Variable exponent Herz spaces

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Abstract. We introduce a new type of variable exponent function spaces $\dot{H}^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ and $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ of Herz type, homogeneous and non-homogeneous versions, where all the three parameters are variable, and give comparison of continual and discrete approaches to their definition. Under the only assumption that the exponents p, q and α are subject to the log-decay condition at infinity, we prove that sublinear operators, satisfying the size condition known for singular integrals and bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, are also bounded in the nonhomogeneous version of the introduced spaces, which includes the case maximal and Calderón-Zygmund singular operators.

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1. Introduction

Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty, 1 \leq q < \infty$. The classical versions of Herz spaces $K^{\alpha}_{p,q}(\mathbb{R}^n)$ ([14]), known under the names of *nonhomogeneous* and *homogeneous* Herz spaces, are defined by the norms

$$\|f\|_{K^{\alpha}_{p,q}} := \|f\|_{L^p(B(0,1))} + \left\{\sum_{k\in\mathbb{N}} 2^{k\alpha q}\left(\int\limits_{2^k<|x|<2^{k+1}} |f(x)|^p dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \tag{1.1}$$

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left(\int\limits_{2^{k-1}<|x|<2^k} |f(x)|^p dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}, \tag{1.2}$$

respectively. They were studied in many papers, see for instance [7], [9], [13], [15], [16], [23].

Last two decades, under the influence of some applications revealed in [27], there was a vast boom of research of the so called variable exponent spaces, and operators in them, where the parameters defining the space or the operator, may depend on the point x of the underlying space. For the time being, the theory of such variable exponent Lebesgue, Orlicz, Lorentz, and Sobolev function spaces is widely developed, we refer to the recent book [5] and surveying papers [4], [19], [22], [28]. For variable exponent Morrey-Campanato spaces we refer to the papers [2], [10], [12], [20], [21] and [26].

Herz spaces with variable exponents have been recently introduced in [1], [15], [16]. In the last two papers the exponent p was variable, the remaining exponents α and q were kept constant. The most general results were obtained in [1], where the variability of α was allowed. The main results obtained, for instance in [1] concern the boundedness of sublinear operators (including the maximal function and Calderón-Zygmund singular operators) and a Spanne type result for the Riesz potential operator. The approach used in [1] allowed to cover the case where p and α are variable and depend on the point x of the underlying set, keeping the exponent q constant.

In this paper, we suggest another approach to introduce variable exponent Herz spaces. The main feature of this approach is that we replace the discrete ℓ^q-norm by the continual L^q-norm with respect to Haar measure (we show that this replacement keeps the norms equivalent in some situations, but this is not always the case). The advantage of this replacement is that all the proofs become shorter and more transparent, and, what is more important, it allows us to admit the variability of the exponent q as well. There is also another modification: we find more natural to introduce the variability of the exponent α not with respect to the point $x \in \mathbb{R}^n$, but the point $t \in \mathbb{R}_+$, where the L^q-integrability is taken (or with respect to the index k in the summation in (1.1)-(1.2)). The advance in covering the case where q is also variable, is based on the fact that the proofs in our approach lead us to Hardy type inequalities in variable exponent $L^{q(\cdot)}(\mathbb{R}_+)$-spaces, which we can derive from results of the paper [6].

Under the only assumption that the exponents p, q and α are subject to the log-decay condition at infinity, we prove that sublinear operators which satisfy the size condition known for singular integrals and bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, are also bounded in the nonhomogeneous version of the introduced spaces. This is applied to the maximal operator and to Calderón- Zygmund singular operators with standard kernel.

The paper is organized as follows. In the preliminary Section 2 we comment the replacement of the discrete ℓ^q-norm by the continual L^q-norm in the case of constant exponents. In Subsection 3.1 we recall some necessary preliminaries on variable exponent Lebesgue spaces, after which in Subsection 3.2 we introduce our definition of variable exponent Herz spaces. In Subsection 3.3 we consider variable exponent Herz spaces with variable p and α but constant q with the goal to show that in this case the discrete and continual norms are equivalent under some natural assumptions on $p(x)$ and $\alpha(t)$ and α_k, and also the norms are equivalent under the change of some other auxiliary parameters. In Subsection 4.1 we prove a statement on the boundedness of Mellin convolution operators in variable exponent Lebesgue spaces on $\mathbb{R}_+$ with Haar measure, important for our goals. Finally, the last section 5 contains the main result on the boundedness of sublinear operators with the size condition in the introduced spaces, and its proof. The section ends with application to maximal and singular operators.

N o t a t i o n:
$B(x,r)$ is the ball of radius r centered at the point x;
$R_{t,\tau} = B(0,\tau)\backslash B(0,t) = \{x : t < |x| < \tau\}$ is a spherical layer;
$R_k = R_{2^{k-1},2^k} = B(0,2^k)\backslash B(0,2^{k-1})$;
$\chi_E(x)$ is the characteristic function of a set E;
$\chi_{t,\tau}(x) = \chi_{R_{t,\tau}}(x)$;
$\bar{d}t = \frac{dt}{t}$ denotes the Haar measure on $\mathbb{R}_+$;
$\mathbb{N}$ is the set of all natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
$\mathbb{Z}$ is the set of all integers;
the equivalence $A \approx B$ for non-negative expressions A and B means that $C_1 A \le B \le C_2 A$ where $C_1 > 0$ and $C_2 > 0$ do not depend on A and B.

2. Preliminaries on Herz spaces with constant exponents

For our goals related to variable exponent spaces, the following lemma is important, which states that the discrete ℓ^q-norm may be equivalently replaced by the integral L^q-norm.

Let $1 \le p < \infty, 1 \le q < \infty$ and $\alpha \in \mathbb{R}$. For $\nu \ge 0$, let

$$k^\alpha_{p,q}(f) := \|f\|_{L^p(B(0,\gamma\nu+\varepsilon))} + \left\{ \int_\nu^\infty t^{\alpha q} \left(\int_{\gamma t<|x|<\delta t} |f(x)|^p \, dx \right)^{\frac{q}{p}} \frac{dt}{t} \right\}^{\frac{1}{q}} \tag{2.1}$$

where $\delta > \gamma > 0$, $\varepsilon > 0$ and the first term is omitted in the case $\nu = 0$. This norm is well known in the case $\nu = 0$ ([8], [14], [17], [18]). These norms are equivalent for different choices of the parameters $\delta > \gamma > 0, \varepsilon > 0$ (and also for different $\nu > 0$, see Lemmas 2.3, 3.5. Note that one cannot take $\varepsilon = 0$ in the above definition when $\nu > 0$, because the finiteness of only the second term in (2.1) implies integrability properties of the function f only for $|x| > \gamma\nu + \varepsilon$ for an arbitrarily small $\varepsilon > 0$, but does not provide any information on integrability of f in the layer $\gamma\nu < |x| < \gamma\nu + \varepsilon$, see the remark below.

Remark 2.1. To clarify the idea, take $n = 1, p = 1, q = 1$ for simplicity. Choose $\gamma = 1, \delta = 2$ and take $f(x) = \frac{1}{x-2}$ for $x \in (2,4)$ and $f(x) \equiv 0$ for $x \in \mathbb{R}\backslash(2,4)$. Then $\int_t^{2t} f(x)\,dx = \ln\frac{2(t-1)}{t-2}$, so that the second term in (2.1) exists, but the function f is not integrable.

The following lemma clarifies the local integrability of f in the layer $\gamma\nu+\varepsilon<|x|<N$. We take $\gamma=1,\delta=2,\nu=2$ for simpicity.

Lemma 2.2. *Let*

$$N(f):=\left\{\int_2^\infty t^{\alpha q}\left(\int_{t<|x|<2t}|f(x)|^p\,dx\right)^{\frac{q}{p}}\frac{dt}{t}\right\}^{\frac{1}{q}}<\infty.$$

Then

$$\|f\|_{L^p(B(0,R)\backslash B(0,2+\varepsilon))}\le C_{\varepsilon,R}N(f)$$

for arbitrarily large $R(\ge 4)$.

Proof. Let $\varepsilon\in(0,2)$. We have

$$N(f)\ge\left\{\int_{2+\varepsilon}^{2+2\varepsilon} t^{\alpha q}\left(\int_{2+2\varepsilon<|x|<2(2+\varepsilon)}|f(x)|^p\,dx\right)^{\frac{q}{p}}\frac{dt}{t}\right\}^{\frac{1}{q}}$$

whence $\|f\|_{L^p(B(0,4+\varepsilon)\backslash B(0,2+2\varepsilon))}\le c(\varepsilon)N(f)<\infty$. Then $f\in L^p(B(0,6-\varepsilon)\backslash B(0,2+\varepsilon))$. Iterating the arguments starting from the point $4+\varepsilon$ instead of $2+\varepsilon$, we can extend this to $f\in L^p(B(0,R)\backslash B(0,2+\varepsilon))$ for an arbitrarily large $R>0$.

□

Lemma 2.3. *I. When* $\nu>0$, *the norms* $\|f\|_{K^\alpha_{p,q}}$ *and* $k^\alpha_{p,q}(f)$ *are equivalent for all finite values of* γ,δ; *the norms* $k^\alpha_{p,q}(f)$ *are also equivalent to each other for different* $\nu=\nu_1>0$ *and* $\nu=\nu_2>0$.
II. When $\nu=0$, *the norms* $\|f\|_{\dot{K}^\alpha_{p,q}}$ *and* $k^\alpha_{p,q}(f)$ *are equivalent to each other for all finite values of* δ,γ.

The statement of Lemma 2.3 was established in [8], see also [17], [18], for $\nu=0$. We give its direct proof in the variable exponent case in Lemma 3.5.

Remark 2.4. Formal particular cases with $\gamma=0$ or $\delta=\infty$ of Herz spaces in integral norm as in (2.1), are known as the so called generalized local Morrey spaces ($\gamma=0,\delta<\infty$) in the case of variable exponents studied in in [2], [10], [12], [20], [19], [21], or complementary generalized local Morrey spaces ($\gamma=1,\delta=\infty$), studied with variable exponents in the case $q=\infty$ in [11]. However, we immediately loose the equivalence of norms, when we pass to $\gamma=0$ or $\delta=\infty$.

3. Herz spaces with variable exponent $p(x), q(t), \alpha(t)$

3.1. Preliminaries on variable exponent Lebesgue spaces

We refer to the book [5] and papers [24], [28], but recall some basics we need. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $p(\cdot)$ be a measurable function on Ω with values in $[1, \infty)$. We suppose that

$$1 \leq p_- \leq p(x) \leq p_+ < \infty, \tag{3.1}$$

where $p_- := \inf_{x\in\Omega} p(x), \quad p_+ := \sup_{x\in\Omega} p(x) < \infty$. By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions f on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : \ I_{p(\cdot)} \left(\frac{f}{\eta} \right) \leq 1 \right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

By $\mathcal{P}^{\log} = \mathcal{P}^{\log}(\Omega)$ we denote the class of functions defined on Ω satisfying the log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \Omega, \tag{3.2}$$

where $A = A(p) > 0$ does not depend on x, y.

We will also work with the variable exponent $L^{q(\cdot)}$-space with the Haar measure $\bar{d}t = \frac{dt}{t}$ on $\mathbb{R}_a+ = (a, \infty)$, where $a \geq 0$, is introduced in the usual way:

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}_a+;\bar{d}t)} = \inf \left\{ \lambda > 0 \int_a^\infty \left| \frac{f(t)}{\lambda} \right|^{q(t)} \bar{d}t \leq 1 \right\}.$$

3.2. Definition of variable exponent Herz spaces

We define the variable exponent Herz spaces as follows.

Definition 3.1. We define the variable exponent Herz space $H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ by the norm

$$\|f\|_{H_\nu^{p,q,\alpha}} := \|f\|_{L^{p(\cdot)}(B(0,\gamma\nu+\varepsilon))} + \left\| t^{\alpha(t)} \left\| f\chi_{R_{\gamma t,\delta t}} \right\|_{L^{p(\cdot)}} \right\|_{L^{q(\cdot)}((\gamma\nu,\infty),\bar{d}t)} < \infty, \tag{3.3}$$

where $0 < \gamma < \delta < \infty$ and $\varepsilon > 0$, and $p : \mathbb{R}^n \to [1, \infty)$, $q : [\gamma\nu, \infty) \to [1, \infty)$ and $\alpha : [\gamma\nu, \infty) \to \mathbb{R}$ are variable exponents. The cases $\nu = 0$ and $\nu > 0$ correspond to homogeneous and inhomogeneous Herz spaces, respectively.

In the notation of the space $H_\nu^{p,q,\alpha}$ we omit the dependence on γ, δ and ε, and distinguish only the cases $\nu = 0$ and $\nu > 0$. By Lemma 3.5, this definition is irrelevant to the choice of γ, δ and ε in the case where the exponent q is constant. However this is no more valid in general when q is variable: then the space may depend on the choice of the parameters γ, δ and ν.

Similarly to Lemma 2.2, the following statement holds.

Lemma 3.2. *Let* $0 < \varepsilon < 2$ *and* $4 \le R < \infty$. *Then*

$$\|f\|_{L^{p(\cdot)}(B(0,R)\setminus B(0,2+\varepsilon))} \le c(\varepsilon,R)\left\| t^{\alpha(t)}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}(2,\infty);\bar{d}t}.$$

The proof follows the same lines as in that of Lemma 2.2.

Denote $\mathbb{R}_{+,\nu} = \{t \in \mathbb{R} : \nu < t < \infty\}$, where $\nu > 0$. By $\mathcal{M}_\infty^{\log}(\mathbb{R}_{+,\nu})$ and $\mathcal{M}_{0,\infty}^{\log}(\mathbb{R}_+)$ we denote the classes of bounded functions on $\mathbb{R}_{+,\nu}$, $\mathbb{R}_+$, respectively, satisfying the decay conditions

$$|\alpha(t)-\alpha(+\infty)| \le \frac{A}{\ln(e+t)}, \quad t\in\mathbb{R}_{+,\nu}; \qquad |\alpha(t)-\alpha_{\pm\infty}| \le \frac{A_\pm}{\ln(e+t)}, t\in\mathbb{R}_+, \tag{3.4}$$

respectively.

Lemma 3.3. *Let* (3.1) *hold. Then the following equivalences of the norms are valid*

$$\|f\|_{H_\nu^{p,q,\alpha}} \approx \|f\|_{L^{p(\cdot)}(B(0,\gamma\nu+\varepsilon))} + \left\|t^{\alpha_\infty}\left\|f\chi_{R_{\gamma t,\delta t}}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}(\mathbb{R}_{+,\nu},\bar{d}t)}, \qquad \nu>0, \tag{3.5}$$

and

$$\|f\|_{H_0^{p,q,\alpha}} \approx \left\|t^{\alpha(0)}(1+t)^{\alpha_\infty-\alpha(0)}\left\|f\chi_{R_{\gamma t,\delta t}}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}(\mathbb{R}_+,\bar{d}t)}, \tag{3.6}$$

if $\alpha \in \mathcal{M}_\infty^{log}(\mathbb{R}_{+,\nu})$ *in the case of* (3.5) *and* $\alpha \in \mathcal{M}_{0,\infty}^{log}(\mathbb{R}_+)$ *in the case of* (3.6).

The proof is direct.

In the sequel, for $\Omega = \mathbb{R}^n$ or $\Omega = (\nu,\infty)$, by $\mathcal{P}_\infty(\Omega)$ we denote the set of exponents $p : \Omega \to [1,\infty)$ which satisfy the decay condition

$$|p(x)-p_\infty| \le \frac{C}{\ln(e+|x|)}.$$

3.3. On Herz spaces with variable $p(x)$ and $\alpha(t)$ and constant q

The variable exponent Herz spaces introduced and studied in [1] were defined in the case of constant q by the norms

$$\|f\|_{L^{p(\cdot)}(B(0,1))} + \left\{\sum_{k\in\mathbb{N}}\left\|2^{k\alpha(\cdot)}f\chi_{R_{k-1}}\right\|_{p(\cdot)}^q\right\}^{\frac{1}{q}} \tag{3.7}$$

and

$$\left\{\sum_{k\in\mathbb{Z}}\left\|2^{k\alpha(\cdot)}f\chi_{R_k}\right\|_{p(\cdot)}^q\right\}^{\frac{1}{q}} \tag{3.8}$$

similar to (1.1)-(1.2), where the variability of α is admitted with respect to the point x of the underlying space. We base ourselves on the idea of expanding the integral type norms (2.1) to the variable exponent case provided by Lemma 2.3 and define the homogeneous and inhomogeneous Herz spaces $H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ in the united way as follows below. With respect to the discretized forms, in the case of constant q they will be related not to the norms (3.7) and (3.8), but norms of the form

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q}} := \|f\|_{L^{p(\cdot)}(B(0,1))} + \left\{\sum_{k\in\mathbb{N}} 2^{qk\alpha_k} \left\|f\chi_{R_{k-1}}\right\|^q_{p(\cdot)}\right\}^{\frac{1}{q}} \tag{3.9}$$

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} := \left\{\sum_{k\in\mathbb{Z}} 2^{qk\alpha_k} \|f\chi_{R_k}\|^q_{p(\cdot)}\right\}^{\frac{1}{q}} \tag{3.10}$$

where the sequence α_k is supposed to be logarithmically stabilizing at $+\infty$ and $-\infty$: there exists $\alpha_\pm := \lim_{k\to\pm\infty} \alpha_k \in \mathbb{R}$ such that

$$|\alpha_k - \alpha_+| \le \frac{A}{\ln(e+k)}, \quad k\in\mathbb{N}; \qquad |\alpha_k - \alpha_-| \le \frac{A}{\ln(e+|k|)}, -k\in\mathbb{N} \tag{3.11}$$

(the second assumption required only in the case of (3.10)). For brevity, by $\mathcal{M}_+^{\log}(\mathbb{N})$ and $\mathcal{M}_\pm^{\log}(\mathbb{Z})$ we denote the classes of bounded sequences satisfying the first of the conditions in (3.11) and both of them, respectively.

Lemma 3.4. *Let* (3.1) *hold. The norms* $\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q}}$ *and* $\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}$ *defined in* (3.9) *and* (3.10) *are equivalent to the norms*

$$\|f\|_{K^{\alpha_+}_{p(\cdot),q}} := \|f\|_{L^{p(\cdot)}(B(0,1))} + \left\{\sum_{k\in\mathbb{N}} 2^{kq\alpha_+} \left\|f\chi_{R_{k-1}}\right\|^q_{p(\cdot)}\right\}^{\frac{1}{q}} \tag{3.12}$$

$$\|f\|_{\dot{K}^{\alpha_\pm}_{p(\cdot),q}} := \left\{\sum_{k\in\mathbb{Z}} \left(1+2^k\right)^{q\alpha_+} \left(\frac{2^k}{1+2^k}\right)^{q\alpha_-} \|f\chi_{R_k}\|^q_{p(\cdot)}\right\}^{\frac{1}{q}}, \tag{3.13}$$

respectively, if $\alpha \in \mathcal{M}_+^{log}(\mathbb{N})$ *in the former case and* $\alpha\in\mathcal{M}_\pm^{log}(\mathbb{Z})$ *in the latter case.*

The proof is a matter of direct verification via the decay conditions (see also Lemma 3.3).

Lemma 3.5. *Let* (3.1) *hold,* $\alpha \in \mathcal{M}^{loc}_\infty$, *if* $\nu > 0$ *and* $\alpha\in\mathcal{M}^{loc}_{0,\infty}$, *if* $\nu = 0$. *The norms* (3.3) *are equivalent to each other for different finite values of* γ *and* δ *(such that* $\gamma < \delta$*). They are also equivalent to each other under different choice of values of* $\nu \ne 0$. *Moreover, the norm* (3.3) *is equivalent to the norm* (3.9) *when* $\nu > 0$ *and norm* (3.10) *when* $\nu = 0$ *under any choice of the sequence* α_k *such that*

$$\alpha_+ = \alpha(+\infty) \quad \textit{and} \quad \alpha_- = \alpha(0). \tag{3.14}$$

Proof. I. *Equivalence between the norms* (3.3) *for different positive* λ, δ *and* ν. By Lemma 3.3 it suffices to consider the norms in the form (3.5)-(3.6). For brevity, keeping in mind that the dependence on λ, δ and ν is now of importance, we denote

$$A_f(\gamma,\delta;\nu) = \|f\|_{L^{p(\cdot)}(B(0,\gamma\nu+\varepsilon))} + \left\{\int\limits_{\nu}^{\infty} t^{q\alpha_\infty} \left\|f\chi_{R_{\gamma t,\delta t}}\right\|^q_{L^{p(\cdot)}} \bar{d}t\right\}^{\frac{1}{q}}, \qquad \nu>0,$$

with the right-hand side replaced by the expression in (3.6) when $\nu = 0$. By the dilation change of the variables t, it is easy to see that

$$A_f(\gamma,\delta;\nu) \approx A_f\left(1,\frac{\delta}{\gamma};\gamma\nu\right), \qquad 0<\gamma<\delta$$

(where the constants in the equivalence relation depend only on γ). Consequently, it suffices to deal only with the case $\gamma = 1$ and $\delta > 1$. For simplicity of calculations, we further consider the case $\nu = 0$. (The case $\nu > 0$ is similarly treated with Lemma 2.2 taken into account, but requires more technical details). We simplify the notation to

$$A_f(1,\delta) := \left\{\int\limits_0^{\infty} t^{\alpha(0)q}(1+t)^{q[\alpha_\infty-\alpha(0)]}\cdot \left\|f\chi_{R_{t,\delta t}}\right\|^q_{L^{p(\cdot)}} \bar{d}t\right\}^{\frac{1}{q}},$$

where we assume that $\delta > 1$. Let $\delta < \lambda$. Then

$$A_f(1,\delta) \le A_f(1,\lambda) \le C_{q,\alpha,\delta}\left[A_f(1,\delta) + A_f\left(1,\frac{\lambda}{\delta}\right)\right],$$

with $C_{q,\alpha,\delta}$ depending only on δ, but not depending on f, where the left-hand side inequality is obvious, and the right-hand side one is easily obtained by splitting $\chi_{(t,\lambda t)} = \chi_{(t,\delta t)} + \chi_{(\delta t,\lambda t)}$.

If $\lambda \le \delta^2$, then $A_f\left(1,\frac{\lambda}{\delta}\right) \le A(1,\delta)$ and the proof of the equivalence $A(1,\lambda) \approx A(1,\delta)$ is over. If $\delta^2 < \lambda \le \delta^3$, we similarly proceed and have $A_f\left(1,\frac{\lambda}{\delta}\right) \le C_{q,\alpha,\delta}\left[A_f(1,\delta) + A_f\left(1,\frac{\lambda}{\delta^2}\right)\right]$. Iterating this N times, $N = \left[\frac{\ln\lambda}{\ln\delta}\right]$, we obtain that $A_f(1,\lambda) \le CA_f(1,\delta)$ with C depending on λ and δ, but not depending on f.

II. *Equivalence of the norms* (3.3) *to the norms* (3.9)-(3.10). By Lemma 3.4, the norms (3.9)-(3.10) may be taken in the form (3.12)-(3.13). We consider the case $\nu = 0$ for simplicity; the case $\nu > 0$ may be similarly treated, with Lemma 2.2 taken into account.

In view of the first part of the lemma we can compare the norm $\|f\|_{\dot{K}^{\alpha\pm}_{p(\cdot),q}}$ with the right-hand side of (3.6) under the concrete choice of γ and δ. With the choice $\gamma = 1$ and $\delta = 2$, $\delta = 4$, for the norm (3.13) we have

$$C_1A_f(1,2) \le \|f\|_{\dot{K}^{\alpha\pm}_{p(\cdot),q}} \le C_2A_f(1,2),$$

where C_1 and C_2 do no depend on f. Indeed, with (3.14) taken into account, and denoting for brevity

$$a_k = \left(1+2^k\right)^{q\alpha_+} \left(\frac{2^k}{1+2^k}\right)^{q\alpha_-}$$

(the coefficient appeared in (3.13)), we have

$$A_f(1,2) = \left\{ \sum_{k\in\mathbb{Z}} \int\limits_{2^k}^{2^{k+1}} t^{\alpha(0)q}(1+t)^{q[\alpha_\infty-\alpha(0)]} \left\| f\chi_{R_{t,4t}} \right\|_{L^{p(\cdot)}}^q \frac{dt}{t} \right\}^{\frac{1}{q}}$$

$$\geq C \left\{ \sum_{k\in\mathbb{Z}} a_k \left\| f\chi_{R_{2^{k+1},2^{k+2}}} \right\|_{L^{p(\cdot)}}^q \int\limits_{2^k}^{2^{k+1}} \frac{dt}{t} \right\}^{\frac{1}{q}} \geq C\|f\|_{\dot{K}^{\alpha_\pm}_{p(\cdot),q}}.$$

Similarly,

$$A_f\,(1,2) \leq C \left\{ \sum_{k\in\mathbb{Z}} a_k \left\| f\chi_{R_{2^k,2^{k+2}}} \right\|_{L^{p(\cdot)}}^q \int\limits_{2^k}^{2^{k+1}} \frac{dt}{t} \right\}^{\frac{1}{q}}$$

$$\leq C \left\{ \sum_{k\in\mathbb{Z}} (a_k + a_{k-1}) \left\| f\chi_{R_{2^k,2^{k+1}}} \right\|_{L^{p(\cdot)}}^q \right\}^{\frac{1}{q}}$$

$$\leq C \left\{ \sum_{k\in\mathbb{Z}} a_k \left\| f\chi_{R_{2^k,2^{k+1}}} \right\|_{L^{p(\cdot)}}^q \right\}^{\frac{1}{q}} = C\|f\|_{\dot{K}^{\alpha_\pm}_{p(\cdot),q}},$$

which completes the proof. □

4. Auxiliary statements

4.1. On Mellin convolutions in variable exponents spaces $L^{q(\cdot)}(\mathbb{R}_+)$

Let

$$K\varphi(t) = \int_0^\infty \mathcal{K}\left(\frac{t}{\tau}\right)\varphi(\tau)\bar{d}\tau$$

be an integral operator with the Haar measure $\bar{d}\tau = \frac{d\tau}{\tau}$ and the kernel homogeneous of order 0, known also as Mellin convolution operator.

In Theorem 4.4 we give conditions for the boundedness of such operators in the variable exponent $L^{q(\cdot)}(\mathbb{R}_+;\bar{d}\tau)$-spaces. Theorem 4.4 will be derived from the following theorem below, which is a particular case of Theorem 4.6 from [6].

Theorem 4.1. *Let* $q \in \mathcal{P}^{log}_\infty(\mathbb{R}^n)$ *and*

$$k \in L^1(\mathbb{R}^n) \cap L^{s_0}(\mathbb{R}^n), \qquad \textit{where} \qquad \frac{1}{s_0} = 1 - \frac{1}{q_-} + \frac{1}{q_+}. \tag{4.1}$$

Then the convolution operator $f \to k * f$ is bounded in the space $L^{q(\cdot)}(\mathbb{R}^n)$.

Let

$$(W_q f)(u) = e^{-\frac{u}{q(\infty)}} f(e^{-u}) \ , \quad -\infty < u < \infty. \tag{4.2}$$

Lemma 4.2. *Let q_∞ be a number in $[1, \infty)$. A Mellin convolution operator K on $\mathbb{R}_+$ reduces to the convolution operator on $\mathbb{R}$ via the relation*

$$(W_q K W_q^{-1} \psi)(u) = \int\limits_{\mathbb{R}} h(u-v)\psi(v)dv, \tag{4.3}$$

where $h(u) = e^{-\frac{u}{q_\infty}} \mathcal{K}\left(e^{-u}\right)$ and $\|h\|_{L^1(\mathbb{R})} = \int\limits_0^\infty \tau^{-\frac{1}{q'_\infty}} |\mathcal{K}(\tau)| d\tau$.

Proof. The proof is a matter of direct verification. □

To be definite with the constants, we adopt the notation

$$A_q^\infty := \sup_{t \in \mathbb{R}_+^1} \left| \frac{1}{q(t)} - \frac{1}{q_\infty} \right| \ln\left(e+t\right), \qquad A_q^0 := \sup_{0<t\leq \frac{1}{e}} \left| \frac{1}{q(t)} - \frac{1}{q_0} \right| \ln \frac{1}{t}. \tag{4.4}$$

It can be easily checked that (4.4) and the condition $q(0) = q(\infty)$ imply that also

$$\sup_{t \in \mathbb{R}_+^1} \left| \left(\frac{1}{q(t)} - \frac{1}{q_\infty} \right) \ln t \right| \leq \max\{A_q^\infty, A_q^0\}. \tag{4.5}$$

We denote

$$q^*(u) = q\left(e^{-u}\right), \quad u \in \mathbb{R}.$$

Note that $q_0 = q_\infty \iff q^*_{-\infty} = q^*_{+\infty}$ and (4.5) is equivalent to

$$\left| \frac{1}{q^*(u)} - \frac{1}{q_\infty} \right| \leq \frac{1}{|u|} \max\{A_q^\infty, A_q^0\}, \quad t \in \mathbb{R}. \tag{4.6}$$

Note also that from (4.6) it follows that

$$\left| \frac{1}{q^*(u)} - \frac{1}{q_\infty} \right| \leq \frac{\max\{A_q^\infty, A_q^0\}}{e \ln(e+|u|)}, \quad t \in \mathbb{R}. \tag{4.7}$$

Lemma 4.3. *Let $u \in \mathcal{P}^{log}_{0,\infty}$ and $q_0 = q_\infty$. Then the operator W_p maps isomorphically the space $L^{q(\cdot)}(\mathbb{R}_+)$ onto the space $L^{q^*(\cdot)}(\mathbb{R})$ and*

$$e^{-A_q} \leq \|W_q\|_{L^{q^*(\cdot)}(\mathbb{R}_+) \to L^{q(\cdot)}(\mathbb{R})} \leq e^{A_q}, \tag{4.8}$$

where $A_q = \max\{A_q^0, A_q^\infty\}$.

Proof. The statement of the lemma was in fact proved in [6], Lemma 5.1. For completeness of presentation we give here its direct proof. We have

$$\int\limits_{\mathbb{R}} \left| \frac{W_q f(u)}{\lambda} \right|^{q^*(u)} du = \int\limits_{\mathbb{R}} \left| \frac{e^{-\frac{u}{q(0)}} f\left(e^{-u}\right)}{\lambda} \right|^{q^*(u)} du = \int\limits_{\mathbb{R}_+} \left| \frac{f(t)}{\lambda t^{\frac{1}{q(t)} - \frac{1}{q_0}}} \right|^{q(t)} dt. \tag{4.9}$$

From (4.5) it follows that $e^{-A_q} \leq t^{\frac{1}{q(t)}-\frac{1}{q_0}} \leq e^{A_q}$. Hence

$$\int_{\mathbb{R}_+} \left|\frac{f(t)}{\|W_q f\|_{q^*} e^{A_q}}\right|^{q(t)} dt \leq 1 = \int_{\mathbb{R}} \left|\frac{W_q f(u)}{\|W_q f\|_{q^*}}\right|^{q^*(u)} du \leq \int_{\mathbb{R}_+} \left|\frac{f(t)}{\|W_q f\|_{q^*} e^{-A_q}}\right|^{q(t)} dt. \tag{4.10}$$

which yields (4.8). □

Theorem 4.4. *Let $q \in \mathcal{P}^{log}_{0,\infty}(\mathbb{R}_+)$, $p_- > 1$ and $q_0 = q_\infty$. Then*

$$\|Kf\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leq C \|f\|_{L^{q(\cdot)}(\mathbb{R}_+)}, \tag{4.11}$$

if

$$\int_0^\infty t^{\frac{s}{q_\infty}} |\mathcal{K}(t)|^s dt < \infty \tag{4.12}$$

for $s = 1$ and $s = s_0$, where $\frac{1}{s_0} = 1 - \frac{1}{q_-} + \frac{1}{q_+}$.

Proof. By Lemmas 4.2 and 4.8, the boundedness of the operator K in $L^{q(\cdot)}(\mathbb{R}_+)$ is equivalent to the boundedness of the convolution operator $h * f = \int_{\mathbb{R}} h(u-v)f(v)dv$ in $L^{q_*(\cdot)}(\mathbb{R})$ with the kernel $h(u) = e^{-\frac{u}{p_0}} \mathcal{K}(e^{-u})$.

By Theorem 4.1, the latter convolution is bounded, if $h \in L^1(\mathbb{R}) \bigcap L^{s_0}(\mathbb{R})$ which is equivalent to (4.12).

□

Corollary 4.5. *Let $q \in \mathcal{P}^{loc}_{0,\infty}(\mathbb{R}_+)$ and $q_o = q_\infty$. The operator K is bounded in the space $L^q(\mathbb{R}_+; \bar{d}t)$, where $\bar{d}t = \frac{dt}{t}$, if*

$$\int_0^\infty |\mathcal{K}(t)|^s \frac{dt}{t} < \infty \quad \textit{for} \quad s = 1 \quad \textit{and} \quad s = s_0. \tag{4.13}$$

Proof. The $L^q(\mathbb{R}_+; \bar{d}t)$-boundedness of the operator K in terms of the corresponding modular means the following:

$$\int_0^\infty \left|\frac{Kf(t)}{t^{1/q(t)}}\right|^{q(t)} dt \leq C \quad \text{as soon as} \quad \int_0^\infty \left|\frac{f(t)}{t^{1/q(t)}}\right|^{q(t)} dt \leq 1.$$

By the decay condition and the assumption $q_0 = q_\infty$, this is equivalent to a similar condition with $q(t)$ in the exponent in the denominator replaced by q_∞. The latter condition means the $L^q(\mathbb{R}_+)$-boundedness of the operator

$$K_1\varphi(t) = \int_0^\infty \mathcal{K}_1\left(\frac{t}{\tau}\right) \varphi(\tau) \bar{d}\tau$$

with the kernel $\mathcal{K}_1(t) = t^{-\frac{1}{q_\infty}} \mathcal{K}(t)$. Applying condition (4.12) to the latter, we arrive at (4.13).

□

4.2. Two auxiliary lemmas

It is known that $\|\chi_{B(0,r)}\|_{p(\cdot)} \approx r^{\frac{n}{p(0)}}$ as $r \to 0$, if $p(x)$ satisfies the local log-condition, and $\|\chi_{B(0,r)}\|_{p(\cdot)} \approx r^{\frac{n}{p_\infty}}$ as $r \to \infty$, if $p(x)$ satisfies the local log-condition and the decay condition at infinity. In [1], Lemma 2.2, it was observed that the validity of similar equivalences for the norm $\|\chi_{B(0,2r)\setminus B(0,r)}\|_{p(\cdot)}$ do not require local log-condition: the decay conditions at the origin and infinity, respectively, are sufficient. In the following lemma we give a simpler proof showing that it follows directly from the definition of the norm.

Recall that from the decay conditions at the origin and infinity it follows that

$$|p(x) - p(0)| \leq \frac{A_0}{|\ln|x||}, \quad |x| \leq 1, \tag{4.14}$$

$$|p(x) - p_\infty| \leq \frac{A_\infty}{|\ln|x||}, \quad |x| > 1. \tag{4.15}$$

Lemma 4.6. *Let $D > 1$ and (4.14) or (4.15) be fulfilled. Then*

$$\frac{1}{c_0} r^{\frac{n}{p(0)}} \leq \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{p(\cdot)} \leq c_0 r^{\frac{n}{p(0)}} \quad \textit{for} \quad 0 < r \leq 1 \tag{4.16}$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{p_\infty}} \leq \|\chi_{B(0,Dr)\setminus B(0,r)}\|_{p(\cdot)} \leq c_\infty r^{\frac{n}{p_\infty}} \quad \textit{for} \quad r \geq 1, \tag{4.17}$$

respectively, where $c_0 \geq 1$ and $c_\infty \geq 1$ depend on D, but do not depend on r.

Proof. We prove (4.17), the arguments for (4.16) are similar. Recall that $\int\limits_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1 \iff \|f\|_{p(\cdot)} \leq \lambda$ and $\int\limits_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \geq 1 \iff \|f\|_{p(\cdot)} \geq \lambda$ for $\lambda > 0$. Therefore, the right-hand side inequality in (4.17) is equivalent to

$$\int\limits_{r<|x|<Dr} \frac{dx}{\left[c_0 r^{\frac{n}{p_\infty}}\right]^{p(x)}} \leq 1. \tag{4.18}$$

The left hand side in (4.18) is estimated as follows:

$$\int\limits_{r<|x|<Dr} \frac{dx}{\left[c_0 r^{\frac{n}{p_\infty}}\right]^{p(x)}} \leq \frac{1}{c_0^{p_-}} \int\limits_{r<|x|<Dr} \frac{dx}{\left(\frac{|x|}{D}\right)^{\frac{np(x)}{p_\infty}}} \leq \frac{D^{\frac{np_+}{p_\infty}}}{c_0^{p_-}} \int\limits_{r<|x|<Dr} \frac{dx}{|x|^{\frac{np(x)}{p_\infty}}}.$$

By the decay condition (4.15) we have $e^{-\frac{A_\infty}{p_\infty}}|x| \leq |x|^{\frac{p(x)}{p_\infty}} \leq e^{\frac{A_\infty}{p_\infty}}|x|, \quad |x| \geq 1.$ Therefore,

$$\int\limits_{r<|x|<Dr} \frac{dx}{\left[c_0 r^{\frac{n}{p_\infty}}\right]^{p(x)}} \leq \frac{D^{\frac{np_+}{p_\infty}} e^{\frac{nA_\infty}{p_\infty}}}{c_0^{p_-}} \int\limits_{r<|x|<Dr} \frac{dx}{|x|^n} = \frac{D^{\frac{np_+}{p_\infty}} e^{\frac{nA_\infty}{p_\infty}}}{c_0^{p_-}} |\mathbb{S}^{n-1}| \ln D, \tag{4.19}$$

which determines the choice of $c_0^p = D^{\frac{np_+}{p_\infty}} \ln D e^{\frac{nA_\infty}{p_\infty}} |\mathbb{S}^{n-1}|$, and proves the right-hand side inequality in (4.17).

Similarly, the left-hand side inequality in (4.17) is checked. □

Lemma 4.7. *The following relations*

$$\int\limits_{2a<|y|<t} |\Phi(y)|\, dy = \frac{1}{\ln 2}\int\limits_{a}^{t} \frac{d\tau}{\tau} \int\limits_{\max(2a,\tau)<|y|<\min(t,2\tau)} \Phi(y)\, dy, \quad t > 2a > 0, \tag{4.20}$$

$$\int_{|y|>2t} |\Phi(y)|\, dy = \frac{1}{\ln 2}\int_{t}^{\infty} \frac{d\tau}{\tau} \int_{\max(\tau,2t)<|y|<2\tau} |\Phi(y)|\, dy, \quad t > 0. \tag{4.21}$$

hold for every measurable function Φ for which the integrals on the left-hand side exists.

Proof. The proof is a matter of direct verification via the change of order of integration on the right-hand side. □

5. Main result

Our main result concerns the boundedness of a sublinear operator T satisfying the well known size condition

$$|Tf(x)| \le C \int\limits_{\mathbb{R}^n} \frac{|f(y)|\, dy}{|x-y|^n}, \qquad x \notin \operatorname{supp} f, \tag{5.1}$$

in the variable exponent non-homogeneous Herz spaces $H_\nu^{p,q,\alpha}$, i.e. in the case $\nu \ne 0$.

Theorem 5.1. *Let $p \in \mathcal{P}_\infty^{log}(\mathbb{R}^n)$ with $1 < p_- < p_+ < \infty$, $q \in \mathcal{P}_\infty^{log}(\nu,\infty)$ with $1 \le q_- \le q_+ < \infty$ and $\alpha \in \mathcal{P}_\infty(\nu,\infty)$. Then every sublinear operator T with the size condition* (5.1)*, bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, is also bounded in the Herz spaces $H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$, $\nu > 0$, if*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty}. \tag{5.2}$$

Proof. The proof does not depend on the concrete choice of $\nu > 0$ and δ, so we choose $\nu = 2$ and $\delta = 2$ for simplicity, but recall that the norms in the space $H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ are not necessarily equivalent for different values of these parameters when q is variable. So in the sequel we will work with the norm

$$\|f\|_{H_2^{p,q,\alpha}} = \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + N(f)_{p,q,\alpha}, \tag{5.3}$$

where we denoted

$$N(f)_{p,q,\alpha} := \left\| t^{\alpha_\infty} \left\| f\chi_{(t,2t)} \right\|_{L^{p(\cdot)}} \right\|_{L^q((2,\infty),\bar{d}t)} \tag{5.4}$$

for brevity.

We start with estimation of the first term $\|Tf\|_{L^{p(\cdot)}(B(0,2+\varepsilon))}$ in the norm (5.3). It suffices to consider $\varepsilon \in (0,1)$. We have

$$|Tf(x)| \le |T(f\chi_{B(0,8)}(x)| + |T(f\chi_{\mathbb{R}^n\setminus B(0,8)})(x)|,$$

where the estimate

$$\left\| T(f\chi_{B(0,8)}) \right\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} \le C\|f\|_{H_2^{p,q,\alpha}}$$

is immediate in view of the assumed boundedness of the operator T in $L^{p(\cdot)}$ and Lemma 3.2. To estimate the second term, we use the relation (4.21):

$$|T(f\chi_{\mathbb{R}^n\setminus B(0,8)})(x)| \le C\int\limits_4^\infty \frac{d\tau}{\tau}\int\limits_{\tau<|y|<2\tau}\frac{|f(y)|\,dy}{|x-y|^n}$$

and observe that for $x\in B(0,2+\varepsilon)\subset B(0,3)$ we have $|x-y|\ge|y|-|x|\ge \tau-3\ge\frac{\tau}{4}$,, so that

$$|T(f\chi_{\mathbb{R}^n\setminus B(0,8)})(x)| \le C\int\limits_4^\infty \tau^{-1-n}d\tau\int_{\tau<|y|<2\tau}|f(y)|dy.$$

By the Hölder inequality and estimate (4.17) we then obtain

$$|T(f\chi_{\mathbb{R}^n\setminus B(0,8)})(x)| \le C\int\limits_4^\infty \tau^{-n-1}\|f\chi_{R_{\tau,2\tau}}\|_{L^{p(\cdot)}}\|\chi_{R_{\tau,2\tau}}\|_{L^{p'(\cdot)}}d\tau$$

$$\le C\int\limits_4^\infty \tau^{-\alpha_\infty-\frac{n}{p}}\left(t^{\alpha_\infty}\|f\chi_{R_{\tau,2\tau}}\|_{L^{p(\cdot)}}\right)\frac{d\tau}{\tau}.$$

Applying the Holder inequality in the form

$$\left|\int\limits_0^\infty \varphi(\tau)\psi(\tau)\bar{d}\tau\right| \le 2\|\varphi\|_{L^{q(\cdot)}(\mathbb{R}_+.\bar{d}\tau)}\|\psi\|_{L^{q'(\cdot)}(\mathbb{R}_+.\bar{d}\tau)},$$

we obtain

$$|T(f\chi_{\mathbb{R}^n\setminus B(0,8)})(x)| \le C\|\tau^{-\alpha_\infty-\frac{n}{p}}\|_{L^{q(\cdot)}((4,\infty);\bar{d}\tau)}\|f\|_{H_2^{p,q,\alpha}} \le C\|f\|_{H_2^{p,q,\alpha}} \tag{5.5}$$

since the norm $\|\tau^{-\alpha_\infty-\frac{n}{p}}\|_{L^{q(\cdot)}((4,\infty);\bar{d}\tau)}$ is finite, or equivalently, the modular $\int_4^\infty \frac{\bar{d}\tau}{\tau^{q(\tau)(\alpha_\infty+\frac{n}{p})}} \approx \int_4^\infty \frac{\bar{d}\tau}{\tau^{q_\infty(\alpha_\infty+\frac{n}{p})}}$ is finite. Consequently,

$$\||T(f\chi_{\mathbb{R}^n\setminus B(0,8)})|\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} \le C\|f\|_{H_2^{p,q,\alpha}}.$$

The main task, however, is to estimate the seminorm $N(Tf)_{p,q,\alpha}$. To this end, we split the function $f(x)$ as

$$f(x)=f_0(x)+f_t(x)+g_t(x)+h_t(x),$$

where

$$f_0(x)=f(x)\chi_{B(0,1)}(x),\quad f_t(x)=f(x)\chi_{B\left((0,\frac{t}{2})\setminus B(0,1)\right.}(x),$$

$$g_t(x)=f(x)\chi_{B(0,8t)\setminus B\left(0,\frac{t}{2}\right)}(x),\quad h_t(x)=f(x)\chi_{\mathbb{R}^n\setminus B(0,8t)}(x),$$

depending on the parameter $t\in(2,\infty)$. This "continual" decomposition is similar to the analogous "discrete" decomposition of such a type used earlier for Herz spaces in [25] in the case of constant exponents and in [1] in the case of variable p and α. Then

$$|Tf(x)| \leq |Tf_0(x)| + |Tf_t(x)| + |Tg_t(x)| + |Th_t(x)|.$$

Estimation of $Tf_0(x)$. For $x \in R_{t,2t}$ and $y \in B(0,1)$ we have $|x-y| \geq |x| - |y| > t - 1 \geq \frac{t}{2}$, so that

$$|Tf_0(x)| \leq \frac{c}{t^n} \int\limits_{B(0,1)} |f(y)|dy \leq \frac{c}{t^n} \|f\chi_{B(0,1)}\|_{p(\cdot)} \|\chi_{B(0,1)}\|_{p'(\cdot)} = \frac{c}{t^n} \|f\chi_{B(0,1)}\|_{p(\cdot)}.$$

Consequently,

$$t^{\alpha_\infty} \|\chi_{R_{t,2t}} Tf_0\|_{p(\cdot)} \leq t^{\alpha_\infty - n} \|f\chi_{B(0,1)}\|_{p(\cdot)} \|\chi_{R_{t,2t}}\|_{p(\cdot)} \leq Ct^{(\alpha_\infty - n + \frac{n}{p_\infty})} \|f\chi_{B(0,2)}\|_{p(\cdot)},$$

by (4.17). Therefore,

$$N(Tf_0)_{p,q,\alpha} \leq C \|t^{\alpha_\infty - \frac{n}{p'_\infty}}\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)} \|f\chi_{B(0,2)}\|_{p(\cdot)} \leq C \|f\chi_{B(0,2)}\|_{p(\cdot)}$$

where the norm $\|t^{\alpha_\infty - \frac{n}{p'_\infty}}\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)}$ is finite, which is justified as in (5.5).

Estimation of $T(f_t)$. We have

$$|Tf_t(x)| \leq C \int\limits_{B\left(0,\frac{t}{2}\right)\setminus B(0,1)} \frac{|f(y)|}{|x-y|^n}\, dy, \quad x \in R_{t,2t}.$$

where $|x-y| \geq |x| - |y| \geq \frac{t}{2}$, so that

$$|Tf_t(x)| \leq \frac{C}{t^n} \int\limits_{1<|y|<\frac{t}{2}} |f(y)|\, dy,$$

We use the relation (4.20) and obtain

$$|Tf_t(x)| \leq \frac{C}{t^n} \int\limits_1^t \|f\chi_{R_{\frac{\tau}{2},\tau}}\|_{p(\cdot)} \|\chi_{R_{\frac{\tau}{2},\tau}}\|_{p'(\cdot)} \frac{d\tau}{\tau}$$

and then

$$|Tf_t(x)| \leq \frac{C}{t^n} \int\limits_1^t \|f\chi_{R_{\frac{\tau}{2},\tau}}\|_{p(\cdot)} \tau^{\frac{n}{p'_\infty} - 1}\, d\tau$$

by (4.17).Therefore,

$$t^{\alpha_\infty} \|\chi_{R_{t,2t}} Tf_t(x)\|_{p(\cdot)} \leq Ct^{\alpha_\infty - n} \|\chi_{R_{t,2t}}\|_{p(\cdot)} \int\limits_1^t \|f\chi_{R_{\frac{\tau}{2},\tau}}\|_{p(\cdot)} \tau^{\frac{n}{p'_\infty} - 1}\, d\tau.$$

Consequently,

$$t^{\alpha_\infty} \|\chi_{R_{t,2t}} Tf_t(x)\|_{p(\cdot)} \leq Ct^{\alpha_\infty - \frac{n}{p'_\infty}} \int\limits_1^t \|f\chi_{R_{\frac{\tau}{2},\tau}}\|_{p(\cdot)} \tau^{\frac{n}{p'_\infty} - 1}\, d\tau$$

by the same formula (4.17). This may be rewritten in the form

$$t^{\alpha_\infty}\|\chi_{R_{t,2t}} Tf_t(x)\|_{p(\cdot)} \le C \int\limits_1^t \left(\frac{t}{\tau}\right)^{\alpha_\infty - \frac{n}{p'_\infty}} \varphi(\tau)\, \bar{d}\tau$$

where $\varphi(\tau) = \tau^{\alpha_\infty}\|f\chi_{R_{\frac{\tau}{2},\tau}}\|_{p(\cdot)}$. We arrived at a Hardy type operator. It is bounded in the space $L^{q(\cdot)}((1,\infty);\bar{d}t)$ with the Haar measure by Corollary 4.5, since $\alpha - \frac{n}{p'_\infty} < 0$: choose $\mathcal{K}(t) = t^{\alpha - \frac{n}{p'_\infty}}$ for $t > 1$ and $\mathcal{K}(t) = 0$ for $0 < t < 1$ in Corollary 4.5. Then

$$N(Tf_t)_{p,q,\alpha} \le C\|\varphi\|_{L^{q(\cdot)}((1,\infty);\bar{d}t)} \le C\|f\|_{L^{p(\cdot)}(B(0,2)} + CN(f)_{p,q,\alpha} \le C\|f\|^q_{H_2^{p,q,\alpha}}$$

Estimation of $T(g_t)$. By the boundedness of the operator T in the space $L^{p(\cdot)}(\mathbb{R}^n)$ we obtain

$$\left\|(Tg_t)\chi_{t,2t}\right\|_{L^{p(\cdot)}} \le C\,\|g_t\|_{L^{p(\cdot)}} = C\left\|f\chi_{\frac{t}{2},8t}\right\|_{L^{p(\cdot)}}$$

$$\le C\left\|f\chi_{\frac{t}{2},t}\right\|_{L^{p(\cdot)}} + C\sum_{j=0}^{2}\left\|f\chi_{2^j t,2^{j+1}t}\right\|_{L^{p(\cdot)}}.$$

Then

$$N(Tg_t)_{p,q,\alpha} \le C\left\|t^{\alpha_\infty}\left\|f\chi_{\frac{t}{2},t}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)} + C\left\|t^{\alpha_\infty}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)}$$

$$\le C\left\|t^{\alpha_\infty}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}((1,2);\bar{d}t)} + C\left\|t^{\alpha_\infty}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)}.$$

Since $\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}} \le \left\|f\chi_{0,4}\right\|_{L^{p(\cdot)}}$, we obtain

$$N(Tg_t)_{p,q,\alpha} \le C\|f\|_{H_\nu^{p,q,\alpha}}$$

with Lemma 3.2 taken into account.

Estimation of $T(h_t)$. We take $x \in R_{t,2t}$ and proceed as follows:

$$|Th_t(x)| \le C\int_{|y|>8t}\frac{|f(y)|\,dy}{|x-y|^n}.$$

Now we use the relation (4.21) and get

$$|Th_t(x)| \le C\int_{4t}^{\infty}\left(\int\limits_{R_{\tau,2\tau}}\frac{\|f\chi_{\tau,2\tau}\|_{L^{p(\cdot)}}\|\chi_{\tau,2\tau}\|_{L^{p'(\cdot)}}}{|x-y|^n}dy\right)\bar{d}\tau.$$

Since $|x-y| \ge |y| - |x| \ge \tau - 2t \ge \tau - 2\frac{\tau}{4} = \frac{\tau}{2}$, by the property (4.17) we obtain

$$t^{\alpha_\infty}\|\chi_{R_{t,2t}}Th_t(x)\|_{L^{p(\cdot)}} \le Ct^{\alpha_\infty}\|\chi_{R_{t,2t}}\|_{p(\cdot)}\int_{2t}^{\infty}\tau^{-\frac{n}{p_\infty}}\|f\chi_{\tau,2\tau}\|_{L^{p(\cdot)}}\bar{d}\tau$$

$$\le C\int_t^{\infty}\left(\frac{t}{\tau}\right)^{\alpha_\infty + \frac{n}{p_\infty}}\psi(\tau)\,\bar{d}\tau,$$

where $\psi(\tau) = \tau^{\alpha_\infty} \|f\chi_{R_{\tau,2\tau}}\|_{p(\cdot)}\chi_{(2,\infty)}(\tau)$. Thus we have arrived at a Hardy type inequality. Since $\alpha_\infty + \frac{n}{p_\infty} > 0$, the operator on the right-hand side is bounded in the space $L^{q(\cdot)}((2,\infty); \bar{d}t)$ by Corollary 4.5. Consequently,

$$\left\| t^{\alpha_\infty} \|\chi_{R_{t,2t}} Th_t(x)\|_{L^{p(\cdot)}} \right\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)} \leq C\|\psi\|_{L^{q(\cdot)}(\mathbb{R}_+;\bar{d}t)} \leq C\|f\|_{H_2^{p,q,\alpha}},$$

which completes the proof. □

The obtained result may be naturally applied to the maximal operator

$$\mathbb{M}f(x) = \sup_{r>0} \frac{1}{B(x,r)} \int\limits_{B(x,r)} |f(y)|\, dy$$

and singular integrals.

Corollary 5.2. *Let the variable exponents satisfy the assumptions:*
i) $p \in \mathcal{P}^{log}(\mathbb{R}^n) \cap \mathcal{P}^{log}_\infty(\mathbb{R}^n), \quad 1 < p_- \leq p(x) \leq p_+ < \infty,$
ii) $q \in \mathcal{P}^{log}_\infty(2,\infty), \quad 1 \leq q_- \leq q(x) \leq q_+ < \infty,$
iii) $\alpha \in \mathcal{M}^{log}_\infty(2,\infty)$. *Then the maximal operator is bounded in the variable exponent non-homogeneous space* $H_2^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ *under the condition* (5.2).

Proof. Since the maximal operator satisfies the size condition (5.1), when applying Theorem 5.1, we only have to refer to conditions which guaranteer the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of the maximal operator. As is known, one of the version of such conditions is that in *i)*, see [3], [5]. □

Another application covers Calderón-Zygmund type singular operators

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} K(x,y)f(y)\, dy,$$

which are bounded in $L^2(\mathbb{R}^n)$ and have a standard singular kernel $K(x,y)$, i.e. $K(x,y)$ is continuous on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ and

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|x-\xi|.$$

Corollary 5.3. *Let the exponents* $p(x), q(t)$ *and* $\alpha(t)$ *satisfy the assumptions i)-iii) of Corollary* 5.2. *Then the singular operator* T *with a standard kernel is bounded in the space* $H_2^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ *under the condition* (5.2).

As in the proof of the previous corollary, we only need to know that T is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. To this end, it suffices to refer for instance to Corollary 7.2.7 (and Theorem 4.4.8) in [5].

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Erratum to "Variable exponent Herz spaces", Mediterr. J. Math. DOI: 10.1007/s00009-013-0285-x, 2013

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Abstract. We fill in a gap in the proof of Theorem 5.1 of [1] on the boundedness of sublinear operators of singular type in variable exponent Herz type spaces $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$. When q is constant, the formulation of Theorem 5.1 from [1] remains the same. In the case where q is variable, Theorem 5.1 needs a more precise formulation with respect to some auxiliary parameters of the space (not reflected in the notation $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ of the space).

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1. Introduction

We use all the definitions and notations from [1] and deal with sublinear operators T satisfying the well known size condition

$$|Tf(x)| \leq C \int\limits_{\mathbb{R}^n} \frac{|f(y)|\,dy}{|x-y|^n}, \qquad x \notin \operatorname{supp} f. \tag{1.1}$$

Theorem 5.1 of [1] was formulated in [1] as follows.

Theorem A. *Let $p \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $1 < p_- < p_+ < \infty$, $q \in \mathcal{P}_\infty^{\log}(\nu,\infty)$ with $1 \leq q_- \leq q_+ < \infty$ and $\alpha \in \mathcal{P}_\infty(\nu,\infty)$. Then every sublinear operator T with the size condition* (1.1)*, bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, is also bounded in the Herz spaces $H_\nu^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$, $\nu > 0$, if*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty}. \tag{1.2}$$

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In the corrected version of this theorem given below, the dependence, on the auxiliary parameters ν, γ, δ is already of importance when q is variable. For simplicity we choose $\nu = 2, \gamma = 1, \delta = 2$, but in the case of variable q we will also have to deal with values of γ and δ different from 1 and 2, respectively. Keeping this in mind, we slightly change the notation of the norm by redenoting the norm as follows

$$\|f\|_{H^{p,q,\alpha}_{(\gamma,\delta)}} := \|f\|_{L^{p(\cdot)}(B(0,2\gamma+\varepsilon))} + \left\| t^{\alpha(t)} \left\| f\chi_{R_{\gamma t,\delta t}} \right\|_{L^{p(\cdot)}} \right\|_{L^{q(\cdot)}((\gamma,\infty),\bar{d}t)}. \quad (1.3)$$

The corrected version of Theorem A runs as follows.

Theorem 1.1. *Let* $p \in \mathcal{P}^{\log}_{\infty}(\mathbb{R}^n)$ *with* $1 < p_- < p_+ < \infty,\ q \in \mathcal{P}^{\log}_{\infty}(\nu,\infty)$ *with* $1 \leq q_- \leq q_+ < \infty$ *and* $\alpha \in \mathcal{P}_{\infty}(\nu,\infty)$. *Let also*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty}. \quad (1.4)$$

Then every sublinear operator T *with the size condition* (1.1), *bounded in* $L^{p(\cdot)}(\mathbb{R}^n)$, *is also bounded within the frameworks of the Herz spaces in the following sense:*

$$\|Tf\|_{H^{p,q,\alpha}_{(\gamma,\delta)}} \leq C\|f\|_{H^{p,q,\alpha}_{(\gamma',\delta')}} \quad (1.5)$$

for any $0 < \gamma' < \gamma$ *and* $\delta < \delta' < \infty$. *In the case where* q *is constant,* (1.5) *holds with* $\gamma' = \gamma$ *and* $\delta' = \delta$.

Proof. We dwell only on the arguments where the changes should be made and skip the parts of the proof, where it remains unchanged.

The proof does not depend on the concrete choice of γ and δ, so we choose $\gamma = 1$ and $\delta = 2$ for simplicity, and work with the norms

$$\|f\|_{H^{p,q,\alpha}_{(1,2)}} = \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + N_{(1,2)}(f)_{p,q,\alpha}, \quad (1.6)$$

and

$$\|f\|_{H^{p,q,\alpha}_{(\gamma',\delta')}} = \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + N_{(\gamma',\delta')}(f)_{p,q,\alpha}, \quad \gamma' < 1,\ \delta' > 2, \quad (1.7)$$

where we denoted

$$N_{(\gamma',\delta')}(f)_{p,q,\alpha} := \left\| t^{\alpha_\infty} \left\| f\chi_{(\gamma' t,\delta' t)} \right\|_{L^{p(\cdot)}} \right\|_{L^{q}((\gamma',\infty),\bar{d}t)} \quad (1.8)$$

for brevity. It is obvious that $\|f\|_{H^{p,q,\alpha}_{(1,2)}} \leq \|f\|_{H^{p,q,\alpha}_{(\gamma',\delta')}}$.

The estimation of $\|Tf\|_{L^{p(\cdot)}(B(0,2+\varepsilon))}$ is the same as in [1].

In estimation of $N_{(1,2)}(Tf)_{p,q,\alpha}$ we use the splitting:

$$f(x) = f_0(x) + f_t(x) + g_t(x) + h_t(x),$$

where

$$f_0(x) = f(x)\chi_{B(0,1/2)}(x), \quad f_t(x) = f(x)\chi_{B((0,\gamma' t)\setminus B(0,1/2)}(x),$$
$$g_t(x) = f(x)\chi_{B(0,\delta' t)\setminus B(0,\gamma' t)}(x), \quad h_t(x) = f(x)\chi_{\mathbb{R}^n\setminus B(0,\delta' t)}(x).$$

The term f_0 is the easiest one and needs only variable exponent Hölder inequality, the terms f_t and h_t are treated with application of the variable exponent Hardy inequality, while for g_t, which is in fact the term where changes should be made, we will use the fact that T is bounded on $L^{p(\cdot)}$.

Note that going out from the interval $(t, 2t)$ to a larger interval $(\gamma' t, \delta' t)$ with $\gamma' < 1$ and $\delta' > 2$ is due to estimations of the term Tg_t, but it does not appear in estimations of the terms Tf_0, Tf_t, Th_t.

Estimation of $Tf_0(x)$ and $Tf_t(x)$ is the same as in [1].

Estimation of $T(g_t)$. By the boundedness of the operator T in the space $L^{p(\cdot)}(\mathbb{R}^n)$ we obtain $\left\|(Tg_t)\chi_{t,2t}\right\|_{L^{p(\cdot)}} \le C\left\|g_t\right\|_{L^{p(\cdot)}} = C\left\|f\chi_{\gamma' t,\delta' t}\right\|_{L^{p(\cdot)}}$. Then

$$N_{(1,2)}(Tg_t)_{p,q,\alpha} \le C\|f\|_{H^{p,q,\alpha}_{(\gamma',\delta')}}.$$

In the case where q is constant, we take γ' sufficiently close to 1 and δ' close to 2, so that $\frac{\delta'}{\gamma'} < 4$ and then

$$\left\|(Tg_t)\chi_{t,2t}\right\|_{L^{p(\cdot)}} \le C\left\|f\chi_{\gamma' t,\delta' t}\right\|_{L^{p(\cdot)}} \le C\sum_{j=-1}^{1}\left\|f\chi_{2^j t,2^{j+1}t}\right\|_{L^{p(\cdot)}}.$$

Consequently,

$$\begin{aligned} &N_{(1,2)}(Tg_t)_{p,q,\alpha} \\ &\le C\left\|t^{\alpha_\infty}\left\|f\chi_{\frac{t}{2},t}\right\|_{L^{p(\cdot)}}\right\|_{L^q((2,\infty);\bar{d}t)} + C\left\|t^{\alpha_\infty}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^q((2,\infty);\bar{d}t)} \\ &\le C\left\|t^{\alpha_\infty}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^q((1,2);\bar{d}t)} + C\left\|t^{\alpha_\infty}\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}}\right\|_{L^q((2,\infty);\bar{d}t)}. \end{aligned}$$

Since $\left\|f\chi_{t,2t}\right\|_{L^{p(\cdot)}} \le \left\|f\chi_{0,4}\right\|_{L^{p(\cdot)}}$, we obtain

$$N_{(1,2)}(Tg_t)_{p,q,\alpha} \le C\|f\|_{H^{p,q,\alpha}_{(1,2)}}$$

with Lemma 3.2 of [1] taken into account.

Estimation of $T(h_t)$. We take $x \in R_{t,2t}$ and have:

$$|Th_t(x)| \le C\int_{|y|>\delta' t}\frac{|f(y)|\,dy}{|x-y|^n}.$$

To proceed, we need the inequality

$$\int\limits_{|y|>t}|\Phi(y)|dy \le \frac{1}{|\ln\lambda|}\int\limits_{\lambda t}^{\infty}\frac{d\tau}{\tau}\int\limits_{\tau<|y|<2\tau}|\Phi(y)|dy, \quad \frac{1}{2} \le \lambda < 1, \tag{1.9}$$

the proof of which is straightforward:

$$\begin{aligned} \int\limits_{\lambda t}^{\infty}\frac{d\tau}{\tau}\int\limits_{\tau<|y|<2\tau}|\Phi(y)|dy &= \int\limits_{|y|>\lambda t}|\Phi(y)|dy\int\limits_{\max(\lambda t,\frac{|y|}{2})}^{|y|}\frac{d\tau}{\tau} \\ &= \int\limits_{\lambda t<|y|<2\lambda t}|\Phi(y)|\ln\frac{|y|}{\lambda t}\,dy + \ln 2\int\limits_{|y|>2\lambda t}|\Phi(y)|dy \\ &\ge \ln\frac{1}{\lambda}\int\limits_{t<|y|<2\lambda t}|\Phi(y)|\,dy + \ln 2\int\limits_{|y|>2\lambda t}|\Phi(y)|dy \end{aligned}$$

$$\geq \ln \frac{1}{\lambda} \int\limits_{|y|>t} |\Phi(y)|dy.$$

By means of (1.9) we then get

$$|Th_t(x)| \leq C \int_{\lambda\delta' t}^{\infty} \left(\int\limits_{R_{\tau,2\tau}} \frac{|f(y)|}{|x-y|^n} dy \right) \bar{d}\tau.$$

Here $|x-y| \geq |y| - |x| \geq \tau - 2t \geq \tau - \frac{2}{\lambda\delta'}\tau = \frac{\lambda\delta'-2}{\lambda\delta'}\tau$. We may choose λ sufficiently close to 1 so that $\lambda\delta' > 2$ and then, with the property (4.17) of [1] taken into account, we obtain

$$t^{\alpha_\infty} \|\chi_{R_{t,2t}} Th_t(x)\|_{L^{p(\cdot)}} \leq C t^{\alpha_\infty} \|\chi_{R_{t,2t}}\|_{p(\cdot)} \int_{2t}^{\infty} \tau^{-\frac{n}{p_\infty}} \|f\chi_{\tau,2\tau}\|_{L^{p(\cdot)}} \bar{d}\tau$$
$$\leq C \int_t^{\infty} \left(\frac{t}{\tau}\right)^{\alpha_\infty + \frac{n}{p_\infty}} \psi(\tau)\, \bar{d}\tau,$$

where $\psi(\tau) = \tau^{\alpha_\infty} \|f\chi_{R_{\tau,2\tau}}\|_{p(\cdot)} \chi_{(2,\infty)}(\tau)$. Thus we have arrived at a Hardy type inequality. Since $\alpha_\infty + \frac{n}{p_\infty} > 0$, the operator on the right-hand side is bounded in the space $L^{q(\cdot)}((2,\infty); \bar{d}t)$ by Corollary 4.5 of [1]. Consequently,

$$\left\| t^{\alpha_\infty} \|\chi_{R_{t,2t}} Th_t(x)\|_{L^{p(\cdot)}} \right\|_{L^{q(\cdot)}((2,\infty);\bar{d}t)} \leq C\|\psi\|_{L^{q(\cdot)}(\mathbb{R}_+;\bar{d}t)} \leq C\|f\|_{H_2^{p,q,\alpha}}.$$

□

A similar modification in the form (1.5) should be also made in the boundedness statements in applications to maximal and singular operators in Corollaries 5.2 and 5.3 of [1].

Finally, we use this opportunity to note a misprint in formula (3.4) of [1]: the inequality $|\alpha(t) - \alpha_{\pm\infty}| \leq \frac{A_\pm}{\ln(e+t)}$ must be replaced by a couple of inequalities $|\alpha(t) - \alpha(+\infty)| \leq \frac{A_{+\infty}}{\ln(e+t)}$ and $|\alpha(t) - \alpha(0)| \leq \frac{A_0}{\ln(e+t)}$.

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