Nonlinear Analysis 78 (2013) 130–140

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Potential operators in generalized Hölder spaces on sets in quasi-metric measure spaces without the cancellation property

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ARTICLE INFO

Article history:
Received 29 February 2012
Accepted 28 September 2012
Communicated by Enzo Mitidieri

MSC:
primary 46E15
secondary 42B35

Keywords:
Potential operators
Quasimetric measure space
Growth condition
Hölder space
Generalized Hölder space

ABSTRACT

We consider potential operators of order α over sets Ω in quasi-metric measure spaces and study their mapping properties from the subspace $H_0^\lambda(\Omega)$ of functions in Hölder space $H^\lambda(\Omega)$ vanishing on the boundary of Ω, into the space $H^{\lambda+\alpha}(\Omega)$, if $\lambda+\alpha < 1$. This is proved in a more general setting of generalized Hölder spaces $H^\omega(\Omega)$ with a given dominant ω of modulus of continuity. Statements of such a kind are known in the Euclidean case or in the case of quasimetric measure spaces with the cancellation property. In the general case, when the cancellation property fails, our proofs are based on a special treatment of the potential of a constant function, which in general has a regularity near the boundary $\partial\Omega$ of the type of the α-th power of the distance to $\partial\Omega$. An application to the case of spatial potentials over domains in $\mathbb{R}^n$ and potentials over spherical caps is given.

1. Introduction

Mapping properties of potential operators within the frameworks of Hölder spaces are well studied in the general setting of quasimetric measure spaces (X, ϱ, μ) under the assumption that X satisfies the so called cancellation property; see [1–4]. The well known examples of underlying spaces X with the cancellation property are the whole space $\mathbb{R}^n$ and the sphere $\mathbb{S}^{n-1}$. We also refer to various more precise specifications and/or generalizations of mapping properties of potential operators in these two model cases presented in the papers [5–13].

In cases where the potential of a constant function on X is well defined, the cancellation property means that the potential of a constant is constant. This property was also used in the recent paper [14], where there were admitted potentials of variable order $\alpha(x)$ with possible degeneration: $\alpha(x) = 0$ on a set of measure zero.

The cancellation property is very restrictive in applications: it fails for domains Ω in $\mathbb{R}^n$. In the case of balls in $\mathbb{R}^n$, for instance, the potential of a constant is constant on the boundary, but is not constant in the ball.

In the Euclidean case for instance, statements of the type

$$I_\Omega^\alpha : H^\lambda(\Omega) \to H^{\lambda+\alpha}(\Omega), \quad \Omega \subset \mathbb{R}^n,$$

for the potential operator

$$I_\Omega^\alpha f(x) := \int_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha}}$$

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doi:10.1016/j.na.2012.09.020

may not be valid for domains, since the potential of a constant has regularity only of order α near the boundary: it behaves in general like $c_1 + c_2[\delta(x)]^\alpha$ near the boundary, where $\delta(x) = \delta(x, \partial\Omega)$ is the distance to the boundary. However, one may expect that there should be a valid statement

$$I^\alpha_\Omega : H^\lambda_0(\Omega) \to H^{\lambda+\alpha}(\Omega) \tag{1.1}$$

for the subspace $H^\lambda_0(\Omega)$ of the Hölder space $H^\lambda(\Omega)$ of functions which vanish at the boundary. Such a mapping is known in the one-dimensional case and goes back to Hardy and Littlewood; see for instance [6, Corollary 1 on p. 56]. A multi-dimensional result of such a kind was recently proved in [15], where in particular the case of uniform domains (Jones domains) was covered. In this paper we develop a similar approach within the framework of general quasimetric measure spaces (X, ϱ, μ) with the growth condition on the measure. We show that a mapping of type (1.1) (and more generally, for spaces of the type $H^\omega(\Omega)$) holds for measurable bounded sets Ω in (X, ϱ, μ) satisfying the so called α-property. Roughly speaking, we can state a result on mapping properties of the potential operator, if we know how the potential operator of the constant, i.e.

$$J_{\Omega,\alpha}(x) = \int_\Omega \frac{d\mu(y)}{\varrho(x, y)^{N-\alpha}}, \quad x \in \Omega, \tag{1.2}$$

where N comes from the growth condition, behaves near the boundary of Ω.

We give the proof of results of such a type in intrinsic terms of the given set $\Omega \subseteq X$. The proof in intrinsic terms allows us to obtain information also about the behaviour of potentials near the boundary $\partial\Omega$ in the cases where $f(x)$ does not vanish at the boundary.

Note that this way was also used in [15] in the case of domains in $\mathbb{R}^n$ and Lebesgue measure, although in this case it is possible to derive just a result of type (1.1) from the estimates of the modulus of continuity of potentials over $\mathbb{R}^n$, obtained in [7], since a function $f \in H^\lambda_0(\Omega)$ may be extended as identical zero outside Ω, which preserves the Hölder behaviour of f. This way was preferred in [15] because it provides information near the boundary, and a derivation of statements even of type (1.1) from [15] is rather artificial: the results in $\mathbb{R}^n$ in [15] were proved in its turn not directly, but by reducing the problem to the case of the unit sphere via the stereographic projection and usage of Fourier–Laplace analysis on the sphere.

The paper is organized as follows. In Section 2 we provide necessary preliminaries related to quasimetric measure spaces (X, ϱ, μ). In Section 3 we study the function $J_{\Omega,\alpha}(x)$, where the main technical statement is Lemma 3.1, and give examples illustrating the behaviour of $J_{\Omega,\alpha}(x)$ near the boundary. In Section 4 we extend the notion of the α-property, introduced in [15] in the Euclidean case, to the general setting. Section 5 contains the main result on the mapping properties. Section 6 contains two applications. The first is related to the case of domains in $\mathbb{R}^n$, where we improve a result from [15] by showing that an arbitrary domain in $\mathbb{R}^n$ satisfies the α-property, introduced in [15]. The second concerns spherical potentials over a spherical cap on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, which is inspired by applications studied in [16]. The final Appendix (Appendix) contains some estimates for the case of spherical potentials on a semisphere.

2. Preliminaries on metric measure spaces

Given a set X, a function $\varrho : X \times X \to [0, \infty)$ is called *quasimetric*, if it satisfies the usual metric axioms with the triangle inequality replaced by the *quasi-triangle inequality*

$$\varrho(x, y) \le K[\varrho(x, z) + \varrho(z, y)], \quad K \ge 1 \tag{2.1}$$

where $x, y, z \in X$. We assume that $\varrho(x, y) = \varrho(y, x)$. Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r)$. Everywhere in the sequel we suppose that all the balls have finite measure for all $x \in X$ and $r > 0$ and that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$.

We assume that X is closed with respect to the metric ϱ, i.e. every fundamental sequence in X has a limit in X. The boundary $\delta(\Omega)$ of an open set Ω in X is interpreted in the usual sense, i.e. as the set of all the points in X, which are limiting points for Ω, but are not inner points of Ω. We always assume that $\mu(\partial\Omega) = 0$.

Let

$$\delta_F(x) = \inf_{y\in F} \varrho(x, y)$$

denote the distance of a point x from the set $F \subseteq X$. By

$$\delta(x) = \delta(x, \partial\Omega) := \inf_{y\in\partial\Omega} \varrho(x, y)$$

we denote the distance of x to the boundary.

We say that the measure μ satisfies *the growth condition* equivalently called the upper Ahlfors N-regular, if

$$\mu B(x, r) \le cr^N, \tag{2.2}$$

where $N > 0$ and $c > 0$ does not depend on x and r.

In this paper we do not assume the measure μ to be doubling, but base ourselves on the growth condition (2.2).

Note that balls in a general space, even of homogeneous type, are not necessarily open, but there exists a continuous quasimetric ϱ' equivalent to ϱ, with respect to which all balls are open.

In the sequel we assume that μ satisfies the growth condition (2.2).

As shown in [17], every quasidistance ϱ on a quasimetric space (X, ϱ) admits an equivalent quasimetric ϱ_1 for which there exists an exponent $\theta \in (0, 1]$ such that the property

$$|\varrho_1(x, z) - \varrho_1(y, z)| \le M\varrho_1^{\theta}(x, y)\, \{\varrho_1(x, z) + \varrho_1(y, z)\}^{1-\theta} \tag{2.3}$$

and

$$\varrho_1(x, y) = d(x, y)^{\frac{1}{\theta}} \tag{2.4}$$

where $d(x, y)$ is a metric (i.e. (2.1) holds for $d(x, y)$ with $K = 1$). By the elementary inequality

$$|a^\beta - b^\beta| \le |\beta||a - b| \max(a^{\beta-1}, b^{\beta-1}), \quad a, b \in \mathbb{R}^1_+, \beta \in \mathbb{R}^1, \tag{2.5}$$

the property (2.3) is an immediate consequence of (2.4) and it holds with

$$M = \frac{1}{\theta}.$$

Definition 2.1. We say that the quasimetric ϱ is regular of order $\theta \in (0, 1]$, if it itself satisfies property (2.4), i.e. $\varrho(x, y) = d(x, y)^{\frac{1}{\theta}}$, where $d(x, y)$ is a distance on X.

Everywhere in the sequel we suppose that the quasimetric is regular of order $\theta \in (0, 1]$.

In this paper we study mapping properties of potential operators

$$(I^\alpha f)(x) = \int_\Omega \frac{f(y)\, d\mu(y)}{\varrho(x, y)^{N-\alpha}}, \quad x \in \Omega \subseteq X, \tag{2.6}$$

for functions f defined on an open set Ω of a quasimetric measure space (X, ϱ, μ), where N is the exponent from the growth condition.

The following estimates are known:

$$\int_{B(x,r)} \frac{d\mu(y)}{\varrho(x, y)^{N-\alpha}} \le cr^\alpha \tag{2.7}$$

and

$$\int_{X\setminus B(x,r)} \frac{d\mu(y)}{\varrho(x, y)^{N+\beta}} \le cr^{-\beta}, \quad \beta > 0, \tag{2.8}$$

under the only condition on (X, ϱ, μ) that μ satisfies the growth condition (2.2); see for instance [2, Lemma 1].

The lemma below supplements (2.8) in the limiting case $\beta = 0$.

Lemma 2.2. *Let μ satisfy the growth condition (2.2) and Ω be bounded. Then*

$$\int_{\Omega\setminus B(x,r)} \frac{d\mu(y)}{\varrho(x, y)^{N}} \le c \ln \frac{D}{r}, \quad D > \operatorname{diam} \Omega. \tag{2.9}$$

Proof. The proof is standard via the dyadic decomposition:

$$\int_{\Omega\setminus B(x,r)} \frac{d\mu(y)}{\varrho(x, y)^{N+\varepsilon}} = \sum_{k=0}^{\infty} \int_{\Omega\cap\{z:2^k r<\varrho(x,y)<2^{k+1}r\}} \frac{d\mu(y)}{\varrho(x, y)^{N+\varepsilon}} \le \frac{c}{r^\varepsilon} \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} \le \frac{C}{\varepsilon r^\varepsilon} \le C \int_r^D \frac{dt}{t^{1+\varepsilon}},$$

after which it remains to pass to the limit when $\varepsilon \to 0$. □

3. Potentials of constant functions

For a bounded, measurable open set $\Omega \subseteq X$ and $\alpha > 0$ we define the potential $J_{\Omega,\alpha}$ by

$$J_{\Omega,\alpha}(x) = \int_X \frac{\chi_\Omega(y)}{\varrho(x, y)^{N-\alpha}}\, d\mu(y) = \int_\Omega \frac{d\mu(y)}{\varrho(x, y)^{N-\alpha}}, \quad x \in \Omega, \tag{3.1}$$

which is well defined in view of (2.7). When Ω is not necessarily bounded, we define the *difference of the potential* by

$$J_{\Omega,\alpha}(x, y) := \int_\Omega \left(\frac{1}{\varrho(x, z)^{N-\alpha}} - \frac{1}{\varrho(y, z)^{N-\alpha}} \right) d\mu(z), \quad x, y \in X. \tag{3.2}$$

If Ω is bounded, then $J_{\Omega,\alpha}(x, y) = J_{\Omega,\alpha}(x) - J_{\Omega,\alpha}(y)$. However, if Ω is not bounded, then $J_{\Omega,\alpha}(x)$ may be not well defined.

Lemma 3.1. *Let (X, ϱ, μ) be a metric measure space with the growth condition (2.2), regular of order $\theta \in (0, 1]$. Then $J_{\Omega,\alpha}(x, y)$ is well defined at the least for $0 < \alpha < \theta$.*

Proof. The following inequality

$$\left| \frac{1}{\varrho(x, z)^{N-\alpha}} - \frac{1}{\varrho(y, z)^{N-\alpha}} \right| \le C \frac{\varrho(x, y)^\theta}{\varrho(x, z)^{N-\alpha+\theta}}, \quad \text{if } \varrho(x, y) \le \frac{1}{2K}\varrho(x, z) \tag{3.3}$$

holds, where K is the constant from (2.1). To prove it, note that the condition $\varrho(x, y) \le \frac{1}{2K}\varrho(x, z)$ involves the equivalence

$$\frac{1}{2K}\varrho(x, z) \le \varrho(y, z) \le \left(K + \frac{1}{2}\right)\varrho(x, z).$$

Hence by (2.5), we obtain

$$\left|\varrho(x, z)^{\alpha-N} - \varrho(y, z)^{\alpha-N}\right| \le C \frac{|\varrho(x, z) - \varrho(y, z)|}{\varrho(x, z)^{N-\alpha+1}}.$$

Then by the θ-regularity (2.3) we arrive at (3.3). The inequality (3.3) with $N - \alpha + \theta > N$ ensures the convergence of the integral in (3.2) by (2.8). □

Thus for $\alpha \in (0, \theta)$, we have

$$J_{\Omega,\alpha}(x, x) = 0, \qquad J_{\Omega,\alpha}(x, y) = -J_{\Omega,\alpha}(y, x) \quad \text{and} \quad J_{\Omega,\alpha}(x, y) = J_{\Omega,\alpha}(x, a) + J_{\Omega,\alpha}(a, y)$$

for all $a, x, y \in X$. This holds for all $\alpha > 0$, if Ω is bounded. If

$$J_{X,\alpha}(x, y) \equiv 0,$$

for all $x, y \in X$, the space X is said to have the *cancellation property*. If X has the cancellation property, then

$$J_{\Omega,\alpha}(x, y) = -J_{X\setminus\Omega,\alpha}(x, y).$$

The spaces $(\mathbb{R}^n, d, dx)$ and $(\mathbb{S}^{n-1}, d, d\sigma)$, where d is the Euclidean distance and $d\sigma$ surface area measure on $(\mathbb{S}^{n-1}, d, d\sigma)$, have the cancellation property.

If Ω is bounded and $\alpha > 0$, then $J_{\Omega,\alpha}(x)$ is continuous in $x \in X$ and $J_{\Omega,\alpha}(x, y)$ is continuous in $x, y \in X$ for every Ω and $0 < \alpha < \theta$.

However, $J_{\Omega,\alpha}(x, y)$ has better properties than just continuity in the inner points of Ω; see Lemma 4.1 in Section 4. These properties may worsen when x or y approaches the boundary of Ω.

Examples. (1) $X = \mathbb{R}^n$, $\Omega = B(0, R)$, $0 < \alpha < n$:
$J_{\Omega,\alpha}(x) = c_0 + c_1(R - |x|)^\alpha + g(x)$, $x \in B(0, R)$,
where $c_0 = 2^{\alpha-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma^{-\frac{1}{2}}\left(\frac{n+\alpha}{2}\right)$, $c_1 = 2^{\alpha-1}R^{-\alpha}\pi^{-\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma^{-1}\left(\frac{n-\alpha}{2}\right)$, $g \in \mathrm{Lip}(\overline{B(0, R)})$ and $g\big|_{|x|=R} = 0$;
(2) $X = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $0 < \alpha < 1$:
$J_{\Omega,\alpha}(x, y) = c_n(\alpha)(\mathrm{sgn}(x_n)|x_n|^\alpha - \mathrm{sgn}(y_n)|y_n|^\alpha)$,
where $x, y \in \mathbb{R}^n$ and $c_n(\alpha) = \frac{\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{\alpha\Gamma(\frac{n-\alpha}{2})}$;
(3) $X = \mathbb{R}^n$, $\Omega = \mathbb{R}^2_{++} = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 > 0, x_2 > 0\}$, $0 < \alpha < 1$:
$J_\Omega(x, y) = \frac{c}{\alpha}\left([\delta(x)]^\alpha - [\delta(y)]^\alpha + x_1^\alpha - y_1^\alpha + x_2^\alpha - y_2^\alpha\right) + U(x) - U(y)$,
where $c = \frac{\sqrt{\pi}}{2\alpha}\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma^{-1}\left(\frac{2-\alpha}{2}\right)$ and $U(x) = |x|tA(t)$, $t = \min\left\{\frac{x_1}{x_2}, \frac{x_2}{x_1}\right\}$ and $A(t)$ is analytic in t.
(4) $X = \mathbb{S}^n = \{\sigma = (\sigma_1, \dots, \sigma_{n+1}) : |\sigma| = 1\}$ with the Euclidean distance between points on $\mathbb{S}^n$, and $\Omega = \mathbb{S}^n_+ := \{\sigma \in \mathbb{S}^n : \sigma_{n+1} > 0\}$; $\delta(\sigma, \partial\Omega) \sim \sigma_{n+1}$; in this case $J_{\mathbb{S}^n_+,\alpha}(\sigma) = c_0 + 2c_1\sigma_{n+1} + k(\sigma)$ for $\sigma_{n+1} > 0$, where c_0 and c_1 are the same as in example (1), and $k \in \mathrm{Lip}(\overline{\mathbb{S}^n_+})$, and $k\big|_{\partial\Omega} = 0$.

Proofs for examples (1)–(3) are given in [15]. Note that in [15] potentials $J_{\Omega,\alpha}(x)$ in those examples were explicitly calculated in terms of special functions. The proof for example (4) is given in the Appendix.

4. On the α-property of sets

In the Euclidean case it is known that the potential of order α of a bounded function on a bounded domain is α-Hölder continuous in Ω, which is a particular case of a Sobolev theorem stating that $I^{\alpha}_{\Omega} : L^p(\Omega) \to H^{\alpha-\frac{n}{p}}(\Omega)$, $1 < p \le \infty$ when $\frac{n}{p} < \alpha < \frac{n}{p} + 1$; see [18, p. 256]. In the following lemma we extend this for sets Ω in (X, ϱ, μ) in the case $p = \infty$, where Ω may be unbounded and we include all $x, y \in X$ into the Hölder condition, not only $x, y \in \Omega$.

Lemma 4.1. *Let $\Omega \subset X$ be measurable and $\alpha \in (0, \theta)$. Then*

$$|\mathcal{J}_{\Omega,\alpha}(x, y)| \le c\, \varrho(x, y)^{\alpha}, \tag{4.1}$$

where c depends on x and y. If Ω is bounded, the case $\alpha = \theta$ may be also admitted with the estimate

$$|\mathcal{J}_{\Omega,\theta}(x, y)| \le c\, \varrho(x, y)^{\theta} \ \ln \frac{D}{\varrho(x, y)}, \quad x, y \in \Omega, D > \operatorname{diam} \Omega. \tag{4.2}$$

Proof. We follow [15, Lemma 3.1], where (4.1) was proved for the case $X = \mathbb{R}^n$. Let $x, y \in X$ and $r := 2K\varrho(x, y)$. Then

$$\begin{aligned} J_{\Omega,\alpha}(x, y) &= \int_{\Omega\setminus B(x,r)} \varrho(x, z)^{\alpha-N} - \varrho(y, z)^{\alpha-N}\, d\mu(z) \\ &\quad + \int_{\Omega\cap B(x,r)} \varrho(x, z)^{\alpha-N}\, d\mu(z) - \int_{\Omega\cap B(x,r)} \varrho(y, z)^{\alpha-N}\, d\mu(z) \\ &=: J_1 + J_2 - J_3. \end{aligned}$$

By (2.7) we have

$$|J_2| \le \int_{B(x,r)} \varrho(x, z)^{\alpha-N}\, d\mu(z) \le c\, r^{\alpha} = C\varrho(x, y)^{\alpha}.$$

For J_3 we similarly have

$$|J_3| \le \int_{B(x,r)} \varrho(y, z)^{\alpha-N}\, d\mu(z) \le \int_{B(y,2Kr)} \varrho(y, z)^{\alpha-N}\, d\mu(z) \le c\varrho(x, y)^{\alpha}.$$

Note that the estimates obtained for J_2 and J_3 hold for all $\alpha > 0$.

Finally, for J_1 we observe that $z \in \Omega \setminus B(x, r)$ implies that $\varrho(z, x) > r = 2K\varrho(x, y)$ and then (3.3) is applicable which yields

$$|J_1| \le c\varrho(x, y)^{\theta} \int_{\Omega\setminus(B(x,r))} \frac{d\mu(z)}{\varrho(x, z)^{N-\alpha+\theta}} \le c\varrho(x, y)^{\alpha}$$

by (2.8). This completes the proof in the case $\alpha < \theta$. When $\alpha = \theta$, use (2.9). □

So the difference of the potential $J_{\Omega,\alpha}(x, y)$ is always Hölder continuous of order α, $0 < \alpha < \theta$, on $X \times X$. In the examples given above, the function $J_{\Omega,\alpha}(x, y)$ is even Lipschitz when x and y are off the boundary $\partial\Omega$ (see details in [15]). In the general setting of quasimetric spaces it is natural to suppose that in many cases the function $J_{\Omega,\alpha}(x, y)$ is Hölderian of order θ when the variables are off the diagonal, the case $\alpha = \theta$ being an analogue of the Lipschitz case.

Definition 4.2. A function $f(x)$ on a quasimetric space (X, ϱ) with the quasimetric ϱ regular of order θ will be called (α, ϱ)-Hölderian on $\Omega \subseteq X$, $0 < \alpha \le \theta$, if

$$|f(x) - f(y)| \le C\varrho(x, y)^{\alpha} \quad \text{for all } x, y \in \Omega.$$

We write $f \in H^{\alpha}(\Omega)$ in this case. When $\alpha = \theta$, we also call the function f Lipschitzian or (θ, ϱ)-Lipschitzian and write $f \in \mathrm{Lip}^{\theta}(\Omega)$.

The next definition is aimed to provide an appropriate language to single out the class of sets $\Omega \subseteq X$, with a prescribed way of how the Lipschitz θ-behaviour worsens to Hölder α-behaviour, $\alpha < \theta$, when x and y approach the boundary.

Definition 4.3. Let $\Omega \subset X$ be a measurable set and $\alpha \in (0, \theta]$. We say that Ω has the α-property, if there exists $c > 0$ such that

$$|J_{\Omega,\alpha}(x, y)| \le c\, \frac{\varrho(x, y)^{\theta}}{\max\{\delta(x), \delta(y)\}^{\theta-\alpha}} \quad \text{if } \varrho(x, y) \le \frac{1}{2^{\frac{1}{\theta}}} \max\{\delta(x), \delta(y)\}, \tag{4.3}$$

for all $x, y \in \Omega$.

Lemma 4.4. *Let Ω be measurable and bounded, ϱ be regular of order $\theta \in (0, 1]$ and $\alpha \in (0, \theta]$. Then*

$$\varrho(x, y) \le \frac{1}{2^{\frac{1}{\theta}}} \max\{\delta(x), \delta(y)\} \Longrightarrow |\delta(x)^\alpha - \delta(y)^\alpha| \le \alpha 2^{\frac{2-\alpha-\theta}{\theta}} \frac{\varrho(x, y)^\theta}{\max\{\delta(x), \delta(y)\}^{\theta-\alpha}}. \tag{4.4}$$

Proof. The case $\alpha = \theta$ is direct: the function $[\delta(x)]^\alpha = \inf_{z\in\partial\Omega} d(x, z)$ is (θ, ϱ)-Lipschitzian: $\delta(x)^\alpha - \delta(y)^\alpha \le d(x, y) = \varrho(x, y)^\theta$. Then $J_{\Omega,\alpha}(x) \in \mathrm{Lip}^\theta(\Omega)$ and consequently $|J_{\Omega,\alpha}(x, y)| \le c\, \varrho(x, y)^\theta$ for all $x, y \in \Omega$.

Let $0 < \alpha < \theta$. We first note that

$$|\delta(x) - \delta(y)| \le 2^{\frac{1}{\theta}-1} \varrho(x, y)^\theta \max\{\delta(x), \delta(y)\}^{1-\theta}. \tag{4.5}$$

Indeed, since $\varrho(x, y) = d(x, y)^{\frac{1}{\theta}}$ by Definition 4.3, where $d(x, y)$ is a distance, we have $|\delta(x) - \delta(y)| = \Big|d(x, \partial\Omega)^{\frac{1}{\theta}} - d(y, \partial\Omega)^{\frac{1}{\theta}}\Big| \le 2^{\frac{1}{\theta}-1} d(x, y) \max\{d(x, \partial\Omega)^{\frac{1}{\theta}-1}, d(y, \partial\Omega)^{\frac{1}{\theta}-1}\}$, from which (4.5) follows.

By (2.5) we have

$$\big|\delta(x)^\alpha - \delta(y)^\alpha\big| \le \alpha \frac{|\delta(x) - \delta(y)|}{[\min\{\delta(x), \delta(y)\}]^{1-\alpha}}.$$

By the condition $\varrho(x, y) \le \frac{1}{2^{\frac{1}{\theta}}} \max\{\delta(x), \delta(y)\}$, we have

$$\min\{\delta(x), \delta(y)\} \ge 2^{-\frac{1}{\theta}} \max\{\delta(x), \delta(y)\}. \tag{4.6}$$

Indeed, suppose that $\delta(y) \le \delta(x)$; to estimate $\delta(x)$, note that $\delta(x) = \inf_{z\in\partial\Omega} d(x, z)^{\frac{1}{\theta}}$ where d is a distance, so that

$$\delta(x)^\theta \le \delta(y)^\theta + \varrho(x, y)^\theta$$

and then $\delta(x) \le 2^{\frac{1}{\theta}-1} [\delta(y) + \varrho(x, y)] \le 2^{\frac{1}{\theta}-1}\delta(y) + \frac{1}{2}\delta(x)$, whence $\delta(x) \le 2^{\frac{1}{\theta}} \delta(y)$, which proves (4.6).

Therefore, by (4.6)

$$\big|\delta(x)^\alpha - \delta(y)^\alpha\big| \le \alpha 2^{\frac{1-\alpha}{\theta}} \frac{|\delta(x) - \delta(y)|}{[\max\{\delta(x), \delta(y)\}]^{1-\alpha}},$$

where it remains to apply (4.5). □

The following corollary provides a sufficient condition for a domain Ω to possess the α-property, this condition being inspired by Examples in Section 3.

Corollary 4.5. *Under the assumptions of Lemma 4.4, if $J_{\Omega,\alpha}(x)$ has the structure*

$$J_{\Omega,\alpha}(x) = c\delta(x)^\alpha + g(x), \quad x \in \Omega, \tag{4.7}$$

where c is a constant and $g \in \mathrm{Lip}^\theta(\overline{\Omega})$, then Ω possesses the α-property.

Remark 4.6. In [15] it was shown that any uniform domain (Jones domain) in $\mathbb{R}^n$ possesses the α-property. In particular, the function $J_{\Omega,\alpha}(x, y)$ was explicitly calculated there for a ball B, the half space $\mathbb{R}^n_+$ and the quarter plane $\mathbb{R}^2_{++}$.

5. Mapping properties of the potential operator I^α in generalized Hölder type spaces

In this section the measurable open set $\Omega \subseteq X$ is supposed to be bounded. We now introduce the generalized Hölder spaces on a set Ω, with the continuity modulus

$$\omega(f, h) = \sup_{\substack{x,y\in\Omega:\\ \varrho(x,y)<h}} |f(x) - f(y)|$$

dominated by a given function ω.

Definition 5.1. Given a continuous semi-additive function ω on $[0, \mathrm{diam}\,\Omega]$, positive for $h > 0$, with $\omega(0) = 0$, by $H^\omega(\Omega)$ we denote the space of functions $f \in C(\overline{\Omega})$ with the finite norm

$$\|f\|_{H^\omega} = \|f\|_{C(\bar{\Omega})} + \sup_{0<h<\mathrm{diam}\,\Omega} \frac{\omega(f, h)}{\omega(h)}.$$

By $H^\omega_0(\Omega)$ we denote the subspace in $H^\omega(\Omega)$ of functions f which vanish on the boundary $\partial\Omega$ of Ω.

Definition 5.2. A non-negative function $\omega(t)$ is called almost increasing (almost decreasing) on $[0,d]$, $0<d\le\infty$, if $\omega(t)\le C\omega(\tau)$ for all $t\le\tau$ ($t\ge\tau$, respectively).

For further goals we need the following auxiliary estimate.

Lemma 5.3. *Let $0<\alpha<\theta$ and $\Omega\subseteq X$ have the α-property. Let $f\in H_0^{\omega}(\Omega)$, where*

$$\omega(h) \text{ is almost increasing} \quad \text{and} \quad \frac{\omega(h)}{h^{\theta-\alpha}} \text{ is almost decreasing}. \tag{5.1}$$

Then

$$\sup_{x,y\in\Omega:\varrho(x,y)<h} \left|f(x)[J_{\Omega,\alpha}(x,y)]\right| \le C\omega_\alpha(h)\|f\|_{H^\omega(\Omega)}, \tag{5.2}$$

where $\omega_\alpha(h)=h^\alpha\omega(h)$. In particular,

$$\sup_{x,y\in\Omega:\varrho(x,y)<h} \left|f(x)[J_{\Omega,\alpha}(x,y)]\right| \le Ch^{\alpha+\lambda}\|f\|_{H^\lambda(\Omega)}, \tag{5.3}$$

when $f\in H_0^\lambda(\Omega)$ and $\lambda+\alpha\le\theta$.

Proof. For $x\in\Omega$, by $\widetilde{x}$ we denote a point of the boundary, such that $\varrho(x,\widetilde{x})=\delta(x)$. Then

$$|f(x)|=|f(x)-f(\widetilde{x})|\le C\omega(\delta(x))\|f\|_{H^\omega} \tag{5.4}$$

and

$$\left|f(x)[J_{\Omega,\alpha}(x,y)]\right| \le C\omega(\delta(x))\|f\|_{H^\omega}|J_{\Omega,\alpha}(x,y)|. \tag{5.5}$$

We distinguish the cases

$$\frac{1}{2^{\frac{1}{\theta}}}\max\{\delta(x),\delta(y)\}\le\varrho(x,y) \quad \text{and} \quad \varrho(x,y)\le\frac{1}{2^{\frac{1}{\theta}}}\max\{\delta(x),\delta(y)\}.$$

In the first case we have $|J_{\Omega,\alpha}(x,y)|\le C\varrho(x,y)^\alpha$ by Lemma 4.1 and then

$$\left|f(x)[J_{\Omega,\alpha}(x,y)]\right| \le C\omega(2^{\frac{1}{\theta}}\varrho(x,y))\varrho(x,y)^\alpha\|f\|_{H^\omega}\le C\omega_\alpha(h)\|f\|_{H^\omega} \tag{5.6}$$

for all x,y such that $\varrho(x,y)<h$. In the second case, by the definition of the α-property we have $|J_{\Omega,\alpha}(x)-J_{\Omega,\alpha}(y)|\le C\frac{\varrho(x,y)^\theta}{(\max\{\delta(x),\delta(y)\})^{\theta-\alpha}}$. Then (5.6), (5.5) and (5.1) yield

$$\begin{aligned}\left|f(x)[J_{\Omega,\alpha}(x,y)]\right| &\le C\|f\|_{H^\omega}\frac{\omega(\delta(x))}{(\max\{\delta(x),\delta(y)\})^{\theta-\alpha}}\varrho(x,y)^\theta\\ &\le C\|f\|_{H^\omega}\frac{\omega(\varrho(x,y))}{\varrho(x,y)^{\theta-\alpha}}\varrho(x,y)^\theta\le C\|f\|_{H^\omega}\omega_\alpha(h),\end{aligned}$$

which completes the proof. □

The result on mapping properties of potentials in the generalized Hölder spaces $H^\omega(\Omega)$, stated in Theorem 5.5 below, was obtained in [14, Theorems 3.8 and 3.9] under the condition that

$$J_{\Omega,\alpha}\in H^{\omega_\alpha(\cdot)}(\Omega), \quad \text{where } \omega_\alpha(h)=h^\alpha\omega(h). \tag{5.7}$$

Note that this condition is obviously satisfied when the cancellation property holds (i.e. $J_{\Omega,\alpha}(x)\equiv const$), but it is restrictive in general, since $J_{\Omega,\alpha}(x)\sim\delta(x)^\alpha$ near the boundary (see for instance, examples (1)–(4) in Section 3) so that (5.7) fails for domains, in general. Note also that in [14] there was proved a more general result than we formulate here in Theorem 5.5: the order $\alpha=\alpha(x)$ and the function $\omega=\omega(x,h)$ were variable in [14]. We need first the following definition.

Definition 5.4. We say that a continuous non-negative function $\omega:\ [0,d)\to[0,\infty)$, $0<d\le\infty$, belongs to a Zygmund class Φ_β, $\beta>0$, if it is almost increasing and $\int_h^d\left(\frac{h}{t}\right)^\beta\frac{w(t)}{t}dt\le cw(h)$, for all $h\in(0,d)$, where $c>0$ does not depend on h.

Theorem 5.5. *Let Ω be a measurable open set in X, $0<\alpha<\theta$. If $\omega\in\Phi_{\theta-\alpha}$ and (5.7) holds. Then the operator I^α is bounded from the space $H^{\omega(\cdot)}(\Omega)$ into the space $H^{\omega_\alpha(\cdot)}(\Omega)$.*

Now, making use of the above arguments, we may avoid condition (5.7) on the set Ω, replacing it by the assumption that ω has the α-property, which holds in many applications, for instance for any domain in $\mathbb{R}^n$ with Lipschitz boundary. Namely, we prove the following theorem.

Theorem 5.6. *Let $0 < \alpha < \theta$ and Ω be a bounded measurable open set in X with α-property. If $\omega \in \Phi_{\theta-\alpha}$, then the potential operator I^α is bounded from $H_0^\omega(\Omega)$ to $H^{\omega_\alpha}(\Omega)$. In particular, it is bounded from $H_0^\lambda(\Omega)$ to $H^{\lambda+\alpha}(\Omega)$ if $\lambda+\alpha < 1$.*

Proof. We only have to adjust the proof in [14] for our goals. That proof was based on the direct estimation of the continuity modulus of the potential (see Theorem 3.4 in [14]) via the representation

$$\begin{aligned}(I^\alpha f)(x) - (I^\alpha f)(y) = & \int_{\varrho(x,z)<2h} [f(z)-f(x)]\varrho(x,z)^{\alpha-N} d\mu(z) - \int_{\varrho(x,z)<2h} [f(z)-f(x)]\varrho(y,z)^{\alpha-N} d\mu(z) \\ & + \int_{\varrho(x,z)>2h} [f(z)-f(x)] \left\{\varrho(x,z)^{\alpha-N} - \varrho(y,z)^{\alpha-N}\right\} d\mu(z) \\ & + f(x) \int_\Omega \left\{\varrho(x,z)^{\alpha-N} - \varrho(y,z)^{\alpha-N}\right\} d\mu(z) \\ =: & \ \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4\end{aligned}$$

for $x, y \in \Omega$ with $\varrho(x,y) < h$. The condition (5.7) was used in the proof of Theorem 3.4 in [14], only in the estimation of the term

$$\Delta_4 = f(x)[J_{\Omega,\alpha}(x) - J_{\Omega,\alpha}(y)],$$

while the estimates

$$|\Delta_k| \le Ch^\alpha \omega(f,h), \quad k = 1,2, \quad \text{and} \quad |\Delta_3| \le Ch^\theta \int_h^d \frac{\omega(f,t)}{t^{1+\theta-\alpha}} dt, d = \operatorname{diam} \Omega, \tag{5.8}$$

of the terms Δ_1, Δ_2 and Δ_3 were proved without the assumption (5.7).

The term Δ_4 is now estimated by means of Lemma 5.3. Note that condition (5.1) assumed in that lemma follows from the assumption $\omega \in \Phi_{\theta-\alpha}$. This completes the proof. □

Remark 5.7. For simplicity, we dealt with the case where $\omega(h)$ does not depend on x. Since in [14] the general case of $\omega(x,h)$ was treated, Theorem 5.6 is extended in the same way to this case. The only changes in the formulation of Theorem 5.6 are that the condition $\omega \in \Phi_{\theta-\alpha}$ now should be interpreted as belongness of $\omega(x,h)$ to $\Phi_{\theta-\alpha}$ in variable h uniformly in x, and we have to write $\omega_\alpha(x,h) = h^\alpha \omega(x,h)$. More interesting is the case where α may depend on $x \in \Omega$ and may vanish at some points of Ω, the effects of which were studied in [14], but this requires more efforts. We do not dwell on this case in this paper.

6. The case of spatial and spherical potentials in $\mathbb{R}^n$

6.1. Any domain in $\mathbb{R}^n$ possesses the α-property

We improve Lemma 3.8 from [15], where it was shown that the α-property holds for the so called uniform domains (Jones domains) by proving that the validity of the α-property does not depend on the structure of the boundary, as stated in the following lemma.

Lemma 6.1. *Every domain in $\mathbb{R}^n$ has the α-property,* $0 < \alpha < 1$.

Proof. To check the condition (4.3), we proceed as follows:

$$J_{\Omega,\alpha}(x) = \int_{|z-x|<\delta(x)} \frac{dz}{|z-x|^{n-\alpha}} + \int_{|z-x|>\delta(x)} \frac{dz}{|z-x|^{n-\alpha}} = c_{n,\alpha}\delta(x)^\alpha + A(x),$$

where $A(x) := \int_{\Omega\setminus B(x,\delta(x))} \frac{dz}{|z-x|^{n-\alpha}}$ and $c_{n,\alpha} = \frac{1}{\alpha}|\mathbb{S}^{n-1}|$. (The above lines are written supposing that $|\Omega| < \infty$, for simplicity; if $|\Omega| = \infty$, one should deal from the very beginning with the differences, as given in the sequel.)

Let for definiteness, $\delta(x) \ge \delta(y)$. We then proceed as follows

$$\begin{aligned}|J_{\Omega,\alpha}(x,y)| \le & \ c_{n,\alpha} |\delta(x)^\alpha - \delta(y)^\alpha| + |A(x) - A(y)| \\ \le & \ c_{n,\alpha} |\delta(x)^\alpha - \delta(y)^\alpha| + \int_{\Omega\setminus B(x,\delta(x))} \left| \frac{1}{|z-x|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \right| dz \\ & + \left| \int_{B(y,\delta(y))} \frac{dz}{|z-y|^{n-\alpha}} - \int_{B(x,\delta(x))} \frac{dz}{|z-y|^{n-\alpha}} \right| =: D_1 + D_2 + D_3.\end{aligned}$$

The term D_1 is estimated by Lemma 4.4. For D_2, by the inequality (2.5) we have

$$D_2 \le c|x-y| \int_{\Omega\setminus B(\Omega,\delta(x))} \frac{dz}{\min\{|z-x|^{n+1-\alpha}, |z-y|^{n+1-\alpha}\}}.$$

Recall that we have to make estimations for $x, y \in \Omega$ such that $|x-y| < \frac{1}{2}\max\{\delta(x), \delta(y)\}$. We then have $|z-y| \geq |z-x| - |x-y| \geq |z-x| - \frac{1}{2}\delta(x) > \frac{1}{2}|x-z|$. Hence,

$$D_2 \leq c|x-y| \int_{|z-x|>\delta(x)} \frac{dz}{|z-x|^{n+1-\alpha}} = c\frac{|x-y|}{\delta(x)^{1-\alpha}}.$$

The term D_3 is equal to

$$D_3 = \left| c_{n,\alpha}\delta(x)^\alpha - \int_{|z|<\delta(x)} \frac{dz}{|z-(x-y)|^{n-\alpha}} \right| = \left| c_{n,\alpha}\delta(x)^\alpha - J_{B(0,\delta(x)),\alpha}(x-y) \right|.$$

For the potential of a constant function over a ball there is known an exact expression in terms of the Gauss hypergeometric function:

$$J_{B(0,R),\alpha}(x) = c_{n,\alpha}R^\alpha F\left(-\frac{\alpha}{2}, \frac{n-\alpha}{2}; \frac{n}{2}; \frac{|x|^2}{R^2}\right) \quad \text{for } x \in B(0,R);$$

see [15, Lemma 2.1]. Therefore,

$$J_{B(0,\delta(x)),\alpha}(x-y) = c_{n,\alpha}\delta(x)^\alpha F\left(-\frac{\alpha}{2}, \frac{n-\alpha}{2}; \frac{n}{2}; \frac{|x-y|^2}{\delta(x)^2}\right) \quad \text{if } |x-y| < \delta(x).$$

The Gauss function $F\left(-\frac{\alpha}{2}, \frac{n-\alpha}{2}; \frac{n}{2}; z\right)$ is analytic in the circle $|z| < 1$, bounded in any closed subcircle $|z| \leq 1-\varepsilon, \varepsilon > 0$. Then, under our condition $|x-y| < \frac{1}{2}\delta(x)$ we have

$$J_{B(0,\delta(x)),\alpha}(x-y) = c_{n,\alpha}\delta(x)^\alpha \left[1 + C(x,y)\frac{|x-y|^2}{\delta(x)^2}\right],$$

where $C(x,y)$ is a bounded function. Therefore,

$$D_3 \leq C\delta(x)^\alpha \frac{|x-y|^2}{\delta(x)^2} \leq C\frac{|x-y|}{\delta(x)^{1-\alpha}},$$

which completes the proof. □

Theorem 5.6 and Lemma 6.1 yield the following statement.

Theorem 6.2. *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$, let $f \in H^\omega(\Omega)$ and $f\big|_{x\in\partial\Omega} \equiv f_0 = const$. If $\omega(h)$ satisfies the assumptions of Theorem 5.6, then the potential $I^\alpha_\Omega f$, $0 < \alpha < 1$, has the following structure*

$$I^\alpha_\Omega f(x) = f_0 a(x) + Kf(x), \quad x \in \Omega,$$

where K is an operator bounded from $H^\omega(\Omega)$ to $H^{\omega_\alpha}(\Omega)$, while the function $a(x)(= J_{\Omega,\alpha}(x))$ is Lipschitz beyond the boundary $\partial\Omega$ and its Hölder properties near the boundary are described by the condition

$$|a(x) - a(y)| \leq c\frac{|x-y|}{\max\{\delta(x), \delta(y)\}^{1-\alpha}}.$$

6.2. The case of spherical potentials over caps

Now let Ω be an arbitrary surface domain on the unit sphere $X = \mathbb{S}^n = \{\sigma = (\sigma_1, \ldots, \sigma_{n+1}) : |\sigma| = 1\}$ in $\mathbb{R}^{n+1}$, we will call it *spherical cap*. An application of Theorem 5.6 in this subsection is inspired by some applications [16] of spherical harmonic analysis to a problem of aerodynamics (the cap Ω in [16] was the semisphere $\mathbb{S}^n_+ := \{\sigma \in \mathbb{S}^n : \sigma_{n+1} > 0\}$). The corresponding potential has the form

$$I^\alpha_\Omega f(\xi) = \int_{\mathbb{S}^n_+} \frac{f(\sigma)\,d\sigma}{|\xi-\sigma|^{n-\alpha}}, \quad \xi \in \Omega, \tag{6.1}$$

where $d\sigma$ is the Lebesgue surface area.

Lemma 6.3. *Every spherical cap has the α-property, with respect to the potential (6.1), $0 < \alpha < 1$.*

Proof. The statement of the lemma may be derived from Lemma 6.1. To this end, we make use of the stereographic projection of $\mathbb{R}^n$ onto $\mathbb{S}^n$ in the space $\mathbb{R}^{n+1}$:

$$\xi = s(x) = \{s_1(x), s_2(x), \ldots, s_{n+1}(x)\}, \quad \xi \in \mathbb{S}^n, x \in \mathbb{R}^n \tag{6.2}$$

where

$$s_k(x) = \frac{2x_k}{1+|x|^2}, \quad k = 1, 2, \ldots, n \quad \text{and} \quad s_{n+1}(x) = \frac{|x|^2-1}{|x|^2+1},$$

so that $\delta(\xi, \partial\mathbb{S}^n_+) \to 0 \Longleftrightarrow |x| \to 1$.

This transforms spherical potential into the space potential and vice versa. Namely, the formula is valid:

$$\int_\Omega \frac{f(\sigma)\, d\sigma}{|\xi-\sigma|^{n-\alpha}} = 2^\alpha (1+|x|^2)^{\frac{n-\alpha}{2}} \int_{\Omega^*} \frac{f[s(y)]\, dy}{|x-y|^{n-\alpha}(1+|y|^2)^{\frac{n+\alpha}{2}}}, \quad x \in \Omega^*, \tag{6.3}$$

(see for instance, [19, Section 2.4]) where Ω^* is the image of Ω under the stereographical mapping. (We use this opportunity to note that the above formula in [19] contains a typo: the factor $(1+|x|^2)^{\frac{n-\alpha}{2}}$ was lost there.) We suppose that the cap Ω does not coincide with the whole sphere, this case being trivial. Then without losing generality, we may assume that the pole $(0, 0, \ldots, 0, 1)$ of the stereographical projection lies outside Ω. Then Ω^* is a bounded domain in $\mathbb{R}^n$ and the power weights appearing in (6.3) are differentiable functions bounded from below and above, so that the α-property of the cap Ω with respect to the spherical potential is reduced to that of the domain Ω^* with respect to the spatial potential and then it remains to apply Lemma 6.1. □

In view of Lemma 6.3, similarly to the previous subsection, from Theorem 5.6 we obtain that Theorem 6.2 remains valid if the spatial potential is replaced by the spherical one and a domain Ω in $\mathbb{R}^n$ by a spherical cap on $\mathbb{S}^n$.

Appendix. The function $J_{\Omega,\alpha}(x)$ in the case of the semisphere $\Omega = \mathbb{S}^n_+$

Let $X = \mathbb{S}^n$, ϱ be the Euclidean distance between points on $\mathbb{S}^n$ and μ the Lebesgue surface measure, and $\Omega = \mathbb{S}^n_+ := \{\sigma \in \mathbb{S}^n : \sigma_{n+1} > 0\}$. In this case the distance $\delta(\sigma) := \delta(\sigma, \Omega)$ of the point $\sigma \in \mathbb{S}^n$ to the boundary of the semisphere is calculated by the formula

$$\delta(\sigma) = C(\sigma_{n+1})|\sigma_n|,$$

where

$$C(\sigma_{n+1}) = \sqrt{\frac{2}{1+\sqrt{1-\sigma_{n+1}^2}}}, \quad 1 \le C(\sigma_{n+1}) \le \sqrt{2}. \tag{A.1}$$

Note that from the calculations below it is seen that $J_{\mathbb{S}^n_+,\alpha}(\xi)$ depends only on the last coordinate ξ_{n+1}. Observe also that $\delta(\xi, \partial\Omega) \to 0 \Longleftrightarrow \xi_{n+1} \to 0$, by (A.1).

Lemma A.1. *Let $0 < \alpha \le 1$. Then the potential $J_{\mathbb{S}^n_+,\alpha}(\xi)$ has the following behaviour near the boundary $\partial\Omega = \{\xi \in \mathbb{S}^n : \xi_{n+1} = 0\}$:*

$$J_{\mathbb{S}^n_+,\alpha}(\xi) = c_0 + 2c_1\xi_{n+1}^\alpha + \mathcal{K}(\xi), \tag{A.2}$$

where c_0 and c_1 are the same as in Example (1) in Section 3, $\mathcal{K} \in \mathrm{Lip}(\overline{\mathbb{S}^n_+})$, $\mathcal{K}(\xi) = k(\xi_{n+1})$ and $k(0) = 0$.

Proof. We use the stereographic projection (6.2) of $\mathbb{R}^n$ onto $\mathbb{S}^n$. By formula (6.3) we have

$$J_{\mathbb{S}^n_+,\alpha}(\xi) = 2^\alpha(1+|x|^2)^{\frac{n-\alpha}{2}} \int_{|y|>1} \frac{dy}{\left(1+|y|^2\right)^{\frac{n+\alpha}{2}} |x-y|^{n-\alpha}} \tag{A.3}$$

$$= \frac{2^\alpha(1+|x|^2)^{\frac{n-\alpha}{2}}}{|x|^{n-\alpha}} \int_{|y|<1} \frac{dy}{\left(1+|y|^2\right)^{\frac{n+\alpha}{2}} |x^*-y|^{n-\alpha}}, \tag{A.4}$$

where $x^* = \frac{x}{|x|^2}$, the last passage being made via the change of variables $y \to \frac{y}{|y|^2}$. Hence

$$J_{\mathbb{S}^n_+,\alpha}(\xi) = \frac{2^\alpha}{\left(1+|x^*|^2\right)^\alpha} \int_{|y|<1} \frac{dy}{|x^*-y|^{n-\alpha}} + h(\xi), \tag{A.5}$$

where the function

$$h(\xi) = \frac{2^\alpha}{\left(1+|x^*|^2\right)^\alpha} \int_{|y|<1} \frac{(1+|x^*|^2)^{\frac{n-\alpha}{2}} - \left(1+|y|^2\right)^{\frac{n+\alpha}{2}}}{\left(1+|y|^2\right)^{\frac{n+\alpha}{2}} |x^*-y|^{n-\alpha}}\, dy$$

is Lipschitzian (even continuously differentiable) for $|x^*| \leq 1 \Longleftrightarrow \xi_{n+1} \geq 0$. For the first term in (A.5), it remains to make use of a result of [15] on estimation of potentials of a constant function over balls (see example (1) in Section 3) which yields (A.2) after direct easy evaluations. □

Remark A.2. The potential $J_{\mathbb{S}^n_+,\alpha}(\xi)$ may be reduced to repeated one-dimensional integration by means of the known representation of potentials of radial functions over balls via the Riemann–Liouville fractional integrals:

$$\frac{1}{\gamma_n(\alpha)} \int_{|y|<a} \frac{f(|y|)dy}{|x-y|^{n-\alpha}} = 2^{-\alpha} r^{2-n} \left(I^{\frac{\alpha}{2}}_{0+} \left[s^{\frac{n-\alpha}{2}-1} \left(I^{\frac{\alpha}{2}}_{a^2-} f^* \right)(s) \right] \right)(r^2); \tag{A.6}$$

see [6, formulae (2.17) and (2.18)] and $f^*(t) = f(\sqrt{t})$ [20], see also [6, p. 590]. This implies

$$J_{\mathbb{S}^n_+,\alpha}(\xi) = \frac{2^\alpha \pi^{\frac{n}{2}} (1+r^2)^{n-\alpha} r^{-2+n}}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n-\alpha}{2}\right)} \int_0^{r^{-2}} \frac{s^{\frac{n-\alpha}{2}-1}}{(r^{-2}-s)^{1-\frac{\alpha}{2}}} ds \int_s^1 \frac{dt}{(1+t)^{\frac{n+\alpha}{2}} (t-s)^{1-\frac{\alpha}{2}}},$$

$$= \frac{2^\alpha \pi^{\frac{n}{2}} (1+r^2)^{n-\alpha} r^{-2+n}}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n-\alpha}{2}\right)} \int_0^{r^{-2}} \frac{s^{\frac{n-\alpha}{2}-1} B_{\frac{1-s}{2}}\left(\frac{\alpha}{2}, \frac{n}{2}\right)}{(1+s)^{\frac{n}{2}} (r^{-2}-s)^{1-\frac{\alpha}{2}}} ds,$$

where $r = |x|$ and $B_x(p, q)$ is the incomplete beta-function.

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