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MAXIMAL, POTENTIAL, AND SINGULAR OPERATORS IN THE GENERALIZED VARIABLE EXPONENT MORREY SPACES ON UNBOUNDED SETS

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We consider generalized Morrey spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ with a variable exponent $p(x)$ and a general function $\omega(x,r)$ defining a Morrey type norm. We extend the results obtained earlier for bounded sets $\Omega \subset \mathbb{R}^n$ by proving the boundedness of the Hardy–Littlewood maximal operator and Calderón–Zygmund singular operators with standard kernels in $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$. We prove a Sobolev type $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega) \to \mathscr{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$-theorem, both the Spanne and Adams versions, for potential operators $I^{\alpha(\cdot)}$, where $\alpha(x)$ can be variable even if Ω is unbounded. The boundedness conditions are formulated either in terms of Zygmund type integral inequalities on $\omega(x,r)$ or in terms of supremal operators. Bibliography: 36 *titles.*

1 Introduction

1.1. Background. The Morrey spaces $\mathscr{L}^{p,\lambda}$ introduced in [1] in relation to the study of partial differential equations are widely presented in the literature (cf., for example, [2]–[4]). We refer also to the recent survey paper [5], where various versions of Morrey type spaces and their generalizations can be found. Many classical operators of harmonic analysis (for example, maximal, singular, and potential operators) were studied in Morrey type spaces during the last decades. The Morrey spaces also attracted attention of researchers in the area of variable exponent analysis. We refer the reader to the recent monographs [6] and [7] for the existing results in variable exponent Lebesgue spaces (cf. also [8]).

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The Morrey spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ with variable exponents $\lambda(\cdot)$ and $p(\cdot)$ were introduced and studied in [9]–[13]. In [9], the space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ was introduced by means of the norm

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \inf\{\nu : I^{p(\cdot),\lambda(\cdot)}(f/\nu) \leqslant 1\} \tag{1.1}$$

via the modular

$$I^{p(\cdot),\lambda(\cdot)}(f) := \sup_{x\in\Omega, r>0} \frac{1}{r^{\lambda(x)}} \int\limits_{\widetilde{B}(x,r)} |f(y)|^{p(y)} dy.$$

In the case of bounded Ω, several equivalent norms can be introduced and embedding theorems for such Morrey spaces were proved under the assumption that $p(x)$ satisfies the log-condition.

In [11] and [10], Morrey type spaces $M^{q(\cdot)}_{p(\cdot)}$ were introduced in the general setting when the underlying space is a homogeneous type space (X, ρ, μ), with the norm

$$\|f\|_{M^{q(\cdot)}_{p(\cdot)}} = \sup_{x\in X, r>0} (\mu(B(x,r)))^{1/p(x)-1/q(x)} \|f\|_{L^{q(\cdot)}(B(x,r))},$$

where

$$1 < \inf_X q \leqslant q(\cdot) \leqslant p(\cdot) \leqslant \sup_X p < \infty.$$

For bounded X the equivalence of norms and embedding theorems were established there.

The spaces considered in [12] were defined by the condition

$$\frac{\varphi(r)}{r^\nu} \int\limits_{B(x,r)} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leqslant 1 \quad \text{for some } \lambda > 0.$$

A more general version $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, of generalized variable exponent Morrey spaces was defined in [14] by means of the norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

The space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is a particular case of such spaces with $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$.

A further generalization was presented in [15], where the L^∞-norm in r was replaced with the L^θ-norm in the definition of Morrey spaces, introduced by means of the norm

$$\sup_{x\in\Omega} \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\ell)},$$

where $\ell = \text{diam } \Omega$. The so-called complementary Morrey spaces of variable order were recently studied (cf. [16]) in the spirit of ideas of [14].

In this paper, we extend the results of [14] on the boundedness of maximal, singular and potential operators to unbounded sets in $\mathbb{R}^n$. Note that the study in [14] was essentially based on estimates for bounded sets. In [17], there was suggested an approach for extending variable order results from bounded sets to unbounded ones by interpreting such spaces over $\mathbb{R}^n$ as the space with mixed norm generated by discrete $\ell^{p(\cdot)}$ with respect to norms over a partition of $\mathbb{R}^n$ in cubes. We do not use this approach, but instead give direct proofs in intrinsic terms of $\mathbb{R}^n$ itself.

1.2. Operators under consideration. We study the following operators: *the maximal operator*

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy, \quad \text{where } \widetilde{B}(x.r) = B(x,r) \cap \Omega,$$

potential type operators

$$I^{\alpha(x)}f(x) = \int\limits_{\Omega} |x-y|^{\alpha(x)-n} f(y)dy, \quad 0 < \alpha(x) < n,$$

the fractional maximal operator of variable order $\alpha(x)$

$$M^{\alpha(x)}f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int\limits_{\widetilde{B}(x,r)} |f(y)|dy, \quad 0 \leqslant \alpha(x) < n,$$

and *Calderón–Zygmund type singular operators*

$$Tf(x) = \int\limits_{\Omega} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel," i.e., a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leqslant C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \leqslant C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leqslant C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|x-\xi|.$$

We emphasize that we prove both Spanne and Adams type theorems for potential operators. Although Adams type theorems provide a stronger estimate, theorems of Spanne type with weaker estimates have wider range of applicability: recall that, in the case of the classical Morrey spaces $\mathscr{L}^{p,\lambda}$, one can take $1 < p < n/\alpha$ for the Spanne estimate and $1 < p < (n-\lambda)/\alpha$ for the Adams estimate, which becomes more essential in the variable exponent setting.

The condition on $\omega(x,r)$ we find for the validity of a Sobolev–Adams type $\mathscr{L}^{p(\cdot),\omega(\cdot)}(\Omega) \to \mathscr{L}^{q(\cdot),\omega(\cdot)}(\Omega)$-theorem recovers the result in the case of the classical Morrey spaces with variable exponents obtained in [9] with the extension of the results of [9] to unbounded sets.

The paper is organized as follows. In Section 2, we recall necessary basic facts about variable exponent Lebesgue spaces and prove some auxiliary assertions. In particular, our estimates for unbounded sets Ω are based on Lemma 2.2 proved in Subsection 2.3. In Section 3, we introduce variable exponent Morrey spaces on unbounded sets. The general case is treated in Subsection 3.2, where we introduce several versions of generalized variable exponent Morrey spaces, discuss the equivalence of norms, and formulate the main results of the paper. The proofs of these results are given in Section 4. Section 5 is devoted to consequences of the main results in the case of classical variable exponent Morrey spaces, i.e., when $\omega(x,r) = r^{\frac{\lambda(x)}{p(x)}}$. One of the consequences (cf. Corollary 5.4) concerns Riesz potentials $I^{\alpha}(\cdot)$ of variable order over $\mathbb{R}^n$ in variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subseteq \mathbb{R}^n$ is an open set, $\ell = \operatorname{diam} \Omega$, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, and c, C, c_1, c_2 etc. are various absolute positive constants which may have different values even in the same line.

2 Preliminaries. Variable Exponent Lebesgue Spaces

2.1. Definitions. Let $p(\cdot)$ be a measurable function on an open set $\Omega \subseteq \mathbb{R}^n$ with values in $[1,\infty)$. We assume that $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$, but in most cases we suppose that $1 < p_- \leqslant p(x) \leqslant p_+ < \infty$. We denote by $L^{p(\cdot)}(\Omega)$ the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int\limits_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\{\eta > 0 : \ I_{p(\cdot)}(f/\eta) \leqslant 1\},$$

this space is a Banach function space. We denote by $p'(\cdot) = p(x)/(p(x)-1)$, $x \in \Omega$, the conjugate exponent.

We use the following notation:

$$p_- = p_-(\Omega) = \inf_{x\in\Omega} p(x), \quad p_+ = p_+(\Omega) = \sup_{x\in\Omega} p(x),$$

$\mathscr{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1,\infty]$, $\mathscr{P}^{\log}(\Omega)$ is the set of exponents $p \in \mathscr{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leqslant \frac{A_p}{-\ln|x-y|}, \quad |x-y| \leqslant \frac{1}{2}, \quad x, y \in \Omega, \tag{2.1}$$

where $A = A(p) > 0$ is independent of x and y, $\mathscr{A}^{\log}(\Omega)$ is the set of bounded exponents $\alpha : \Omega \to \mathbb{R}$ satisfying the condition (2.1), and $\mathbb{P}^{\log}(\Omega)$ is the set of exponents $p \in \mathscr{P}^{\log}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$. For Ω which can be unbounded we denote by $\mathscr{P}_\infty(\Omega), \mathscr{P}^{\log}_\infty(\Omega), \mathbb{P}^{\log}_\infty(\Omega), \mathscr{A}^{\log}_\infty(\Omega)$ the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded);

$$|p(x) - p(\infty)| \leqslant A_\infty \ln(2 + |x|), \quad x \in \mathbb{R}^n. \tag{2.2}$$

2.2. Basic theorems for operators in variable exponent Lebesgue spaces. We use the following boundedness result for the maximal operator (cf. [18]).

Theorem 2.1. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open set and $p \in \mathbb{P}^{\log}_\infty(\Omega)$. Then*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

For singular operators the following result is known (cf., for example, [19, 20, 6]).

Theorem 2.2. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $p \in \mathbb{P}^{\log}_\infty(\Omega)$. Then every Calderón–Zygmund singular operator with standard kernel and of weak $(1,1)$ type is bounded in $L^{p(\cdot)}(\Omega)$.*

The following theorem for fractional integrals was proved in [21].

Theorem 2.3. *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, $\alpha \in \mathscr{A}^{\log}(\Omega)$, $p \in \mathfrak{P}(\Omega)$, and*

$$\alpha_- := \inf_{x\in\Omega} \alpha(x) > 0, \quad (\alpha p)_+ := \sup_{x\in\Omega} \alpha(x)p(x) < n. \tag{2.3}$$

Then

$$\|I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)},$$

where $1/q(x) = 1/p(x) - \alpha(x)/n$ and $C = C(\Omega, p)$ depends only on $p_-(\Omega)$, $p_+(\Omega)$, A_p, α_-, $(\alpha p)_+$, and diam Ω.

For unbounded sets, say $\Omega = \mathbb{R}^n$, and constant orders α the corresponding Sobolev theorem proved in [18] runs as follows.

Theorem 2.4. *Suppose that $0 < \alpha < n$, $\Omega \subset \mathbb{R}^n$ is an open unbounded set, and $p \in \mathbb{P}^{\log}_\infty(\Omega)$. Let $p_+ < n/\alpha$. Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ with $1/q(x) = 1/p(x) - \alpha/n$.*

Such a Sobolev type theorem on $\mathbb{R}^n$ also holds for variable $\alpha(x)$ with an additional weight at infinity, as asserted by the next theorem proved in [22].

Theorem 2.5. *Suppose that $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$ and (2.3) holds. Then*

$$\|(1+|x|)^{-\gamma(x)} I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{2.4}$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}, \quad \gamma(x) = A_\infty \alpha(x)\left[1 - \frac{\alpha(x)}{n}\right] \leqslant \frac{n}{4} A_\infty$$

with A_∞ coming from (2.2).

2.3. Estimates of norms of truncated potentials. The following assertion with the estimation of the norms of potential kernels truncated to a ball is known (cf. Corollary to Lemma 3.22 in [23]; we also refer to [24] for a simpler proof). The boundedness of Ω is essentially used here. In Lemma 2.2, we extend this estimate to the case of unbounded sets Ω.

Lemma 2.1. *Suppose that Ω is a bounded domain and $p \in \mathscr{P}^{log}(\Omega)$. We also assume that $\nu_+ := \sup \nu(x) < \infty$ and $(n+\nu p)_- := \inf[n + \nu(x)p(x)] > 0$. Then*

$$\||x-y|^{\nu(x)} \chi_{B(x,r)}(y)\|_{p(y)} \leqslant C r^{\nu(x)+\frac{n}{p(x)}}, \tag{2.5}$$

where $x \in \Omega$, $0 < r < \ell =$ diam Ω, and C depends only on $p_-(\Omega)$, $p_+(\Omega)$, $A_p, \nu_+, (n+\nu p)_-$, and diam Ω.

Remark 2.1. It can be shown that the constant C in (2.5) can be estimated as $C = C_0\, \ell^{n(1/p_- - 1/p_+)}$, where C_0 is independent of Ω.

Let p satisfy the log-condition (2.1). The inequality

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\Omega)} \leqslant C r^{\frac{n}{p(x)}}, \tag{2.6}$$

for bounded open sets Ω is a particular case of (2.5). If Ω is unbounded and, in addition to (2.1), the exponent p satisfies the decay condition (2.2), then

$$\|\chi_{B(x,r)}\|_{p(\cdot)} \leqslant c r^{\theta_p(x,r)}, \quad x \in \Omega \subseteq \mathbb{R}^n, \quad p \in \mathscr{P}_\infty^{\log}, \tag{2.7}$$

where (cf. [6, Corollary 4.5.9])

$$\theta_p(x,r) = \begin{cases} n/p(x), & r \leqslant 1, \\ n/p(\infty), & r \geqslant 1. \end{cases}$$

Lemma 2.2. *Suppose that Ω is an unbounded open set, $p \in \mathscr{P}_\infty^{\log}(\Omega)$, and the function $\nu(x)$ satisfies the assumptions of Lemma 2.1 and additionally*

$$\inf_{x\in\Omega} [n + \nu(x) p(\infty)] > 0.$$

Then

$$\| \, |x-y|^{\nu(x)} \chi_{B(x,r)}\|_{p(\cdot)} \leqslant c r^{\nu(x)+\theta_p(x,r)}, \quad r > 0, \tag{2.8}$$

where $c > 0$ is independent of r and x.

Proof. Let $B_k(x,r) := B(x, 2^{-k}) \backslash B(x, 2^{-k-1})$. We have

$$\| \, |x-y|^{\nu(x)} \chi_{B(x,r)}\|_{p(\cdot)} \leqslant \sum_{k=0}^{\infty} \| \, |x-y|^{\nu(x)} \chi_{B_k(x,r)}\|_{p(\cdot)} \leqslant C \sum_{k=0}^{\infty} (2^{-k}r)^{\nu(x)} \|\chi_{B_(x,2^{-k}r)}\|_{p(\cdot)},$$

where $C = \max\{1, \sup_{x\in\Omega} 2^{-\nu(x)}\} < \infty$. By (2.7),

$$\|\chi_{B_(x,2^{-k}r)}\|_{p(\cdot)} \leqslant c (2^{-k}r)^{\theta_p(x,2^{-k}r)}.$$

Therefore,

$$\| \, |x-y|^{\nu(x)} \chi_{B(x,r)}\|_{p(\cdot)} \leqslant C \sum_{k=0}^{\infty} (2^{-k}r)^{\nu(x)+\theta_p(x,2^{-k}r)} \leqslant \frac{C}{\ln 2} \int_0^r t^{\nu(x)+\theta_p(x,t)} \frac{dt}{t}, \tag{2.9}$$

where the last passage to the integral is verified in the standard way with the monotonicity of the function $t^{\nu(x)+\theta_p(x,t)}$ in t taken into account:

$$\int_0^r t^{\nu(x)+\theta_p(x,t)} \frac{dt}{t} = \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} t^{\nu(x)+\theta_p(x,t)} \frac{dt}{t}$$

$$\geqslant \sum_{k=0}^{\infty} (2^{-k}r)^{\nu(x)+\theta_p(x,2^{-k}r)} \int_{2^{-k-1}r}^{2^{-k}r} \frac{dt}{t} = \ln 2 \sum_{k=0}^{\infty} (2^{-k}r)^{\nu(x)+\theta_p(x,2^{-k}r)}.$$

It remains to note that

$$\int_0^r t^{\nu(x)+\theta_p(x,t)} \frac{dt}{t} = \frac{r^{\nu(x)+\frac{n}{p(x)}}}{\nu(x)+\frac{n}{p(x)}}$$

if $0 < r \leqslant 1$ and

$$\int_0^r t^{\nu(x)+\theta_p(x,t)} \frac{dt}{t} = \frac{1}{\nu(x)+\frac{n}{p(x)}} + \frac{r^{\nu(x)+\frac{n}{p(\infty)}} - 1}{\nu(x)+\frac{n}{p(\infty)}}$$

if $r \geqslant 1$, so that (2.9) implies (2.8). □

3 Generalized Variable Exponent Morrey Spaces. Definitions and the Main Results

3.1. Classical variable exponent Morrey spaces. Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ can be introduced via the norm

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 3.1. Such spaces were defined in [9] by means of the norm

$$\sup_{x\in\Omega,\, r>0} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f\chi_{\widetilde{B}(x,r)} \right\|_{p(\cdot)}$$

which is equivalent to the above norm when Ω is bounded and p satisfies the log-condition.

The following theorems were proved in [9] for bounded sets Ω.

Theorem 3.1. *Suppose that Ω is bounded, $p \in \mathbb{P}^{\log}(\Omega)$, and $\lambda(x) \geqslant 0$, $\sup_{x\in\Omega} \lambda(x) < n$. Then the maximal operator M is bounded in $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

Theorem 3.2 (Spanne type result)**.** *Suppose that Ω is bounded, $p \in \mathbb{P}^{\log}(\Omega)$, and $\alpha, \lambda \in \mathscr{A}^{\log}(\Omega)$. We also assume that $\lambda(x) \geqslant 0$ and the condition (2.3) holds. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathscr{L}^{q(\cdot),\mu(\cdot)}(\Omega)$, where $1/q(x) = 1/p(x) - \alpha(x)/n$ and*

$$\frac{\mu(x)}{q(x)} = \frac{\lambda(x)}{p(x)}. \tag{3.1}$$

Theorem 3.3 (Adams type result)**.** *Suppose that Ω is bounded, $p \in \mathbb{P}^{\log}(\Omega)$, and $\alpha, \lambda \in \mathscr{A}^{\log}(\Omega)$. We aslo assume that $\lambda(x) \geqslant 0$ and*

$$\inf_{x\in\Omega} \alpha(x) > 0, \quad \sup_{x\in\Omega} [\lambda(x) + \alpha(x)p(x)] < n. \tag{3.2}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathscr{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$, where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}. \tag{3.3}$$

3.2. Generalized variable exponent Morrey spaces. In order to avoid confusions, we use the letter $\mathscr{M}$ for generalized Morrey spaces defined via a general function $\omega(x,r)$ (cf. (3.4)) and the letter $\mathscr{L}$ for classical Morrey spaces when $\omega(x,r)=r^{\lambda(x)/p(x)}$.

Hereinafter, $\omega(x,r)$, $\omega_1(x,r)$, and $\omega_2(x,r)$ are nonnegative measurable functions on $\Omega\times[0,\ell]$, where $\Omega\subseteq\mathbb{R}^n$ is an open set and $\ell=\operatorname{diam}\Omega$.

Definition 3.1. Let $p\in\mathscr{P}(\Omega)$. The generalized Morrey space $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ is defined by means of the norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}=\sup_{x\in\Omega,r>0}\frac{\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}{\omega(x,r)}. \tag{3.4}$$

Hereinafter, we assume that

$$\inf_{x\in\Omega}\omega(x,r)>0 \tag{3.5}$$

for every $r>0$, which makes the space $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ nontrivial.

By the definition of the $L^{p(\cdot)}$-norm,

$$\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}=\sup_{x\in\Omega,r>0}\inf\left\{\eta=\eta(x,r):\int\limits_{\Omega}\left|\frac{f(y)\chi_{\widetilde{B}(x,r)}(y)}{\eta\omega(x,r)}\right|^{p(y)}dy\leqslant 1\right\}. \tag{3.6}$$

The spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ include, in particular, classical type Morrey spaces with different measuring of the Morrey property for small and large values of r, i.e., the spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot),\lambda_\infty(\cdot)}(\Omega)$ defined by means of the norm

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot),\lambda_\infty(\cdot)}}=\sup_{x\in\Omega}\left(\sup_{0<r<1}r^{-\frac{\lambda(x)}{p(x)}}\|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}+\sup_{r>1}r^{-\frac{\lambda_\infty(x)}{p(x)}}\|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}\right)$$

corresponding to the choice

$$\omega(x,r)=\begin{cases}r^{\lambda(x)}, & r\leqslant 1,\\ r^{\lambda_\infty(x)}, & r\geqslant 1.\end{cases}$$

The norm (3.6) prompts us to introduce the norm

$$\|f\|^*_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}=\sup_{x\in\Omega,r>0}\inf\left\{\eta=\eta(x,r):\frac{1}{\omega(x,r)^{p(x)}}\int\limits_{\Omega}\left|\frac{f(y)}{\eta}\right|^{p(y)}dy\leqslant 1\right\}. \tag{3.7}$$

The norms $\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}$ and $\|f\|^*_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}$ are not equivalent in general, and we denote by $\mathscr{M}^{p(\cdot),\omega(\cdot)}_*(\Omega)$ the space of functions f equipped with finite norm $\|f\|^*_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}$. Lemma 3.2 contains conditions under which the spaces $\mathscr{M}^{p(\cdot),\omega(\cdot)}_*(\Omega)$ and $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ coincide.

In the spirit of (3.7), we can introduce the corresponding versions $\mathscr{L}^{p(\cdot),\lambda(\cdot)}_*(\Omega)$ of classical type Morrey spaces, defined similarly to (3.7) by means of the norm

$$\|f\|^*_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}}=\sup_{x\in\Omega,r>0}\inf\left\{\eta=\eta(x,r):\frac{1}{r^{\lambda(x)}}\int\limits_{\widetilde{B}(x,r)}\left|\frac{f(y)}{\eta}\right|^{p(y)}dy\leqslant 1\right\}. \tag{3.8}$$

We single out the case where

$$\omega(x,r)\equiv\text{const}\quad\text{for } r\geqslant 1, \tag{3.9}$$

i.e., the case where the "Morrey regularity" is measured only for small r. The Morrey space with a function $\omega(x,r)$ satisfying (3.9) can be called a *locally introduced* Morrey space.

In Lemma 3.2, we use the log-condition in the form

$$|p(x)-p(y)|\cdot|\ln\omega(x,r)|\leqslant C,\quad x,y\in\Omega,\quad |x-y|\leqslant r\leqslant 1, \tag{3.10}$$

where c is independent of x, y, and r. The following lemma provides us with a sufficient condition for the validity of (3.10).

Lemma 3.1. *The condition* (3.10) *is satisfied if* $p\in\mathscr{P}(\Omega)$, *the function* w *is bounded, fulfills the condition* (3.5), *and satisfies the inequality* $\omega(x,r)\geqslant C_0r^a$, $a\geqslant 0$, *in a neighborhood* $0\leqslant r\leqslant\varepsilon$ *of the origin.*

Proof. It suffices to consider the case $\omega(x,r)\leqslant 1/2$ (otherwise, there is nothing to prove in (3.10)). We can assume that $C_0=1$ and $a>0$. Then

$$\ln\frac{1}{\omega(x,r)}\leqslant a\ln\frac{1}{r}\leqslant a\ln\frac{1}{|x-y|},$$

so that the usual log-condition for p implies (3.10). □

Lemma 3.2. *Suppose that* $p(x)\in\mathscr{P}$ *and* $\omega(x,r)$ *satisfies* (3.9) *and* (3.10). *Then the norms* $\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}$ *and* $\|f\|^*_{\mathscr{M}^{p(\cdot),\omega(\cdot)}}$ *are equivalent.*

Proof. It suffices to prove that $c_1\omega(x,r)^{p(x)}\leqslant\omega(x,r)^{p(y)}\leqslant c_2\omega(x,r)^{p(x)}$, which follows from (3.10) under the condition (3.9). □

3.2.1. Theorems for the maximal operator.

Theorem 3.4. *Suppose that* $p\in\mathbb{P}^{\log}_\infty(\Omega)$. *Then*

$$\|Mf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\leqslant Ct^{\theta_p(x,t)}\sup_{r>2t}r^{-\theta_p(x,r)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},\quad t>0, \tag{3.11}$$

for every $f\in L^{p(\cdot)}(\Omega)$, *where* C *is independent of* f, $x\in\Omega$, *and* t.

Theorem 3.5. *Suppose that* $p\in\mathbb{P}^{\log}_\infty(\Omega)$ *and*

$$\sup_{t>r}\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\omega_1(x,s)}{t^{\theta_p(x,t)}}\leqslant C\,\frac{\omega_2(x,r)}{r^{\theta_p(x,r)}}, \tag{3.12}$$

where C *is independent of* x *and* r. *Then the maximal operator* M *is bounded from the space* $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ *to the space* $\mathscr{M}^{p(\cdot),\omega_2(\cdot)}(\Omega)$.

Note that supremal estimates for the maximal operators in generalized Morrey spaces were obtained in [25] in the case of constant exponents.

3.2.2. Theorems for singular operators.

Theorem 3.6. *Suppose that* $p\in\mathbb{P}^{\log}_\infty(\Omega)$. *Then*

$$\|Tf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\leqslant Ct^{\theta_p(x,t)}\int_t^\infty r^{-\theta_p(x,r)-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr, \tag{3.13}$$

where C *is independent of* f *and* $t>0$.

Theorem 3.7. *Suppose that* $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$ *and*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \omega_1(x,s)}{t^{1+\theta_p(x,t)}} dt \leqslant C \frac{\omega_2(x,r)}{r^{\theta_p(x,r)}}, \tag{3.14}$$

where C *is independent of* x *and* r. *Then the singular integral operator* T *is bounded from the space* $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ *to the space* $\mathscr{M}^{p(\cdot),\omega_2(\cdot)}(\Omega)$.

Remark 3.2. The condition (3.12) is weaker than the condition (3.14). Indeed, if the condition (3.14) holds, then for all $s \in (r,\infty)$

$$\frac{\omega_2(x,r)}{r^{\theta_p(x,r)}} \gtrsim \int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \omega_1(x,\tau)}{t^{\theta_p(x,t)}} \frac{dt}{t} \gtrsim \int_s^{\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \omega_1(x,\tau)}{t^{\theta_p(x,t)}} \frac{dt}{t}$$

$$\gtrsim \operatorname*{ess\,inf}_{s<\tau<\infty} \omega_1(x,\tau) \int_s^{\infty} \frac{ds}{s^{\theta_p(x,s)+1}} \approx \frac{\operatorname*{ess\,inf}_{s<\tau<\infty} \omega_1(x,\tau)}{s^{\theta_p(x,s)}}.$$

Then

$$\sup_{s>r} \frac{\operatorname*{ess\,inf}_{s<\tau<\infty} \omega_1(x,\tau)}{s^{\theta_p(x,s)}} \lesssim \frac{\omega_2(x,r)}{r^{\theta_p(x,r)}}.$$

On the other hand, the functions $\omega_1(x,t) = \omega_2(x,t) = t^{\theta_p(x,t)}$ satisfy (3.12), but not (3.14).

3.2.3. Theorems for the fractional operator.

Theorem 3.8. *Assume that* $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$ *and a constant* α *satisfies* (2.3). *Then*

$$\|I^{\alpha} f\|_{L^{q(\cdot)}(\widetilde{B}(x,t))} \leqslant C t^{\theta_q(x,t)} \int_t^{\infty} r^{-\theta_q(x,r)-1} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} dr, \quad t>0, \tag{3.15}$$

where $1/q(x) = 1/p(x) - \alpha/n$ *and* C *is independent of* f, x, *and* t.

Theorem 3.9 (Spanne type result)**.** *Suppose that* α *and* p *satisfy the assumptions of Theorem* 3.8 *and*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \omega_1(x,s)}{t^{1+\theta_p(x,t)}} dt \leqslant C \frac{\omega_2(x,r)}{r^{\theta_q(x,r)}}, \tag{3.16}$$

where $1/q(x) = 1/p(x) - \alpha/n$ *and* C *is independent of* x *and* r. *Then the operators* M^{α} *and* I^{α} *are bounded from the space* $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ *to the space* $\mathscr{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 3.10 (Spanne type result with variable $\alpha(x)$)**.** *Suppose that* $p \in \mathbb{P}^{\log}_{\infty}(\mathbb{R}^n)$, α *satisfies* (2.3), *and*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \omega_1(x,s)}{t^{1+\theta_p(x,t)}} dt \leqslant C \frac{\omega_2(x,r)}{r^{\theta_q(x,r)}}, \tag{3.17}$$

where $1/q(x) = 1/p(x) - \alpha(\cdot)/n$ *and* C *is independent of* x *and* r. *Then the operators*

$$\frac{1}{(1+|x|)^{\gamma(x)}} M^{\alpha} \quad \textit{and} \quad \frac{1}{(1+|x|)^{\gamma(x)}} I^{\alpha(\cdot)}$$

are bounded from $\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\mathbb{R}^n)$ *to* $\mathscr{M}^{q(\cdot),\omega_2(\cdot)}(\mathbb{R}^n)$, *where* $\gamma(x)$ *comes from* (2.4).

Theorem 3.11. *Suppose that $p \in \mathscr{P}^{\log}_\infty(\Omega)$ and $\alpha(x)$ is a measurable function satisfying* (2.3)*. Then*

$$|I^{\alpha(\cdot)}f(x)| \leqslant Ct^{\alpha(x)}\, Mf(x) + C\int\limits_t^\infty r^{\alpha(x)-\theta_p(x,r)-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}dr, \tag{3.18}$$

where t is an arbitrary positive number and C is independent of f, x, and t.

In the following theorem with variable $\alpha(x)$, we establish the mapping property of the algebraic sum of two Morrey spaces with variable and constant exponents $q(x)$ and $q(\infty)$. Recall that the sum $X+Y$ of two Banach spaces is defined via the norm

$$\|f\|_{X+Y} := \inf_{\substack{f=g+h\\ g\in X,\, h\in Y}} (\|g\|_X + \|h\|_Y)$$

We use the notation

$$p_r = \begin{cases} p(x), & r\leqslant 1,\\ p(\infty), & r>1, \end{cases} \qquad q(x,r) = \begin{cases} q(x), & r\leqslant 1,\\ q_\infty(x), & r>1, \end{cases}$$

where $q(x) > p(x)$ and $q_\infty(x) > p(\infty)$.

Theorem 3.12 (Adams type result)**.** *Suppose that $p \in \mathbb{P}^{\log}_\infty(\Omega)$, $\alpha(x)$ satisfies* (2.3)*, and $\omega(x,t)$ satisfies the following conditions*:

$$\int\limits_r^1 \frac{\omega(x,t)}{t^{1+\frac{n}{p(x)}}}dt \leqslant C\,\frac{\omega(x,r)}{r^{\frac{n}{p(x)}}}, \quad r\in(0,1), \tag{3.19}$$

$$\int\limits_r^\infty \frac{\omega(x,t)}{t^{1+\frac{n}{p(\infty)}}}dt \leqslant C\,\frac{\omega(x,r)}{r^{\frac{n}{p(\infty)}}}, \quad r\in\mathbb{R}_+, \tag{3.20}$$

$$\int\limits_r^\infty \frac{\omega(x,t)}{t^{1+\theta_p(x,t)-\alpha(x)}}dt \leqslant Cr^{-\frac{\alpha(x)p_r(x)}{q(x,r)-p_r(x)}}, \tag{3.21}$$

where C is independent of $x\in\Omega$ and $r\in\mathbb{R}_+$. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathscr{M}^{p(\cdot),\omega(\cdot)}_*(\Omega)$ to $\mathscr{M}^{q(\cdot),\omega^*(\cdot)}_*(\Omega) + \mathscr{M}^{q_\infty(x),\omega_1(\cdot)}(\Omega)$, where $\omega_1(x,r) := [\omega(x,r)]^{\frac{p(x)}{q(x)}}$.

If Ω is bounded and $q(r,x) = q(x)$ is independent of r, then from (3.21) it follows that

$$\frac{1}{q(x)} \geqslant \frac{1}{p(x)} - \frac{\alpha(x)}{m(x)}, \quad m(x) = p(x)\left[\alpha(x) - \varlimsup_{r\to 0}\frac{\ln\int\limits_r^\ell t^{\alpha(x)-1}w(r,t)\,dt}{\ln r}\right].$$

The corresponding exponent $q^\#(x)$ given by the equality

$$\frac{1}{q^\#(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{m(x)} \tag{3.22}$$

can be called the *Sobolev-Adams type exponent* corresponding to the space $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\Omega)$. In particular, for the classical variable exponent Morrey space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ (the case $\omega(x,r) = r^{\frac{\lambda(x)}{p(x)}}$) from (3.22) we recover the Adams exponent defined by $1/q(x) = 1/p(x) - \alpha(x)/(n-\lambda(x))$ under the assumption (3.2).

Remark 3.3. For $p(x) = \text{const}$ the boundedness results in classical Morrey spaces go back to [26]–[28] for singular operators, [29] for the maximal operator, and [27, 30] for fractional integrals. For Morrey spaces with constant p, but a general function $\omega(x,r)$ such results under certain assumptions were obtained in [31]–[34].

4 Proofs

We begin with auxiliary assertions for supremal and Hardy operators. Denote by $L^\infty(\mathbb{R}_+, v)$ the space of all functions $g(t)$, $t>0$, equipped with finite norm

$$\|g\|_{L^\infty(\mathbb{R}_+,v)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t).$$

As usual, $L^\infty(\mathbb{R}_+) = L^\infty(\mathbb{R}_+,1)$. Let $\mathfrak{M}(\mathbb{R}_+)$ be the set of all Lebesgue measurable functions on $\mathbb{R}_+$, and let $\mathfrak{M}^+(\mathbb{R}_+)$ be its subset of all nonnegative functions. We denote by $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ the cone of all nondecreasing functions in $\mathfrak{M}^+(\mathbb{R}_+)$ and set $\mathscr{A} = \big\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+}\varphi(t) = 0\big\}$. Let u be a continuous nonnegative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ for $g \in \mathfrak{M}(\mathbb{R}_+)$ as follows: $(\overline{S}_u g)(t) := \|u\, g\|_{L^\infty(t,\infty)}$, $t \in \mathbb{R}_+$.

The following theorem was proved in [35].

Theorem 4.1. *Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L^\infty(t,\infty)} < \infty$ for every $t>0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L^\infty(\mathbb{R}_+,v_1)$ to $L^\infty(\mathbb{R}_+,v_2)$ on the cone $\mathscr{A}$ if and only if*

$$\big\|v_2\overline{S}_u\big(\|v_1\|^{-1}_{L^\infty(\cdot,\infty)}\big)\big\|_{L_\infty(\mathbb{R}_+)} < \infty. \tag{4.1}$$

Let w be a weight on $\mathbb{R}_+$. The following assertion concerning the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

was proved in [36] in the case $w \equiv 1$.

Theorem 4.2. *Suppose that v_1, v_2, and w are weights on $\mathbb{R}_+$. Then the inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w^* g(t) \leqslant C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t) \tag{4.2}$$

holds with some $C>0$ for all nonnegative and nondecreasing g on $\mathbb{R}_+$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t)\int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}\limits_{s<\tau<\infty} v_1(\tau)} < \infty \tag{4.3}$$

and $C = B$ is the best constant in (4.2).

Proof. *Sufficiency.* Assume that (4.3) holds. If F, G are nonnegative functions on $\mathbb{R}_+$ and F is nondecreasing, then

$$\operatorname*{ess\,sup}_{t>0} F(t)G(t) = \operatorname*{ess\,sup}_{t>0} F(t) \operatorname*{ess\,sup}_{t>0} G(s), \quad t > 0. \tag{4.4}$$

By (4.4), we have

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w^* g(t) = \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty g(s)w(s) \frac{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} ds$$

$$\leqslant \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} \operatorname*{ess\,sup}_{t>0} g(t) \operatorname*{ess\,sup}_{t<\tau<\infty} v_1(\tau)$$

$$= \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} \operatorname*{ess\,sup}_{t>0} g(t)v_1(t) \leqslant B \operatorname*{ess\,sup}_{t>0} g(t)v_1(t).$$

Necessity. Assume that (4.2) holds. The function

$$g(t) = \frac{1}{\operatorname*{ess\,sup}_{t<\tau<\infty} v_1(\tau)}, \quad t > 0,$$

is nonnegative and nondecreasing on $(0, \infty)$. Thus,

$$B = \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} \leqslant C \operatorname*{ess\,sup}_{t>0} \frac{v_1(t)}{\operatorname*{ess\,sup}_{t<\tau<\infty} v_1(\tau)} \leqslant C.$$

Hence $B < \infty$ and B is the best constant. □

In the proofs below, we take $\Omega = \mathbb{R}^n$ without losing generality.

Lemma 4.1. *Assume that* $p \in \mathscr{P}_\infty$ *and* $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$\|f\|_{L^{p(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,t)} \sup_{r>t} r^{-\theta_p(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))}, \tag{4.5}$$

$$\|f\|_{L^{p(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,t)} \int_t^\infty r^{-\theta_p(x,r)-1} \|f\|_{L^{p(\cdot)}(B(x,r))} dr. \tag{4.6}$$

Proof. It suffices to observe that $\|f\|_{L^{p(\cdot)}(B(x,t))}$ is nondecreasing in t and

$$1 \leqslant Ct^{\theta_p(x,r)} \sup_{r>t} r^{-\theta_p(x,r)}$$

in the case of (4.5) and

$$1 \leqslant Ct^{\theta_p(x,r)} \int_t^\infty r^{-\theta_p(x,r)-1} dr$$

in the case of (4.6). □

Proof of Theorem 3.4. We split f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\mathbb{R}^n\setminus B(x,2t)}(y), \quad t > 0. \tag{4.7}$$

By Theorem 2.1,

$$\|Mf_1\|_{L^{p(\cdot)}(B(x,t))} \leqslant \|Mf_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} = C\|f\|_{L^{p(\cdot)}(B(x,2t))}, \tag{4.8}$$

where C is independent of f. By (4.5),

$$\|Mf_1\|_{L^{p(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,r)} \sup_{r>t} r^{-\theta_p(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))}. \tag{4.9}$$

Let y be an arbitrary point in $B(x,r)$. If $B(y,t) \cap \complement(B(x,2r)) \neq \varnothing$, then $t > r$. Indeed, if $z \in B(y,t) \cap \complement(B(x,2r))$, then $t > |y-z| \geqslant |x-z| - |x-y| > 2r - r = r$. On the other hand, $B(y,t) \cap \complement(B(x,2r)) \subset B(x,2t)$. Indeed, for $z \in B(y,t) \cap \complement(B(x,2r))$ we get $|x-z| \leqslant |y-z| + |x-y| < t + r < 2t$. Hence

$$\begin{aligned} Mf_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|} \int\limits_{B(y,t)\cap \complement(B(x,2r))} |f(z)|dz \\ &\leqslant 2^n \sup_{t>r} \frac{1}{|B(x,2t)|} \int\limits_{B(x,2t)} |f(z)|dz = 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(z)|dz. \end{aligned}$$

Therefore, for all $y \in B(x,r)$

$$\begin{aligned} Mf_2(y) &\leqslant 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(z)|dz \leqslant 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \|f\|_{L^{p(\cdot)}(B(x,t))} \|1\|_{L^{p'(\cdot)}(B(x,t))} \\ &\leqslant C \sup_{t>2r} \|f\|_{L^{p(\cdot)}(B(x,t))}\, t^{-n+\theta_{p'}(x,t)} = C \sup_{t>2r} \|f\|_{L^{p(\cdot)}(B(x,t))}\, t^{-\theta_p(x,t)}. \end{aligned}$$

Thus, the function $Mf_2(y)$, with fixed x and t, is dominated by the expression independent of y. Then

$$\|Mf_2\|_{L^{p(\cdot)}(B(x,t))} \leqslant C \sup_{t>2r} \|f\|_{L^{p(\cdot)}(B(x,t))}\, t^{-\theta_p(x,t)}\, \|1\|_{L^{p(\cdot)}(B(x,t))}. \tag{4.10}$$

Since $\|1\|_{L^{p(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,t)}$, we obtain (3.11) from (4.9) and (4.10). □

Proof of Theorem 3.5. By Theorem 3.4, for the norm

$$\|Mf\|_{\mathscr{M}^{p(\cdot),\omega_2}(\Omega)} = \sup_{x\in\Omega,\, t>0} \omega_2^{-1}(x,t)\|Mf\|_{L^{p(\cdot)}(B(x,t))}$$

we have

$$\begin{aligned} \|Mf\|_{\mathscr{M}^{p(\cdot),\omega_2(\cdot)}(\Omega)} &\leqslant C \sup_{x\in\Omega,\, t>0} \omega_2^{-1}(x,t) t^{\theta_p(x,t)} \sup_{r>t} r^{-\theta_p(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))} \\ &\leqslant C \sup_{x\in\Omega,\, t>0} \omega_1^{-1}(x,t)\|f\|_{L^{p(\cdot)}(B(x,t))} = \|f\|_{\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)} \end{aligned}$$

in view of (3.12), which completes the proof. □

Proof of Theorem 3.6. Representing f as in (4.7), we have

$$\|Tf\|_{L^{p(\cdot)}(B(x,t))} \leqslant \|Tf_1\|_{L^{p(\cdot)}(B(x,t))} + \|Tf_2\|_{L^{p(\cdot)}(B(x,t))}.$$

By Theorem 2.2,

$$\|Tf_1\|_{L^{p(\cdot)}(B(x,t))} \leqslant \|Tf_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_1\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

so that

$$\|Tf_1\|_{L^{p(\cdot)}(B(x,t))} \leqslant C\|f\|_{L^{p(\cdot)}(B(x,2t))}$$

and from (4.6) it follows that

$$\|Tf_1\|_{L^{p(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,r)} \int\limits_t^\infty r^{-\theta_p(x,r)-1}\|f\|_{L^{p(\cdot)}(B(x,r))}dr. \tag{4.11}$$

To estimate $\|Tf_2\|_{L^{p(\cdot)}(B(x,t))}$, we observe that

$$|Tf_2(z)| \leqslant C \int\limits_{\mathbb{R}^n\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$, and the inequalities $|x-z| \leqslant t$ and $|z-y| \geqslant 2t$ imply the inequality $\frac{1}{2}|z-y| \leqslant |x-y| \leqslant \frac{3}{2}|z-y|$. Therefore,

$$\|Tf_2\|_{L^{p(\cdot)}(B(x,t))} \leqslant C \int\limits_{\mathbb{R}^n\setminus B(x,2t)} |x-y|^{-n}|f(y)|dy\|\chi_{B(x,t)}\|_{L^{p(\cdot)}\mathbb{R}^n)}.$$

By the Hölder inequality and the estimate (2.7), we get

$$\|Tf_2\|_{L^{p(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,r)} \int\limits_t^\infty r^{-\theta_p(x,r)-1}\|f\|_{L^{p(\cdot)}(B(x,r))}dr. \tag{4.12}$$

From (4.11) and (4.12) we arrive at (3.13). □

Proof of Theorem 3.7. For the norm

$$\|Tf\|_{\mathscr{M}^{p(\cdot),\omega_2(\cdot)}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\omega_2(x,t)}\|Tf\chi_{B(x,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{4.13}$$

we estimate $\|Tf\chi_{B(x,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ by means of Theorem 3.6 and obtain

$$\begin{aligned}\|Tf\|_{\mathscr{M}^{p(\cdot),\omega_2(\cdot)}(\mathbb{R}^n)} &\leqslant C \sup_{x\in\mathbb{R}^n,\, t>0} \frac{t^{\theta_p(x,t)}}{\omega_2(x,t)} \int\limits_t^\infty r^{-\theta_p(x,t)-1}\|f\|_{L^{p(\cdot)}(B(x,r))}dr \\ &\leqslant C \sup_{x\in\mathbb{R}^n,\, t>0} \omega_1(x,t)^{-1}\|f\|_{L^{p(\cdot)}(B(x,t))} = \|f\|_{\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

It remains to use the condition (3.12). □

Proof of Theorem 3.8. Representing f in the form (4.7), we have

$$I^\alpha f(x) = I^\alpha f_1(x) + I^\alpha f_2(x).$$

By Theorem 2.4,

$$\|I^\alpha f_1\|_{L_{q(\cdot)}(B(x,t))} \leqslant \|I^\alpha f_1\|_{L_{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_1\|_{L_{p(\cdot)}(\mathbb{R}^n)} = C\|f\|_{L_{p(\cdot)}(B(x,2t))}.$$

By Lemma 4.1,

$$\|I^\alpha f_1\|_{L_{q(\cdot)}(B(x,t))} \leqslant Ct^{\theta_q(x,t)} \int\limits_{2t}^{\infty} r^{-\theta_q(x,t)-1}\|f\|_{L_{p(\cdot)}(B(x,r))}dr. \tag{4.14}$$

If $|x-z| \leqslant t$ and $|z-y| \geqslant 2t$,, we have $\frac{1}{2}|z-y| \leqslant |x-y| \leqslant \frac{3}{2}|z-y|$. Therefore,

$$\begin{aligned}\|I^\alpha f_2\|_{L_{q(\cdot)}(B(x,t))} &\leqslant \Bigg\| \int\limits_{\mathbb{R}^n\setminus B(x,2t)} |z-y|^{\alpha-n} f(y)dy \Bigg\|_{L_{q(\cdot)}(B(x,t))} \\ &\leqslant C \int\limits_{\mathbb{R}^n\setminus B(x,2t)} |x-y|^{\alpha-n}|f(y)|dy \ \|\chi_{B(x,t)}\|_{L_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Choosing $\beta > n/q_-$, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^n\setminus B(x,2t)} |x-y|^{\alpha-n}|f(y)|dy &= \beta \int\limits_{\mathbb{R}^n\setminus B(x,2t)} |x-y|^{\alpha-n+\beta}|f(y)| \Bigg(\int\limits_{|x-y|}^{\infty} s^{-\beta-1}ds \Bigg) dy \\ &= \beta \int\limits_{2t}^{\infty} s^{-\beta-1} \Bigg(\int\limits_{\{y\in\mathbb{R}^n : 2t\leqslant|x-y|\leqslant s\}} |x-y|^{\alpha(x)-n+\beta}|f(y)|dy \Bigg) ds \\ &\leqslant C \int\limits_{2t}^{\infty} s^{-\beta-1}\|f\|_{L_{p(\cdot)}(B(x,s))}\||x-y|^{\alpha-n+\beta}\|_{L_{p'(\cdot)}(B(x,s))}ds \\ &\leqslant C \int\limits_{2t}^{\infty} s^{\alpha-\theta_p(x,s)-1}\|f\|_{L_{p(\cdot)}(B(x,s))}ds,\end{aligned}$$

where the estimate (2.8) was taken into account in the last passage. Therefore,

$$\|I^\alpha f_2\|_{L_{q(\cdot)}(B(x,t))} \leqslant Ct^{\theta_p(x,t)} \int\limits_{2t}^{\infty} s^{-\theta_q(x,s)-1}\|f\|_{L_{p(\cdot)}(B(x,s))}ds$$

which, together with (4.14), yields (3.15). ☐

Proof of Theorem 3.9. By Theorem 3.8,

$$\begin{aligned}\|I^\alpha f\|_{\mathscr{M}^{q(\cdot),\omega_2(\cdot)}(\mathbb{R}^n)} &\leqslant C \sup_{x\in\mathbb{R}^n,\, t>0} \frac{t^{\theta_q(x,t)}}{\omega_2(x,t)} \int\limits_t^{\infty} r^{-\theta_q(x,r)-1}\|f\|_{L^{p(\cdot)}(B(x,r))}dr \\ &\leqslant C \sup_{x\in\mathbb{R}^n,\, t>0} \omega_1(x,t)^{-1}\|f\|_{L^{p(\cdot)}(B(x,t))} = \|f\|_{\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

It remains to use the condition (3.17). $\square$

Proof of Theorem 3.10. The proof is the same as that of Theorem 3.9, i.e., it is obtained in the same way from the estimate of Theorem 3.8 because the estimate (3.15) remains valid for variable $\alpha(x)$ if we replace I^α by $\frac{1}{(1+|x|)^{\gamma(x)}} I^{\alpha(\cdot)}$. For this purpose, it suffices to use Theorem 2.5 instead of Theorem 2.4 at the beginning of the proof of Theorem 3.9. $\square$

Proof of Theorem 3.11. As usual, we represent the function f in the form (4.7) and have

$$I^{\alpha(\cdot)} f(x) = I^{\alpha(\cdot)} f_1(x) + I^{\alpha(\cdot)} f_2(x).$$

For $I^{\alpha(\cdot)} f_1(x)$, following Hedberg's trick (cf., for example, [21, p. 278] in the case of variable exponents), we obtain

$$\left| I^{\alpha(\cdot)} f_1(x) \right| \leqslant C_1 t^{\alpha(x)} M f(x).$$

For $I^{\alpha(\cdot)} f_2(x)$ we have

$$\left| I^{\alpha(\cdot)} f_2(x) \right| \leqslant \int\limits_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{\alpha(x)-n} |f(y)| dy = \frac{1}{n-\alpha(x)} \int\limits_{\mathbb{R}^n \setminus B(x,2t)} |f(y)| dy \int\limits_{|x-y|}^{\infty} r^{\alpha(x)-n-1} dr.$$

Hence

$$|I^{\alpha(\cdot)} f_2(x)| \leqslant C \int\limits_{2t}^{\infty} \Bigg(\int\limits_{2t<|x-y|<r} |f(y)| dy \Bigg) r^{\alpha(x)-n-1} dr \leqslant C \int\limits_{t}^{\infty} \|f\|_{L^{p(\cdot)}(B(x,r))} r^{\alpha(x)-\theta_p(x,r)-1} dr,$$

where (2.7) was taken into account. This proves (3.18). $\square$

Proof of Theorem 3.12. Since $M^{\alpha(\cdot)} f(x) \leqslant C(I^{\alpha(\cdot)}|f|)(x)$, it suffices to consider only the operator $I^{\alpha(\cdot)}$. In view of Theorem 3.11 and the assumption (3.21), we get

$$|I^{\alpha(\cdot)} f(x)| \leqslant C r^{\alpha(x)} M f(x) + C r^{-\frac{\alpha(x) p_r(x)}{q_r(x)-p_r(x)}} \|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)} \tag{4.15}$$

for every $f \in \mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)$ and $r > 0$. We split the function $f(x) = f_\ell(x) + f_s(x)$ into "large" and "small" parts $f_\ell(x)$ and $f_s(x)$ as follows. Let

$$E = E_f := \left\{ x \in \mathbb{R}^n : \ M f(x) \geqslant \|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)} \right\},$$

$$\complement E := \left\{ x \in \mathbb{R}^n : \ M f(x) < \|f\|_{\mathscr{M}^{p(\cdot)\omega(\cdot)}(\mathbb{R}^n)} \right\}.$$

We put $f_\ell(x) = f(x)\chi_E(x)$ and $f_{\mathrm{s}}(x) = f(x)\chi_{\complement E}(x)$. The inequality (4.15) holds for both f_ℓ and f_s and arbitrary $r > 0$. With the goal of minimization, we choose r such that

$$r^{\alpha(x)} M f(x) = r^{-\frac{\alpha(x) p_r(x)}{q_r(x)-p_r(x)}} \|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)}$$

with different choice of r for f_ℓ and f_s:

$$r = \left(\frac{\|f\|}{M f(x)} \right)^{\frac{1}{\alpha(x)} \left(1 - \frac{p(x)}{q(x)} \right)} \leqslant 1$$

for f_ℓ and

$$r = \left(\frac{\|f\|}{Mf(x)}\right)^{\frac{1}{\alpha(x)}\left(1-\frac{p(\infty)}{q_\infty(x)}\right)} > 1$$

for f_s, where $\|f\|$ stands for $\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)}$, under the assumption that f is not identically equal to zero. Then

$$|I^{\alpha(\cdot)}f_\ell(x)| \leqslant C(Mf_\ell(x))^{\frac{p(x)}{q(x)}} \|f\|^{1-\frac{p(x)}{q(x)}}_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)},$$

$$|I^{\alpha(\cdot)}f_s(x)| \leqslant C(Mf_s(x))^{\frac{p(\infty)}{q(\infty)}} \|f\|^{1-\frac{p(\infty)}{q(\infty)}}_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)}.$$

Since the operator $I^{\alpha(\cdot)}$ is linear, it suffices to show the boundedness of the corresponding modulars under the assumption that $\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)} \leqslant 1$. We have

$$\int\limits_{B(x,r)} \left|I^{\alpha(\cdot)}f_\ell(y)\right|^{q(y)} dy \leqslant C \int\limits_{B(x,r)} |Mf_\ell(y)|^{p(y)}\, dy,$$

$$\int\limits_{B(x,r)} \left|I^{\alpha(\cdot)}f_s(y)\right|^{q_\infty(y)} dy \leqslant C \int\limits_{B(x,r)} |Mf_s(y)|^{p(\infty)}\, dy.$$

Hence

$$\|I^{\alpha(\cdot)}f_\ell\|_{\mathscr{M}^{q(\cdot),\omega_1(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_\ell\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)}$$

because of the boundedness of the maximal operator M in the space $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)$ provided by Theorem 3.5. To apply this theorem, we need the condition (3.12). It is satisfied in view of the assumptions (3.19) and (3.20). Finally, the estimate

$$\|I^{\alpha(\cdot)}f_s\|_{\mathscr{M}^{p(\cdot),\omega_1(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_s\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)}$$

follows from Theorem 3.5 in view of the condition (3.20). □

5 Corollaries

Corollary 5.1. *Let $p \in \mathbb{P}^{\log}_\infty(\Omega)$. If*

$$\lambda(x) \geqslant 0, \quad \sup_{x\in\Omega} \lambda(x) < n, \quad \sup_{x\in\Omega} \frac{\lambda(x)}{p(x)} < \frac{n}{p(\infty)}, \tag{5.1}$$

then the maximal operator M and the Calderón-Zygmund operator T are bounded in $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

It suffices to apply Theorems 3.5 and 3.7 with $\omega_1(x,r) = \omega_2(x,r) = r^{\frac{\lambda(x)}{p(x)}}$.

Corollary 5.1 was proved in [9] for bounded sets Ω where the last condition in (5.1) does not appear.

Corollary 5.2. *Let α and p satisfy the assumptions of Theorem 3.8. If*

$$\lambda(x) \geqslant 0, \quad \sup_{x\in\Omega}[\lambda(x) + \alpha p(x)] < n, \quad \sup_{x\in\Omega} \frac{\lambda(x)}{p(x)} < \frac{n}{p(\infty)} - \alpha, \tag{5.2}$$

then the fractional maximal operator M^α and the fractional integration operator I^α are bounded from $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathscr{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$, where $1/q(x) = 1/p(x) - \alpha/n$.

It suffices to apply Theorem 3.9 with $\omega_1(x,r) = \omega_2(x,r) = r^{\frac{\lambda(x)}{p(x)}}$.

Corollaries 5.1 and 5.2 were proved in [9] for bounded sets Ω, where the last condition in (5.1) and (5.2) does not appear.

Corollary 5.3. *Suppose that* $p \in \mathbb{P}^{\log}_\infty(\Omega)$ *and* $\alpha(x)$ *satisfies* (2.3). *If*

$$\lambda(x) \geqslant 0, \quad \sup_{x\in\Omega} \frac{\lambda(x)}{p(x)} < \frac{n}{p(\infty)}, \tag{5.3}$$

$$\sup_{x\in\Omega}[\lambda(x) + \alpha(x)p(x)] < n, \quad \sup_{x\in\Omega}[\lambda(x)\frac{p(\infty)}{p(x)} + \alpha(x)p(\infty)] < n, \tag{5.4}$$

then the fractional maximal operator M^α *and the fractional integration operator* I^α *are bounded from* $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *to* $\mathscr{L}^{q(\cdot),\lambda(\cdot)}(\Omega) + \mathscr{L}^{q_\infty(\cdot),\lambda_\infty(\cdot)}(\Omega)$, *where*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}, \quad \frac{1}{q_\infty(x)} = \frac{1}{p(\infty)} - \frac{\alpha(x)}{n-\lambda_\infty(x)}, \quad \lambda_\infty(x) = \frac{p(\infty)}{p(x)}\lambda(x).$$

It suffices to apply Theorem 3.12 with $\omega(x,r) = r^{\frac{\lambda(x)}{p(x)}}$. The relation $\lambda_\infty(x) = \frac{p(\infty)}{p(x)}\lambda(x)$ is obtained from (3.21) with $r > 1$.

In particular, from Corollary 5.3 for variable exponent spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (the case $\lambda(x) = \lambda_\infty(x) \equiv 0$) we obtain the following assertion.

Corollary 5.4. *Suppose that* $p \in \mathbb{P}^{\log}_\infty(\Omega)$ *and* $1/q(x) = 1/p(x) - \alpha(x)/n$. *If*

$$\inf_{x\in\Omega} \alpha(x) > 0, \quad \sup_{x\in\Omega} p(x)\alpha(x) < n, \quad p(\infty)\sup_{x\in\Omega} \alpha(x) < n,$$

then the fractional maximal operator M^α *and the fractional integration operator* I^α *are bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to the algebraic sum* $\mathscr{L}^{q(\cdot)}(\mathbb{R}^n) + \mathscr{L}^{q(\infty)}(\mathbb{R}^n)$.

Corollary 5.4 seems to be never mentioned earlier in variable exponent analysis.

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