_1 > 1, k_2 > 1$, up to equivalence of norms. More precisely

$$\|f\|_{\mathscr{M}^p_q(k_1,\mu)} \le \|f\|_{\mathscr{M}^p_q(k_2,\mu)} \le C_n \left(\frac{k_1-1}{k_2-1} \right)^n \|f\|_{\mathscr{M}^p_q(k_1,\mu)} \tag{16}$$

for $1 < k_1 < k_2 < \infty$, see formula (3) in [96]. In [96] there was also made a comparison of the space $\mathscr{M}^p_q(2,\mu)$ with the space $\mathscr{M}^p_q(1,\mu)$, the latter being defined with the usage of cubes Q which only satisfy the condition $\mu(kQ) \le \beta\mu(Q)$ with $\beta > k^{\frac{npq}{p-q}}$ where $k > 1$ is fixed and the measure μ does not necessarily satisfies the growth condition or the doubling condition. This comparison includes also the case of vector-valued Morrey spaces $\mathscr{M}^p_q(\ell^r,\mu)$ defined by

$$\|f_j\|_{\mathscr{M}^p_q(\ell^r,\mu)} := \sup_{Q\in\mathcal{Q}(\mu;k;\beta)} \mu(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|f_j\|^q_{\ell^r} \, \mathrm{d}\mu \right)^{1/q} < \infty.$$

For similar results on Campanato spaces, we refer to Section 4.

For Morrey spaces in a more general setting of abstract quasimetric measure spaces see Subsection 3.1.

3. Generalized Morrey spaces

Recall that the classical Morrey space is defined by the norm.

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{x\in\Omega} \left\| \frac{1}{r^{\frac{\lambda}{p}}} \|f\|_{L^p(\widetilde{B}(x,r))} \right\|_{L^\infty(0,d)}, \quad d = \operatorname{diam}\Omega. \tag{17}$$

There are known two types of generalizations of Morrey spaces. The first is to replace the power function r^λ by a function $\varphi(r)$ (or more generally $\varphi(x,r)$), usually with some quasi monotonicity type conditions with respect to r. Another way is to replace the $L^\infty(0,d)$-norm by $L^\theta(0,d)$-norm, $0<\theta<\infty$. For brevity, we will call these by φ-generalizations and θ-generalizations. Both ways may be naturally mixed.

3.1. φ-generalizations

Let X be a quasimetric space with a Borel measure μ. The generalized Morrey space is defined by the (quasi)norm

$$\|f\|_{p,\varphi} = \sup_{x,r}\left(\frac{1}{\varphi(x,r)}\int_{B(x,r)}|f(y)|^p\,\mathrm{d}\mu(y)\right)^{1/p}, \quad 0<p<\infty, \tag{18}$$

where $B(x,r)$ is a ball in X and the non-negative function φ is subject to some restrictions, usually related to monotonicity-type conditions in r. Generalized Morrey spaces, $L^{p,\varphi,S}$, of such a type seem to first appear in the paper G.T. Dzhumakaeva and K. Zh. Nauryzbaev [31] (1982), where the norm is introduced by

$$\|f\|_{p,\varphi,S} = \sup_{E\in S}\frac{1}{\varphi(|E|)}\left(\int_{E\cap\Omega}|f(y)|^p\,\mathrm{d}y\right)^{1/p}<\infty,$$

$1\le p<\infty$, Ω is a domain of finite measure in $\mathbb{R}^n$, S is the family of all measurable subsets of Ω and $\varphi(r)$ is a positive nondecreasing function on $\mathbb{R}^1_+$. Under the assumption that $\varphi(r)=1$ for $r\ge 1$ and that $\varphi^p(r)$ is concave in $(0,1)$, in [31] there was proved that $L^{p,\varphi,S}(\Omega)\subset L^q(\Omega)$, $p<q\le\infty$, if and only if $\int_0^1 r^{-q/p}\varphi^q(r)\,\mathrm{d}r<\infty$, with the corresponding interpretation for $q=\infty$.

The generalized Morrey spaces $L^{p,\varphi}(\Omega)$ defined by the norm

$$\|f\|_{p,\varphi} = \sup_{x,r}\left(\frac{1}{\varphi(r)}\int_{\widetilde{B}(x,r)}|f(y)|^p\,\mathrm{d}\mu(y)\right)^{1/p}, \quad 1\le p<\infty, \tag{19}$$

were studied in the paper C. Zorko [115] (1986) in a more general setting of Campanato spaces, see Section 4. We mention the result from [115, Prop. 2] stating that the zero continuation of a function $f\in L^{p,\varphi}(\Omega)$ belongs to $L^{p,\varphi}(\mathbb{R}^n)$ under the assumption that the function φ is nondecreasing. In [115, Prop. 3] there was also shown a possibility to approximate by nice functions in the subspace of $L^{p,\varphi}(\mathbb{R}^n)$ defined by the condition $\lim_{y\to 0}\|f(\cdot-y)-f(\cdot)\|_{L^{p,\varphi}}=0$ (recall that Morrey spaces are not separable).

Often the (quasi)norm in such a generalized Morrey space is taken in the form

$$\|f\|_{L^p_\psi} = \sup_{B(x,r)}\frac{1}{\psi(x,r)}\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)|^p\,\mathrm{d}\mu(y)\right)^{1/p}, \quad 0<p<\infty, \tag{20}$$

in particular in the form

$$\|f\|_{L^p_\psi(w)} = \sup_B \frac{1}{\psi(|B|)} \left(\frac{1}{|B|} \int_B |f(y)|^p \, w(y) \, \mathrm{d}y \right)^{1/p}, \quad 0 < p < \infty, \tag{21}$$

in the case $X = \mathbb{R}^n$.

With the norm of form (18), such spaces appeared in E. Nakai [72] (1994) for $X = \mathbb{R}^n$, and the spaces $L^p_\psi(X)$ with the (quasi)norm (21) in J. Alvarez and C. Pérez [9] (1994) and with the norm (20) in E. Nakai [73] (1997).

In [73] there were studied the pointwise multipliers from such a space $L^p_\psi(X)$ to another one of similar type. Let $\mathrm{PWM}(E, F)$ denote the set of pointwise multipliers from E to F. Under some assumptions on ψ_1 and ψ_2, it was proved that

$$\mathrm{PWM}(L^{p_1}_{\psi_1}, L^{p_2}_{\psi_2}) = L^{p_3}_{\psi_3}, \tag{22}$$

where $1/p_1 + 1/p_3 = 1/p_2$, $0 < p_2 < p_1 < \infty$ and $\psi_3 = \psi_2/\psi_1$. In E. Nakai [74] (2000) there were obtained necessary conditions on p_i and ψ_i for (22) to be valid, and sufficient conditions for $\mathrm{PWM}(L^{p_1}_{\psi_1}, L^{p_2}_{\psi_2}) = \{0\}$.

In the paper H. Arai and T. Mizuhara [10] (1997) the generalized Morrey spaces with the norm of the type (18) were considered within the framework of homogeneous underlying space, normal in the sense of Macías and Segovia [67], under the assumption that $\varphi(x, r)$ is increasing in r and satisfies the doubling condition uniformly in x. There was proved a general theorem which allows to obtain estimates of the form

$$\|F\|_{L^{p,\varphi}} \leq C\|G\|_{L^{q,\varphi}}$$

from estimates of the form $\int F^p w d\mu \leq C \int G^q w d\mu$, where w ranges some subclasses of the Muckenhoupt class $A_1(\mu)$. This important result was used to obtain Morrey space estimates for various classical operators.

Relations between the generalized Morrey spaces with the norm (21) and the corresponding Stummel classes (see section 2.1.9) were studied in Eridani and H. Gunawan [33] (2005), the results adjoin to those for the case where ψ is a power function.

In E. Nakai [75] (2006) the generalized Morrey spaces, with the norm defined as in (20), appeared in the case where the underlying space X was a homogeneous metric measure space.

In L. Softova [97] (2006) and [98] (2007) there were introduced the generalized anisotropic Morrey spaces with the aim to study anisotropic singular integrals. Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a given vector with $\alpha_i \geq 1, i = 1, \ldots, n$, and $\mathscr{E}_{\overline{\alpha}}(x, r)$ the ellipsoid defined in (11). Then the anisotropic space $L^{p,\varphi,\overline{\alpha}}(\mathbb{R}^n)$ is defined by the norm

$$\|f\|_{p,\varphi,\overline{\alpha}} = \sup_{x,r} \left(\frac{1}{\varphi(x,r)} \int_{\mathscr{E}_{\overline{\alpha}}(x,r)} |f(y)|^p \, \mathrm{d}y \right)^{1/p} < \infty.$$

As a generalization of results from Y. Sawano and H. Tanaka [95] (see Subsection 2.2), Y. Sawano in [94] (2008) dealt with the generalized Morrey spaces defined by the condition

$$\sup_Q \left(\frac{1}{\varphi(\mu(kQ))} \int_Q |f|^p \,\mathrm{d}\mu \right)^{1/p} < \infty,$$

where $1 \leq p < \infty, k > 1$, φ is an increasing function, Q is a cube with edges parallel to the coordinate axes, and μ is a positive Radon measure, non necessarily satisfying the doubling condition. The independence of such spaces on the choice of $k > 1$, as in (15)–(16), is extended to this setting.

Y. Komori and S. Shirai [60] (2009) considered the generalized Morrey spaces $L^{p,\kappa}(w)$, defined by the norm

$$\|f\|_{L^{p,\kappa}(w)} = \sup_Q \left(\frac{1}{w(Q)^\kappa} \int_Q |f(x)|^p w(x) \,\mathrm{d}x \right)^{1/p}, \quad w(Q) = \int_Q w(x) \,\mathrm{d}x, \quad (23)$$

where $0 < \kappa < 1$ and the supremum is taken over all cubes in $\mathbb{R}^n$, which is nothing else, but the usual Morrey space with respect to the measure $\mu(E) = \int_E w(x) \,\mathrm{d}x$; the authors called this space *weighted*. Note that if we interpret the space $L^{p,\kappa}(w)$ as a weighted generalized Morrey space, then given the function w, the function $\varphi = w^\kappa$ already defines the generalized Morrey space, this meaning that the space $L^{p,\kappa}(w)$, introduced in this way, is not a space with an arbitrary weight, but with a special weight equal to a power of the function φ.

3.2. θ-generalizations

A Morrey-type space with $\sup_{r>0}$ replaced by the $\|\cdot\|_{L^\theta}(0,\infty)$-norm first appeared in D.R. Adams [5], p. 44 (1981) with the norm defined by

$$\|f\|_{L^{p,\theta,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \left(\int_0^\infty \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \,\mathrm{d}y \right)^{\theta/p} \frac{\mathrm{d}r}{r} \right)^{1/\theta} \quad (24)$$

where the corresponding Sobolev type theorem for the Riesz potential operator was stated. Spaces with both θ- and φ-generalization, but “localized” to the point $x = 0$, with the norm

$$\|f\|_{L^{p,\theta,\varphi}_{\mathrm{loc},0}(\mathbb{R}^n)} := \left(\int_0^\infty \left(\frac{1}{\varphi(r)} \int_{B(0,r)} |f(y)|^p \,\mathrm{d}y \right)^{\theta/p} \frac{\mathrm{d}r}{r} \right)^{1/\theta} \quad (25)$$

were introduced and intensively studied by V.S. Guliyev [47] (1994) together with the study of the classical operators in these spaces, see also the books V.S. Guliyev [50] (1996) and [51] (1999) where these results were presented for the case when the underlying space is the Heisenberg group or a homogeneous group, respectively. Note that these investigations appeared in fact independently of the development of the main trends in the theory of Morrey spaces and their applications. They had as a background the usage of the local characteristics

$$\Omega(f,r) = \int_{\mathbb{R}^n\setminus B(x,r)} |f(y)|^p \,\mathrm{d}y \quad \text{and} \quad \Omega^*(f,r) = \int_{B(x,r)} |f(y)|^p \,\mathrm{d}y$$

widely used in Baku mathematical school (A.A. Babaev and his students) for a characterization of weighted Hölder and other spaces, we refer for instance to the papers [11] and [1], [2].

In the case $\theta = p$ the spaces $L^{p,\theta,\varphi}_{\mathrm{loc},0}(\mathbb{R}^n)$ coincide with a certain weighted Lebesgue spaces:

$$L^{p,p,\varphi}_{\mathrm{loc},0}(\mathbb{R}^n) = L^p(\mathbb{R}^n, w), \quad w(x) = \int_{|x|}^\infty \frac{\mathrm{d}r}{r\varphi(r)}.$$

In a series of papers by V. Burenkov, H. Guliyev and V. Guliyev related to such spaces, this "localized" version with the norm (25), where $p,\theta \in (0,\infty)$, was called "local Morrey-type space" and the version with the norm

$$\|f\|_{L^{p,\theta,\varphi}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \left(\int_0^\infty \left(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p \,\mathrm{d}y \right)^{\theta/p} \frac{\mathrm{d}r}{r} \right)^{1/\theta}, \tag{26}$$

the "global Morrey-type space", with $p,\theta \in (0,\infty)$. As shown in V.I. Burenkov and H. Guliyev [18] (2004), such space $L^{p,\theta,\varphi}(\mathbb{R}^n)$ is "reasonable" under the assumptions

$$\left\| \frac{1}{\varphi^{1/p}} \right\|_{L^\theta(t_1,\infty)} < \infty \quad \text{and} \quad \left\| \frac{r^{\frac{n}{p}}}{\varphi^{1/p}} \right\|_{L^\theta(0,t_2)} < \infty$$

for some $t_1, t_2 \in (0,\infty)$, being trivial ($L^{p,\theta,\varphi}(\mathbb{R}^n) = \emptyset$) if one of these conditions is violated; the space $L^{p,\theta,\varphi}_{\mathrm{loc},0}$ is also trivial if the second condition is violated, and the function in $L^{p,\theta,\varphi}_{\mathrm{loc},0}$ must vanish in a sense at the origin, if the first condition does not hold.

4. Campanato spaces

Campanato spaces, also referred to sometimes as Morrey-Campanato spaces, were introduced by S. Campanato [19] (1963) (in the case of bounded domains in $\mathbb{R}^n$); in 1964 they also appeared in the paper of G. Stampacchia [101]. They are a generalization of the BMO spaces of functions of bounded mean oscillation introduced by F. John and L. Nirenberg [56] (1961) and defined, for open sets $\Omega \subseteq \mathbb{R}^n$, by the seminorm

$$[f]_{\mathrm{BMO}} := \sup_{x,r} \frac{1}{|\widetilde{B}(x,r)|} \int_{\widetilde{B}(x,r)} \left| f(y) - f_{\widetilde{B}(x,r)} \right| \mathrm{d}y.$$

4.1. Definitions and basic facts

Definition 4.1 (Campanato spaces). Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 \le p < \infty$ and $\lambda \ge 0$. The *Campanato space* $\mathscr{L}^{p,\lambda}(\Omega)$ is defined as

$$\mathscr{L}^{p,\lambda}(\Omega) := \left\{ f \in L^p(\Omega) : [f]_{\mathscr{L}^{p,\lambda}(\Omega)} < \infty \right\} \tag{27}$$

the *Campanato seminorm* being given by

$$[f]_{\mathscr{L}^{p,\lambda}(\Omega)} := \sup_{x\in\Omega; r>0} \left(\frac{1}{r^\lambda} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|^p \, \mathrm{d}y \right)^{1/p}$$

or equivalently

$$\sup_{x\in\Omega; r>0} \left(\frac{1}{r^\lambda} \inf_{c\in\mathbb{R}^1} \int_{\widetilde{B}(x,r)} |f(y) - c|^p \, \mathrm{d}y \right)^{1/p}. \tag{28}$$

The embedding theorem for Campanato spaces reads as follows (see [63, p. 217])

Theorem 4.2. *Let $1 \le p \le q < \infty$ and let λ, ν be non-negative numbers. If $|\Omega|$ is finite then*

$$\mathscr{L}^{q,\nu}(\Omega) \hookrightarrow \mathscr{L}^{p,\lambda}(\Omega) \tag{29}$$

under the condition

$$\frac{\lambda - n}{p} \le \frac{\nu - n}{q}. \tag{30}$$

In G. Stampacchia [102] (1965) there was introduced Campanato-type space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$ where Q_0 is a cube in $\mathbb{R}^n$ defined by the set of seminorms

$$K(Q_j) := \sup_{Q\subset Q_j} \left(\frac{1}{|Q|^{1-\lambda/n}} \int_Q |u(x) - u_Q|^p \, \mathrm{d}x \right)^{1/p} \tag{31}$$

where $\{Q_j : \cup Q_j \subset Q_0\}$ is a given family of cubes parallel to the cube Q_0, no two of which have common interior points, and the condition

$$\sup_{\{Q_j\}} \left(\sum_j |K(Q_j)|^r \right)^{\frac{1}{r}} < \infty \tag{32}$$

holds, where the supremum is taken with respect to all admissible families of cubes. In some papers such spaces were called strong Campanato spaces, see, e.g., [79, 84].

The importance of Campanato spaces stems from the fact that, for λ greater than n (and less than $n + p$), they coincide with the spaces of Hölder continuous functions, providing an integral characterization of such functions, while in the case $\lambda < n$ they coincide with Morrey spaces, as the theorem below states, proved in S. Campanato [19] (1963) (in [19] the domain was supposed to satisfy the condition (A) and have Lipschitz boundary; for the proof under the only condition (A) we refer to Section 4.3 of the book by A. Kufner et al. [63]), where the proof of the coincidence of the Campanato spaces with the BMO space in the case $\lambda = n$ may be also found.

We say that an open set $\Omega \subset \mathbb{R}^n$ is of type (A), if there exists a constant $A > 0$ such that

$$|\widetilde{B}(x,r)| \geq Ar^n, \tag{33}$$

and by $H^\alpha(\overline{\Omega})$ we denote the space of functions satisfying the Hölder condition in $\overline{\Omega}$.

Theorem 4.3. *Let* $1 \leq p < \infty$ *and* Ω *be a bounded domain of type* (A). *Then*

1. $\mathscr{L}^{p,\lambda}(\Omega) \cong L^{p,\lambda}(\Omega)$, *when* $\lambda \in [0, n)$,
2. $\mathscr{L}^{p,\lambda}(\Omega) \cong BMO(\Omega)$ *when* $\lambda = n$,
3. $\mathscr{L}^{p,\lambda}(\Omega) \cong H^\alpha(\overline{\Omega})$ *with* $\alpha = \frac{\lambda-n}{p}$, *when* $\lambda \in (n, n+p]$.

Note that the statement (3) of Theorem 4.3 for the case $p = 1$ was also proved in N. Meyers [69] (1964).

For strong Campanato spaces defined by (31) and (32), in A. Ono [79] (1970) there were obtained relations with Lipschitz spaces $\mathrm{Lip}(\alpha, p)$ of functions Hölder continuous in L^p-norm, and in A. Ono [83] (1978) in the final form as the statement

$$\mathscr{L}_r^{(p,\lambda)}(Q_0) \cong \mathrm{Lip}\left(\frac{n}{r} - \frac{n-\lambda}{p}, r\right),$$

with $1 \leq r < \infty$ and $0 < n/r - (n-\lambda)/p < 1$.

We refer also to A. Ono [80] (1972), A. Ono and Y. Furusho [84], A. Ono [82] (1977/1978), and A. Ono [81] (1977/1978) with regards to other results around the strong Campanato spaces.

In [20] (1964) S. Campanato introduced spaces $\mathscr{L}_k^{p,\lambda}(\Omega)$ of "higher order" defined by the seminorm

$$[f]_{\mathscr{L}_k^{p,\lambda}} := \sup_{x\in\Omega; r>0} \left(\frac{1}{r^\lambda} \inf_{P\in\mathcal{P}_k} \int_{\widetilde{B}(x,r)} |f(y) - P(y)|^p \, \mathrm{d}y \right)^{1/p} \tag{34}$$

where $\mathcal{P}_k$ is the class of polynomials of degree at most k and proved the following generalization of Theorem 4.3, where $C^{m,\alpha}(\Omega), m \geq 0, 0 < \alpha \leq 1$, stands for the class of functions continuous in $\overline{\Omega}$ with all the derivatives up to the order m and with the derivatives of order m in $H^\alpha(\overline{\Omega})$.

Theorem 4.4. *Let* $1 \leq p < \infty, k \geq 0$ *and* Ω *be a bounded domain of type* (A). *Then*

1. $\mathscr{L}_k^{p,\lambda}(\Omega) \cong L^{p,\lambda}(\Omega)$, *when* $\lambda \in [0, n)$,
2. $\mathscr{L}_k^{p,\lambda}(\Omega) \cong C^{m,\alpha}(\overline{\Omega})$ *with* $m = \left[\frac{n-\lambda}{p}\right], \alpha = \frac{\lambda-n}{p} - m$, *when* $n + mp < \lambda < n + (m+1)p,\ m = 0, 1, 2, \ldots, k$.

We refer to S. Janson et al. [55] (1983) for the alternative proof of Theorem 4.4 in the case $\Omega = \mathbb{R}^n$, which includes also the case $p = \infty$.

Note that the condition (A) is not necessary for the validity of the embedding $\mathscr{L}_k^{p,\lambda}(\Omega) \hookrightarrow C^{m,\alpha}(\overline{\Omega})$ but the inverse embedding in equivalence (2) in Theorem 4.4

essentially uses this condition. We refer to D. Opěla [85] (2003) for the study of the influence of the geometry of Ω on the inverse embedding.

4.2. DeVore-Sharpley-Christ versions of Campanato-type spaces

In R.A. DeVore and R.C. Sharpley [28] and M. Christ [25] there was introduced a version of Campanato-type spaces in which the L^∞-norm in x is replaced by L^p-norm, namely they introduced the space C_p^α defined for $1 \le q \le p$, by the norm

$$\|f\|_{C_p^\alpha} := \left[\int_\Omega \sup_{Q \ni x} \inf_{P \in \mathcal{P}_{[\alpha]}} \frac{1}{|Q|^{\frac{\alpha p}{n} + \frac{p}{q}}} \left(\int_Q |f(y) - P(y)|^q \, \mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right]^{1/p}, \tag{35}$$

where $\mathcal{P}_k$ stands for the class of polynomials of degree at most k, $k \ge 0$. This norm does not depend on $q \in [1, p]$, see [28, p. 36]. We refer to [28] for the study of various properties of these spaces such as comparison with Besov spaces, interpolation, embeddings, extension theorem, etc. These spaces may be also found in H. Triebel [109, Subsection 1.7.2.]. They are also known as *local approximation Campanato spaces*. In the case $p = 2$ we refer also to a paper [32] (2006) on a characterization of such spaces when α may be negative $(\alpha > -\frac{n}{2})$.

Spaces of the type $C_p^\alpha(X)$ were studied in D. Yang [113] (2005) in the case where the underlying space was a homogeneous metric measure spaces. A comparison of such spaces and some other Campanato related spaces with Besov and Triebel-Lizorkin spaces may be also found in that paper. We also mention a characterization of the Hajłasz-Sobolev spaces in terms of the Calderón-Scott maximal function $f_\alpha^\sharp$, obtained in [113].

4.3. φ-generalization

Following the long-standing traditions in the study of Campanato spaces, we use two forms to define them. Namely

$$\mathscr{L}_k^{p,\varphi} := \left\{ f \in L^p : \sup_{x,r} \frac{1}{\varphi(r)} \inf_{P \in \mathcal{P}_k} \int_{\widetilde{B}(x,r)} |f(y) - P(y)|^p \, \mathrm{d}y < \infty \right\} \tag{36}$$

and

$$\mathsf{L}_k^{p,\psi} = \left\{ f \in L^p : \sup_{x,r} \frac{1}{r^n \psi(r)} \inf_{P \in \mathcal{P}_k} \int_{\widetilde{B}(x,r)} |f(y) - P(y)|^p \, \mathrm{d}y < \infty \right\}. \tag{37}$$

Such a generalized Campanato space $\mathsf{L}^{1,\psi}(Q) := \mathsf{L}_0^{1,\psi}(Q)$, over cubes $Q \subset \mathbb{R}^n$, defined by the seminorm

$$[f]_{\mathsf{L}^{1,\psi}(\Omega)} := \sup_{x,r} \frac{1}{r^n \psi(r)} \int_{I(x,r) \subset Q} |f(y) - f_{I(x,r)}| \, \mathrm{d}y,$$

with $I(x,r) = \{u : |y - x| < r/2\}$, appeared in S. Spanne [99] (1965), where $\mathsf{L}^{1,\psi}(Q)$ was characterized in terms of rearrangements of the function $|f - f_{I(x,r)}|$, restricted to $I(x,r)$. Under the assumption that the function ψ is increasing on $(0,\infty)$ and the integral $\int_0^\varepsilon \frac{\psi(t)}{t}\,\mathrm{d}t$ converges, he proved the embedding

$$\mathsf{L}^{1,\psi}(Q) \hookrightarrow H^{\psi_1}(Q), \tag{38}$$

where H^{ψ_1} is the generalized Hölder space

$$H^{\psi_1} = \{f : |f(x+h) - f(x)| \le C\psi_1(h)\}, \quad \psi_1(h) = \int_0^h \frac{\psi(t)}{t}\,\mathrm{d}t. \tag{39}$$

The generalized Campanato space $\mathscr{L}_k^{p,\varphi}(\Omega)$ of higher order defined by the seminorm

$$[f]_{\mathscr{L}_k^{p,\varphi}} := \sup_{x,r} \left(\frac{1}{\varphi(r)} \inf_{P\in\mathcal{P}_k} \int_{\widetilde{B}(x,r)} |f(y) - P(y)|^p\,\mathrm{d}y \right)^{1/p} \tag{40}$$

where $\mathcal{P}_k$ is the class of polynomials of degree at most k, $k \ge 0$ and Ω is an open set in $\mathbb{R}^n$, was studied by S. Spanne [100] (1966) who gave its equivalent characterization in terms of the seminorm

$$\sup_{x,r} \left(\frac{1}{\varphi(r)} \|f - \mathsf{P}_k(f)\|^p_{L^p(\widetilde{B}(x,r))} \right)^{1/p} \tag{41}$$

where P_k is the orthogonal projection of $L^2(\widetilde{B}(x,r))$ onto the space of restrictions of polynomials of order k on $\widetilde{B}(x,r)$, under the assumption that Ω is of type (A). He also considered weak generalized Morrey-type spaces with the L^p-norm in (41) replaced by the weak L^p-norm.

As shown in J. Alvarez [7] (1981) the generalized Campanato spaces $\mathscr{L}_0^{p,\varphi}$ are not better than the L^p space if one admits the function φ such that $\varphi(t) \to \infty$ as $t \to 0$. More precisely, let φ be a nonnegative function such that $\varphi(t)$ is nonincreasing and $t\varphi^p(t)$ is nondecreasing near zero and $\varphi(0) = \infty$; suppose also that $g : (0,1) \to \mathbb{R}$ is a nonnegative, nonincreasing p-integrable function such that $g(t) \to \infty$ as $t \to 0$. Then there exist a cube Q_0, a function $f \in \mathscr{L}_0^{p,\varphi}(Q_0)$ and two constants $C, t_0 > 0$ such that

$$\lambda_f(t) \ge C\lambda_g(t_0)$$

where $\lambda_f(t) = |\{x : |f(x)| > t\}|$ is the distribution function, so that $\mathscr{L}_0^{p,\varphi}(Q_0)$ contains functions whose distribution functions exceed that of any given function in $L^p(Q_0)$.

In the case where $\Omega \subset \mathbb{R}^n$ is a bounded open set, generalized Campanato spaces $\mathscr{L}_k^{p,\varphi}(\Omega)$ defined by condition (40), appeared in C. Zorko [115] (1986). As a generalization of the statement 1. of Theorem 4.4, there was proved that

$$\mathscr{L}_k^{p,\varphi}(\Omega) \cong L^{p,\varphi}(\Omega)$$

under the condition (A), see (33), and the following assumptions: $\varphi(r)$ is nondecreasing, $\varphi(r)r^{-n}$ is nonincreasing and $\varphi(2r) \le c\varphi(r)$ with $0 < c < 2^{\frac{n}{p}}$, with the generalized Morrey space $L^{p,\varphi}(\Omega)$ defined by the norm (19).

We refer also to Proposition 5 of [115] where the reader can find a statement on preduals of type of Theorem 2.4 for Campanato spaces.

As a generalization of Spanne's result (38), J. Kovats [61] (1999) proved the embedding

$$\mathsf{L}_k^{p,\psi}(\Omega) \hookrightarrow C^{k,\psi_1}(\Omega), \quad \psi_1(t) = \int_0^t \frac{\psi(r)^{1/p}}{r^{1+k}}\,\mathrm{d}r \tag{42}$$

where Ω is a domain of type (A) and C^{k,ψ_1} is the space of functions differentiable up to order k with the last derivative satisfying the Hölder condition as in (39), under the assumption that the integral defining the function ψ_1 converges.

The generalized Campanato spaces, in the case where the underlying space X was a normal homogeneous metric measure space, defined for $1 \le p < \infty$ by

$$\|f\|_{\mathcal{L}^{p,\phi}} := \sup_{x,r} \frac{1}{\phi(x,r)} \left(\frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \,\mathrm{d}\mu(y) \right)^{1/p}$$

were introduced in E. Nakai [75] (2006). Recall that a homogeneous metric measure space is called *normal* if

$$K_1 r \le \mu B(x,r) \le K_2 r. \tag{43}$$

There were given relations between such generalized Campanato spaces and Morrey and Hölder spaces, the latter defined by the norm

$$\|f\|_{\Lambda_\phi} := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{2|f(x) - f(y)|}{\phi(x, d(x,y)) + \phi(y, d(y,x))},$$

including necessary and sufficient conditions on the function ϕ for the relations

$$\mathcal{L}^{p,\phi}(X)/\mathcal{C} \cong L^{p,\phi}(X), \quad \mathcal{L}^{p,\phi}(X) \cong L^{p,\phi}(X), \quad \mathcal{L}^{p,\phi}(X) \cong \Lambda_\phi(X).$$

A modified version of (vector-valued) Campanato spaces, with non-doubling measures, in the language of the *RBMO* spaces of X. Tolsa [107] (2001) was introduced and studied in Y. Sawano and H. Tanaka [96] (2006).

P. Górka [41, Theor. 3.1] (2009) gave a simple proof of a statement of type (3) of Theorem 4.3 in the general setting of homogeneous metric measure spaces (X, ρ, μ), for the Campanato spaces defined by the condition

$$\frac{1}{\mu B(x,r)} \int_{\widetilde{B}(x,r)} \left|f(y) - f_{B(x,r)}\right|^p \mathrm{d}\mu(y) \le C^p r^{\alpha p},$$

not requiring the space (X, ρ, μ) to be normal. A local version of this theorem was used in [41, Theor. 3.3] to prove some embeddings of Hajłasz-Sobolev space $M^{1,p}(X)$, $1 < p < \infty$, into Hölder spaces.

4.4. Interpolation results

G. Stampacchia [101] (1964), [102] (1965) and S. Campanato and M. Murthy [21] (1965) proved a Riesz-Thorin-type interpolation theorem for operators acting from L^p into Campanato spaces $\mathscr{L}^{q,\lambda}$ (at that time, Morrey and Campanato spaces were simply called Morrey spaces). The result in a more complete form obtained in S. Campanato and M. Murthy [21] (1965) is the following, where $\mathscr{L}_k^{p,\lambda}(\Omega)$ is the space defined by (34) and Ω is a bounded open set in $\mathbb{R}^n$.

Theorem 4.5. *Let* $1 \le p_i \le \infty, 1 \le q_i \le \infty,\ 0 \le \lambda_i < n+p,\ i = 1,2,$ *and for* $0 < \theta < 1$ *define* p, q *and* λ *by*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{\lambda} = \frac{1-\theta}{\lambda_1} + \frac{\theta}{\lambda_2}. \tag{44}$$

If T *is a bounded linear operator from* $L^{q_i}(\Omega)$ *to* $\mathscr{L}_k^{p_i,\lambda_i}(\Omega)$, $i = 1,2$ *with the operator norm* K_i, *then* T *is bounded from* $L^q(\Omega)$ *to* $\mathscr{L}_k^{p,\lambda}(\Omega)$ *with the norm at most* $CK_1^{1-\theta}K_2^{\theta}$, *with* C *depending only on* θ, λ_i, p_i *and* q_i.

Interpolation in the other direction fails, as first shown by E. Stein and A. Zygmund [103] (1967) who constructed a bounded linear operator on H^α and L^2 but not on $L^q, q > 2$ and BMO. Further results on such a failure may be found in the papers by A. Ruiz and L. Vega [93] (1995) and O. Blasco et al. [17] (1999), where there were given examples of operators bounded from $L^{p_i,\lambda}$ to L^{q_i}, which are not bounded in the intermediate spaces.

Note that a version of Marcinkiewicz type theorem was obtained in G. Stampacchia [101] (1964) for spaces $\mathscr{L}^{p,\lambda}(Q_0)$, where Q_0 is a cube in $\mathbb{R}^n$. The linear operator T was defined to be of strong type (p,q,λ), if $\|Tf\|_{\mathscr{L}^{q,\lambda}} \le K\|f\|_{L^p}$ and of weak type (p,q,λ), if

$$\sup_Q r^{-\lambda} |\{x \in Q : |Tf - (Tf)_Q| > \sigma\}| \le \left(\frac{K}{\sigma}\|f\|_{L^p}\right)^q,$$

where Q is a cube with sides parallel to Q_0, and the following interpolation theorem was proved

Theorem 4.6. *If* T *is of weak types* (p_1,q_1,λ_1) *and* (p_2,q_2,λ_2), *where* $p_i \ge 1$, $p_i \le q_i$, $i = 1,2$, $q_1 \ne q_2$, $p_1 \ne p_2$, *then* T *is of strong type* (p,q,λ) *with* p,q,λ *defined in* (44).

For some related interpolation statements we also refer to the thesis of P. Grisvard [44] (1965), published in [45, 46] (1966) and the paper J. Peetre [87] (1966). S. Spanne [100] (1966) generalized and simplified the proofs of the interpolation theorem in the setting of generalized Campanato space. In fact, he reduced the validity of the interpolation to the L^p case. Namely, let

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \varphi(r) = \varphi_0(r)^{1-\theta}\varphi_1(r)^{\theta}, \quad 0 < \theta < 1$$

and let A_0, A_θ, A_1 be normed spaces such that the interpolation theorem is valid for the two triplets (A_0, A_θ, A_1) and (L^{p_0}, L^p, L^{p_1}). Then the interpolation theorem is valid also for (A_0, A_θ, A_1) and $(\mathscr{L}_k^{p_0,\varphi_0}, \mathscr{L}_k^{p,\varphi}, \mathscr{L}_k^{p_1,\varphi_1})$, with the same convexity constant. A similar result holds for the corresponding weak Campanato spaces.

4.5. Other characterizations of Campanato spaces

B. Grevholm [43] (1970) used the interpolation theorem for Campanato spaces to characterize the Campanato spaces as the Besov spaces, namely

$$\mathscr{L}_k^{p,\lambda}(\Omega) = B^\alpha(\Omega), \quad 0 < \alpha = \frac{\lambda - n}{p} < k,$$

where Ω is an open set in $\mathbb{R}^n$ satisfying some conditions and $B^\alpha(\Omega)$, in the case $\Omega = \mathbb{R}^n$, is defined by the seminorm

$$\sup_{t>0,|y|<1} \frac{\|\Delta_{ty}^k f\|_{L^\infty}}{t^\alpha}$$

while in the case $\Omega \neq \mathbb{R}^n$ the space $B^\alpha(\Omega)$ is defined as the interpolation space

$$B^\alpha(\Omega) = \big(C^0(\Omega), C^k(\Omega)\big)_{\frac{\alpha}{k},\infty}$$

under a certain interpolation method.

A result similar in a sense was obtained by different means in H.C. Greenwald [42] (1983) who proved the coincidence of the Campanato space $\mathscr{L}_k^{p,\lambda}(\mathbb{R}^n)$ with the Lipschitz-type space $\Lambda(\alpha, k)$ defined in terms of Gauss-Weierstrass integral:

$$\|f\|_{\alpha,k+1} = \sum_{|\nu|=k} \sup_{t\in\mathbb{R}^+} \sup_{x\in\mathbb{R}^n} t^{(k-\alpha)/2} |D^\nu f(x,t)| < \infty,$$

where $f(x,t)$ is the Gauss-Weierstrass integral of f and D stands for the differentiation with respect to x.

Consider also the space $L(\alpha, p, k-1)$ of equivalence classes modulo P_{k-1} of locally integrable functions f for which

$$\|f\|_{L(\alpha,p,k-1)} = \sup_{Q\subset\mathbb{R}^n} |Q|^{-\alpha/n} \left[\frac{1}{|Q|}\int_Q |f(x) - P_Q f(x)|^p \,\mathrm{d}x\right]^{1/p} < \infty, \tag{45}$$

where Q is a ball and $P_Q f$ is the unique element of P_{k-1} such that

$$\int_Q [f(x) - P_Q f(x)] x^\nu \,\mathrm{d}x = 0, \quad 0 \le |\nu| \le k-1. \tag{46}$$

Such spaces occur in the duality theory of Hardy spaces as discussed by M. Taibleson and G. Weiss [105] (1980); we refer also to a related paper M.H. Taibleson and G. Weiss [104] (1979). The main result of [42] asserts that the spaces $\Lambda_{\alpha,k}$ and $L(\alpha, p, k-1)$ coincide and that their norms are equivalent. An earlier result of similar nature was obtained by B. Grevholm [43] (1970) for p in the range $1 \le p < \infty$ using interpolation theory. The result in [42] is valid for $1 \le p \le \infty$ and is proved by elementary methods.

X.T. Duong and L.X. Yan [30] (2005) studied identity approximations adapted to Morrey-Campanato spaces on quasimetric measure spaces.

In D. Deng, X.T. Duong and L. Yan [27], the authors gave an equivalent characterization of the spaces $L(\alpha, p, k-1)$ by using the identity approximations instead of the minimizing polynomial in the definition of the norm (45) in the case $\alpha > 0$, $k > [n\alpha] + 1$ when these spaces do not depend on $p \in [1, \infty]$.

X.T. Duong, J. Xiao and L. Yan [29] (2006) studied the Morrey-Campanato spaces defined with the constant $c = f_B$ in the definition in (28) replaced by a semigroup of operators. They studied relations with the usually defined Morrey-Campanato spaces and showed that under appropriate choice of a semigroup, the new definition coincides with the old one.

L. Tang [106] (2007) used the ideas of [30] to define the Campanato spaces by the norm

$$\sup_B \frac{1}{\mu(B)^{\alpha+1}} \int_B |f(x) - A_B(f)| \, \mathrm{d}x,$$

where $A_B(f)$ is an identity approximation from [30]. There is shown that in some cases such different norms are equivalent but there were also given examples where they are not.

4.6. Miscellaneous

The *central mean oscillation space* CMO^q, introduced in Y.Z. Chen and K.S. Lau [24] (1989) and J. García-Cuerva [37] (1989), defined by

$$\|f\|_{\mathrm{CMO}^q} = \sup_{r \geq 1} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q \, \mathrm{d}x \right)^{1/q}$$

was shown to be the dual space of an atomic space HA^q associated with the Beurling algebra. The *central bounded mean oscillation space* CBMO^q introduced in S. Lu and D. Yang [65] (1992) and S. Lu and D. Yang [66] (1995) is defined by

$$\|f\|_{\mathrm{CBMO}^q} = \sup_{r > 0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q \, \mathrm{d}x \right)^{1/q}.$$

A generalization of CMO^q and CBMO^q, introduced in J. García-Cuerva and M.J.L. Herrero [38] (1994) and J. Alvarez, M. Guzmán-Partida and J. Lakey [8] (2000), are the so-called λ-*central mean oscillation spaces* $\mathrm{CMO}^{q,\lambda}$ and λ-*central bounded mean oscillation spaces* $\mathrm{CBMO}^{q,\lambda}$, defined by

$$\|f\|_{\mathrm{CMO}^{q,\lambda}} = \sup_{r \geq 1} \frac{1}{|B(0,r)|^\lambda} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q \, \mathrm{d}x \right)^{1/q}$$

and

$$\|f\|_{\mathrm{CBMO}^{q,\lambda}} = \sup_{r > 0} \frac{1}{|B(0,r)|^\lambda} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q \, \mathrm{d}x \right)^{1/q}.$$

M. Kronz [62] (2001) introduced Morrey and Campanato spaces for elements which are mappings between metric measure spaces.

A classical Morrey inequality states that in the case $p > n$, the following embedding of a Sobolev space into Hölder space holds

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega).$$

In the paper A. Cianchi and L. Pick [26] (2003), in the case $p = 1$, there was given a detailed study of more general embeddings of Sobolev spaces into Morrey and Campanato spaces for the case Ω is a cube in $\mathbb{R}^n$. For a weakly differentiable function f on Q they gave optimal integrability conditions on the gradient of f, to belong to Morrey or Campanato space. More generally they gave a characterization of the rearrangement-invariant Banach function spaces such that the corresponding Sobolev space $W^1X(Q)$ is continuously embedded into Morrey or Campanato space. This enabled the authors to find the largest space $X(Q)$ for which such an embedding holds (the so-called optimal range partner). Such an optimal space is of Marcinkiewicz type in the case of Campanato spaces and have a different nature in the case of Morrey spaces. In particular, the following theorem was proved in [26], where $M_\psi(Q)$ is the Marcinkiewicz space defined by the norm

$$\|f\|_{M_\psi(Q)} = \sup_{0<t<1} \psi(t) f^{**}(t), \quad \psi(t) = \frac{t^{\frac{1}{n}+1}}{\varphi(t^{\frac{1}{n}})}.$$

Theorem 4.7. *Let φ be a strictly positive continuous function on $(0,\infty)$. Then the space $X(Q) = M_\psi(Q)$ is the largest rearrangement invariant space for which the embedding*

$$\|f\|_{\mathscr{L}^{1,\varphi}(Q)} \le C \|\nabla f\|_{X(Q)}$$

holds.

A version of grand Morrey spaces $L^{p),\lambda}(X)$ over homogeneous-type space X, which turns into the grand Lebesgue space $L^{p)}(X)$ introduced in T. Iwaniec and C. Sbordone [54] (1992) when $\lambda = 0$, was suggested in A. Meskhi [68] (2009). It is defined by the norm

$$\|f\|_{L^{p),\lambda}(X)} := \sup_{0<\varepsilon<p-1} \left(\sup_{x\in X, r>0} \frac{\varepsilon}{(\mu(B(x,r)))^\lambda} \int_{B(x,r)} |f(y)|^{p-\varepsilon} \, \mathrm{d}\mu(y) \right)^{1/(p-\varepsilon)}.$$

5. Variable exponent Morrey and Campanato spaces

The Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ with variable exponents $\lambda(\cdot)$ and $p(\cdot)$ over an open set $\Omega \subset \mathbb{R}^n$, were recently introduced almost simultaneously by different authors in A. Almeida, J. Hasanov and S. Samko [6] (2008), V. Kokilashvili and A. Meskhi [58] (2008), [59] (2010), T. Ohno [77] (2008), X. Fan [35] (2010).

In A. Almeida, J. Hasanov and S. Samko [6] (2008) the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ was introduced as the space of functions with the finite norm

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Omega)} = \inf\left\{\nu : I^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\nu}\right) \leq 1\right\}$$

and the modular $I^{p(\cdot),\lambda(\cdot)}(f)$ defined by

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in\Omega, r>0} \frac{1}{r^{\lambda(x)}} \int_{\widetilde{B}(x,r)} |f(y)|^{p(y)}\, \mathrm{d}y.$$

In the case of a bounded Ω they gave several equivalent norms and proved embedding theorems for such Morrey spaces under the assumption that $p(x)$ satisfies the log-condition well known in the variable exponent analysis. Similar embedding theorem for variable Campanato spaces may be found in [90] (2011) within the frameworks of the general setting of metric measure spaces.

V. Kokilashvili and A. Meskhi [58] (2008), see also [59] (2010), introduced Morrey-type spaces $M^{q(\cdot)}_{p(\cdot)}$ in the general setting when the underlying space is a homogeneous-type space (X, ρ, μ), with the norm defined by

$$\|f\|_{M^{q(\cdot)}_{p(\cdot)}} = \sup_{x\in X, r>0} (\mu(B(x,r)))^{1/p(x)-1/q(x)} \|f\|_{L^{q(\cdot)}(B(x,r))}$$

where $1 < \inf_X q \leq q(\cdot) \leq p(\cdot) \leq \sup_X p < \infty$. In the case where X is bounded, some equivalence of norms and embedding theorems were obtained.

A φ-generalization $L^{p(\cdot),\nu,\varphi}(\mathbb{R}^n)$ of Morrey spaces with variable exponent $p(x)$ and constant $0 \leq \nu \leq n$, was given in T. Ohno [77] (2008) by the condition

$$\frac{\varphi(r)}{r^\nu} \int_{B(x,r)} \left|\frac{f(y)}{\lambda}\right|^{p(y)} \mathrm{d}y \leq 1$$

for some $\lambda > 0$.

A more general version $\mathcal{M}^{p(\cdot),\omega}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ of such generalized variable exponent Morrey spaces was introduced in V. Guliev, J. Hasanov and S. Samko [49] (2010), defined by the norm

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

They recover the space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$.

Both φ- and θ-generalizations of Morrey spaces of variable order were introduced in V. Guliev, J. Hasanov and S. Samko [48] (2010), as the space of functions with the finite norm

$$\sup_{x\in\Omega} \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\ell)},$$

where $\ell = \operatorname{diam} \Omega$.

The corresponding variable exponent Campanato spaces are interesting because they in general contain functions which are locally in $L^{p(\cdot),\lambda(\cdot)}$ on one subset, BMO-functions locally on another subset and variable order Hölder continuous on the third one.

Such spaces appeared in X. Fan [35] (2010), where besides variable exponent Morrey spaces there were also introduced Campanato spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}$ of variable order, in the Euclidean case, via the norm

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \sup_{x_0\in\Omega, r>0} \left\| r^{-\frac{\lambda(\cdot)}{p(\cdot)}} (f - f_{B(x_0,r)}) \right\|_{L^{p(\cdot)}(B(x_0,r))},$$

where $f_B = |B|^{-1}\int_B f(x)\,\mathrm{d}x$. The equivalence of such Campanato spaces to variable exponent Hölder spaces is shown when $\inf_{x\in\Omega}\lambda(x) > n$ and to variable exponent Morrey spaces, when $\sup_{x\in\Omega}\lambda(x) < n$. In the latter result, the proof of the embedding of Morrey spaces into Campanato spaces was based on the notion of $p(\cdot)$-average of a function introduced in this paper.

Similar results for variable exponent Campanato spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(X)$ in a more general setting of metric measure spaces were obtained in H. Rafeiro and S. Samko [90] (2011). In [90], in the setting of an arbitrary quasimetric measure spaces, the log-Hölder condition for $p(x)$ is introduced with the distance $d(x, y)$ replaced by $\mu B(x, d(x, y))$, which provides a weaker restriction on $p(x)$ in the general setting. Some initial basic facts for variable exponent Lebesgue spaces hold without the assumption that X is homogeneous or even Ahlfors lower or upper regular, but the main results for Campanato spaces are proved in the case of homogeneous spaces X.

In E. Nakai [76] (2010) there were introduced φ-generalizations of such spaces on a space of homogeneous-type, normal in the sense of Macías and Segovia. In [76] φ was admited to be variable, but p constant and the norm defined by

$$\|f\|_{\mathscr{L}^{p,\varphi}} = \sup_{x,r>0} \frac{1}{\varphi(B(x,r))} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \,\mathrm{d}\mu(y) \right)^{1/p}.$$

We note also the embedding $L^{p(\cdot)}(X) \hookrightarrow L^{1,\varphi} \hookrightarrow \mathscr{L}^{1,\varphi}$ proved in [76], where $L^{1,\varphi}$ stands for the corresponding Morrey space and $\varphi(B(x,r)) = r^{-\frac{1}{p_*(x)}}$, where $p_*(x) = p(x)$ when $0 < r < 1/2$ and $p_*(x) = p_+$ when $1/2 \le r < \infty$.

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